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THEORY OF HIGH POWER ELECTRON
CYCLOTRON RESONANCE HEATING

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THESIS SUBMITTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
OF THE UNIVERSITY OF ST ANDREWS

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THEORY OF HIGH POWER ELECTRON CYCLOTRON RESONANCE HEATING

Abstract

Electron cyclotron resonance heating has been successfully used on a series of experiments in an attempt to raise plasma temperatures beyond the constraints of the resistive dissipation which occurs with ohmic heating. Recently progress in gyrotron design has allowed for significant increases in applied microwave power and for the first time a free electron laser will generate high power pulsed radio-frequency waves in the MTX experiment at Lawrence Livermore Laboratory in 1987.

Classically the theory of ECRH has been considered by a Fokker-Planck approach and by a quasilinear approach. Both lead to a diffusion equation in velocity space for the distribution function but as the applied power increases the approximations made in these approaches are likely to become unsatisfactory. Adopting a test particle approach we firstly consider modifications to the velocity space diffusion co-efficient at high powers and then dispense with the diffusion equation completely.

We begin by deriving averaged particle equations from a Lagrangian formulation which require less computer processor time to integrate than the exact Lorentz-force equations. These have been incorporated in a particle code to simulate ECRH in a tokamak. The results for this code are compared with analytic expressions derived for a modified diffusion coefficient and a probability function $P(\underline{v}, \Delta \underline{v})$.

We show that for low fields the diffusive form is correct but for higher fields nonlinear effects become important.

Acknowledgements

It is with regret and sadness that I must thank those below. Not because I deserve all the credit myself, the reverse is true, but because I know in the future, I shall see too few of them, too infrequently for my liking.

Firstly I must thank my supervisor Dr Alan Cairns for his expert guidance and advice. My thanks also go to the rest of the Plasma Physics Group, Professor Jeffrey Sanderson, Dr Lindsay Wood, Dr Anthony Kay, Morag McMinn and Alastair McGowan for their helpful support. I would also like to thank the rest of the Mathematics Institute staff for their encouragement and infectious enthusiasm in particular the Maths Institute 5-a-side football team who have left me with many happy memories and bruises!

Secondly my thanks go to my colleagues and friends at Culham Laboratory, Dr Chris Lashmore-Davies, Dr Richard Dendy, Martin Cox, and Giles Pickering for their assistance and especially Martin O'Brien for many illuminating and informative discussions and comments.

I must also thank Miss Shiela Wilson for deciphering my handwriting and typing the thesis and Mrs Anne Cunningham and Mrs Georgina Rumgay for their cheery conversation on many a dreich St Andrews morning.

I must also express my gratitude to the Science and Engineering Research Council and the United Kingdom Atomic Energy Authority for their financial support.

Finally I would like to thank all my flatmates and friends for their support, encouragement and cooking! Especially Alan, Janet, Bob and Katie, and "The Feen", "Meat", "Spankie" (forget the cooking!), "Mince", "The Chief", "The McCann Man" and "Le Jour Oiseau", otherwise known as Ally Feeney, Alastair Gray, Ivan Smith, Nick Warburton, Stephen Baxter, John McCann and Jon Day.

A.W.T.

Certificate

I certify that Allan W Taylor has satisfied the conditions of the Ordinance and Regulations and is thus qualified to submit the accompanying application for the Degree of Doctor of Philosophy.

Postgraduate Career

I was admitted into the University of St Andrews as a research student under Ordinance General No. 12 in October 1984 to work on the theory of electron cyclotron heating under the supervision of Dr R A Cairns. I was admitted under the above resolution as a candidate for the Degree of Ph.D. in October 1985.

Declaration

I declare that the following thesis is a record of research work carried out by me, that the thesis is my own composition, and that it has not previously been submitted in application for a higher degree.

FOR MUM, DAD AND ROBERT

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CHAPTER 1

Introduction

1.1 An Historical Perspective of Fusion

Nuclear fusion energy was discovered by F W Aston, who published his pioneering measurements of the nuclear masses of the light elements in 1919 Aston (1919), (1920). These showed the mass of a helium-4 atom to be about 1% less than that of 4 hydrogen atoms which were, at that time, believed to be its essential constituents. The excess mass of the hydrogen, which can be equated to a source of energy by Einsteins relation $E = mc^2$, would be released when hydrogen is fused into helium. When Rutherford, at about the same time, showed that light elements colliding together can undergo nuclear transformations, astrophysicists suggested not only that this fusion could occur but deduced it to be the power-house of the stars. Thus began the quest by man to harness the energy source which had, since the beginning of the universe, sustained him. The aspirations of those early physicists were embodied by Eddington (1920) in his British Association Lecture:

"we sometimes dream that man will one day learn to release it (sub-atomic energy) and use it for his service".

The theoretical possibility of helium being formed in thermo-nuclear fusion reactions in high temperature hydrogen, was shown by Atkinson and Houtermans (1929). The demonstration of the first fusion reaction with practical potential, the deuterium-deuterium reaction,

was achieved by Oliphant (1934), and the first between deuterium and tritium at Purdue University in 1942 by Schrieber and King. Subsequent measurements by Bretscher and French (1948) of the large deuterium-tritium cross-section led this reaction to be adopted as the most promising reaction for the practical release of controlled nuclear energy. Uncontrolled nuclear fusion energy has of course been available since the early 1950's with the development of the hydrogen bomb, so in an ironic sense Eddington's prophecy if not his dream has been fulfilled already.

It was the major international conference on the Peaceful Uses of Atomic Energy in 1958 that showed the idea of producing high temperature plasmas and confining them with magnetic fields was plausible and exploratory investigations were begun in the USA, USSR and Western Europe. In the United Kingdom the story began with the patent, filed by G P Thomson and M Blackman (1946), for a toroidal discharge as a basis of a nuclear fusion reactor with the first experiments on toroidal discharges taking place at Imperial College and Oxford, see Thonemann (1958).

In one of the first studies on fusion and energy confinement by Adams (1959) the three main methods of containment were identified.

1. Mirror Machines

These open-ended devices depend on the reflection of charged particles by regions of strong magnetic field. Early experiments with radioactive sources in such arrangements of strong magnetic fields showed that individual particles could be readily contained for a million transits. However on larger experiments magnetic instabilities

such as the flute instability increased particle diffusion across field lines. These effects together with anomolous transport effects make Mirror Machines unlikely to ever reach reactor conditions. An overview of the confinement loss problems is given by Baldwin (1977).

2. Stellarators

The stellarator concept was devised at Princeton and its early development has been recorded by Rose and Clark (1961), Artsimovich (1964) and Miyamoto (1978). In order to combat the $\underline{E} \times \underline{B}$ drift helical conductors twist the magnetic field lines so that free electron movement along the field lines cancels the electric field. It is possible to run stellarators in a currentless mode as the field lines act as a trap for plasma even without a current. Several stellarators are under investigation at the present time including CLEO at Culham Laboratory, HELIOTRON-E at Kyoto, Japan and WVIIA at the Max-Planck Institute, Garching. However because of the complication of the helical winding large stellarators have proved more difficult to build than other toroidal devices like Tokamaks and Reverse Field Pinches.

3. Tokamaks and Reverse Field Pinches

The atmosphere of euphoria which surrounded the announcement that neutrons had been observed from the ZETA device at Harwell in 1957 was soon dispelled when it became clear that the plasmas were not hot enough to sustain thermonuclear reactions. The shadows over fusion research lengthened as problems with plasma instabilities set serious barriers to obtaining high temperatures and it was not until 1969 that neutrons of demonstrably thermonuclear origin were produced in a

Russian toroidal pinch or "tokamak" device. This led to a rapid expansion of fusion research in particular with tokamak devices which is still continuing today.

1.2 Cold Plasma Theory

A hot magnetised plasma supports a wide variety of wave modes. For modes with phase velocity well above the plasma thermal velocity the behaviour of waves is predicted by the cold plasma dispersion relation see Boyd and Sanderson (1969). Many properties of this relation are summarised in the CMA diagram, see Figure (1.1). The cold plasma dispersion relation is derived from the linearised fluid equations with zero plasma pressure and through the Lorentz force to Maxwells equations. In Figure (1.1)

$$R = 1 - \sum_{s=e,i} \frac{w_{ps}^2}{w^2} \left(\frac{w}{w + \Omega_s} \right) \quad (1)$$

$$L = 1 - \sum_{s=e,i} \frac{w_{ps}^2}{w^2} \left(\frac{w}{w - \Omega_s} \right) \quad (2)$$

$$P = 1 - \sum_{s=e,i} \frac{w_{ps}^2}{w^2} \quad (3)$$

$$S = \frac{1}{2} (R + L) \quad (4)$$

where

w = wave frequency

w_{ps} = plasma frequency for species s

Ω_s = cyclotron frequency for species s

$$w_{ps} = \left(\frac{n_s q_s^2}{\epsilon_0 m_s} \right)^{\frac{1}{2}} \quad \Omega_s = \frac{q_s B}{m_s}$$

with

q_s = charge on species s

n_s = density of species s

m_s = mass of species s

(after Allis et al. (1963)).

and ϵ_0 = permittivity of free space

B = magnetic field strength

and α^2 and β are proportional to the plasma density and magnetic field strength respectively.

In cold plasma theory wave modes are either propagating or evanescent. A propagating wave will travel through the plasma until its refractive index becomes zero or infinite, corresponding to a cut-off or resonance. In the neighbourhood of a cold plasma resonance the phase velocity of the wave drops to thermal levels and substantial interaction between the wave and plasma can occur. The cold plasma dispersion relation can be written in the form

$$\tan^2 \vartheta = - \frac{P(n^2 - R)(n^2 - L)}{(Sn^2 - RL)(n^2 - P)} \quad (5)$$

(see Boyd and Sanderson (1969)) where ϑ is the angle between the background magnetic field and the wave propagation vector. If we consider modes propagating parallel ($\vartheta = 0$) and perpendicular ($\vartheta = \frac{\pi}{2}$) to the background magnetic field equation (5) reduces to

$$\vartheta = 0 \quad P = 0 \quad n^2 = R \quad n^2 = L$$

$$\vartheta = \frac{\pi}{2} \quad n^2 = \frac{RL}{S} \quad n^2 = P.$$

For $\vartheta = 0$ we have the plasma oscillation ($P = 0$) which is an electrostatic mode, and two circularly polarised modes $n^2 = R$, the right circularly polarised mode and $n^2 = L$, the left circularly polarised mode. These electromagnetic modes have resonances at $R = \infty$ (the electron cyclotron resonance) and $L = \infty$ (the ion cyclotron resonance) and cut-offs at $R = 0$ and $L = 0$ as shown in Figure (1.1). Similarly

for perpendicular propagation we have the ordinary mode ($n^2 = P$) and the extraordinary mode ($n^2 = RL/S$). The ordinary mode is a plane polarised mode with a cut-off at $P = 0$ and no cold plasma resonance. The extraordinary mode is elliptically polarised and has cut-offs at $R = 0$ and $L = 0$ and resonances at $R = \infty$ (electron cyclotron), $L = \infty$ (ion cyclotron) and $S = 0$ (upper hybrid).

To describe wave-particle interactions in experimental regimes a more sophisticated approach is required.

1.3 Warm Plasma Theory

The Vlasov-Maxwell equations of Stix (1962) and Clemmow and Dougherty (1969) reveal many new waves which propagate near the ion and electron cyclotron frequencies. Kinetic theory also introduces the theory of collisionless damping of waves, with the resonance condition

$$\omega - n\Omega_s - k_{\parallel}v_{\parallel s} = 0 \quad n = 0, 1, 2, \dots \quad (6)$$

ω - wave frequency

n - harmonic number

Ω_s - cyclotron frequency for species s

$k_{\parallel}$ - wave number parallel to the background magnetic field

$v_{\parallel s}$ - free streaming velocity for species s .

$n = 0$ corresponds to Landau damping. It was in 1946 that L. D. Landau first realised the existence of a damping mechanism which would absorb energy from oscillations in a collision free plasma.

In this regime the resonance condition (6) reduces to the Doppler resonance condition. For longitudinal waves propagating obliquely to the magnetic field energy can be transferred to and from the wave to particles whose velocity is close to the phase velocity of the wave. Particles whose velocity is slightly less than the phase velocities of the wave, "surf the wave", increasing their velocity in the direction of wave propagation and consequently reducing the magnitude of the wave. Particles whose velocity is slightly greater than that of the wave give up energy to the wave and the magnitude of the wave grows. Consequently for equilibrium plasmas with a

Maxwellian distribution longitudinal waves are damped and those plasmas with a distribution function with a bump in the tail near the phase velocity of the wave excite the wave by inverse Landau damping or Landau growth, see Chen (1984), Boyd and Sanderson (1969) and Stix (1962).

Setting $n \neq 0$ in (6) gives rise to the resonance condition for cyclotron damping. For cyclotron damping the right or left hand circularly polarised electric field component drives the ions and electrons as will be discussed in more detail later. Suffice to say that warm plasma theory introduces many new modes and many new resonances to cold plasma theory. For example the O-mode which had no resonances in cold plasma theory is resonant at the cyclotron frequency and all of its harmonics when warm plasma effects are considered. Important accessibility conditions do carry through from cold plasma theory.

If we consider an X-mode wave propagating from the outside of a tokamak into the plasma, then, as is shown by the bottom left hand corner of the CMA diagram, the X-mode is reflected at the low-density cyclotron cut-off ($R = 0$) before it reaches the upperhybrid resonance. Whereas the O-mode wave injected similarly is unaffected by these cut-offs and resonances passing through to the cyclotron resonances at the electron cyclotron resonance. Launching the X-mode from the centre of the torus overcomes this problem as the upper hybrid resonance is accessible from the high field side.

1.4 Tokamaks and Additional Heating Schemes

The magnetic field required for stable plasma confinement has two components in a tokamak. A toroidal field, B_ϕ , provided by external coils and a weaker poloidal field B_θ which is generated by the toroidal current in the plasma. This current is induced by transformer action and heats the plasma by resistive dissipation, see Figures (1.2) and (1.3). Typically currents of up to 5MA have resulted in ion and electron temperatures of 2.9 and 3.8 keV respectively in pulses lasting a few seconds, see Bickerton (1986). However the plasma resistivity is typically of the form

$$\eta = \frac{Z e^2 m_e^{1/2} \log \Lambda}{3^{1/2} 12 \pi \epsilon_0^2 (K T_e)^{3/2}} \quad (7)$$

see Cairns (1985) and consequently resistive heating is less efficient at temperatures of 2-3 keV, necessitating the use of auxilliary heating to reach the temperatures required for fusion (~ 10 keV). These auxilliary heating schemes are of two types.

1. Neutral Beam Injection

Powerful ion beams are extracted and accelerated from multi-aperture ion sources. The ion beams are neutralised by passage through gas cells to give beams of neutral atoms, see Figure (1.4). These cross the confining magnetic fields and enter the plasma where they are ionised by collisions. The resulting trapped fast ions (energies 30-80 keV) then transfer their energy to the bulk plasma by collisional processes. Although plasma temperatures can be raised significantly by this technique, for example electron heating

Figure 1.2. Schematic of a Tokamak.

Figure 1.3. Distribution of Toroidal and Poloidal Fields in a Tokamak.

Figure 1.4. Diagram of PLT Tokamak showing
Neutral Beam Injector.

(after Fisch (1987))

of the order of 0.5 eV kW^{-1} of beam power has been recorded at PDX by Eubank (1982) this has usually been accompanied by degradation of energy confinement and enhanced sawteeth activity, Bickerton (1986), Gill (1986).

2. Radio Frequency Heating

In radio frequency heating energy is transported into the plasma by high frequency electromagnetic waves. These waves propagate through the plasma to a localised region where they are absorbed by one of several mechanisms. The four main r.f. heating schemes are, in order of increasing frequency, Alfven wave heating, ion cyclotron frequency heating (ICRF), lower hybrid resonance heating (LHRH) and electron cyclotron resonance heating (ECRH). Alfven wave heating, in the range of a few MHz relies on the excitation of a localised Alfven wave where the local Alfven velocity v_A is given by

$$v_A = \frac{w}{k_{||}}$$

where w and $k_{||}$ are the frequency and parallel wavenumber of the applied wave. The energy of the wave is damped in the plasma as it travels along the magnetic field lines. Recent reviews of theory and experimental results for Alfven wave heating include Appert et al (1986) and Behn et al (1984).

In ICRF the applied radiation lies in the frequency range 25-100 MHz. The power is absorbed at the ion cyclotron resonance $15.2 \text{ MHz (Tesla)}^{-1}$ and its harmonics, and the ion-ion hybrid resonance which can exist if there are two ion species in the plasma with differing mass charge ratio. The energy is fed into the plasma by means of large

antennae connected to an amplifier through co-axial cables, see Figure (1.5). The electromagnetic waves are converted to electrostatic waves near the hybrid resonance and their energy is subsequently absorbed by Landau damping or cyclotron damping see Stix (1975). Recently up to 3MW in 2 second pulses have been used on the JET tokamak, see Jacquinet et al (1985) and this will be increased to 20MW by July 1987. Although electron temperatures have been raised to 3 keV giant sawteeth disruptions of the central electron temperature have been seen together with a drop in ohmic energy confinement, Figure (1.6). A study of particle confinement under ICRH has been made by Whang and Morales (1983).

Lower hybrid resonance heating (LHRH) uses frequencies at the lower hybrid resonance which lies between the ion and electron cyclotron frequencies. The main attraction of heating in this frequency range $\sim 1-5$ GHz is that no antenna coil is required (as in ICRH) and waveguide techniques can be used to couple the power to the plasma. At the lower hybrid resonance

$$w_{LH}^2 = w_{pi}^2 / (1 + w_{pe}^2 / w_{ce}^2) \quad (8)$$

cold plasma slow waves are mode converted into ion plasma or Bernstein waves which are subsequently Landau damped or absorbed at the ion cyclotron harmonics, see Stix (1965). Both the electron and ion temperatures have been raised using LHRH, typically $\Delta T_e = 400$ eV see Gormezano et al (1982) on the WEGA tokamak and Porkolab et al (1984) on ALCATOR C.

The highest frequency range r.f. heating is electron cyclotron resonance heating (ECRH).

Figure 1.5. Diagram of JET ICRF System.
(after Jacquinot et al. (1985))

Figure 1.6. Sawteeth in Electron
Temperature during an ICRF pulse on JET.
(after Jacquinot et al. (1985))

1.5 Review of ECRH and Current Drive Theory

Electron cyclotron resonance heating is the simplest form of radio frequency heating and uses frequencies in the region 25-150 GHz. Until recent times the main drawback with ECRH was the lack of high power r.f. sources at these frequencies. However over the past decade both the frequency and output power of gyrotron oscillations have increased rapidly. Earlier results include cw devices that have generated powers ranging from 200 kW at 28 GHz (Jory et al (1980)) to 22 kW at 150 GHz (Andronov et al (1978)). In addition short pulse gyrotrons have produced powers in excess of 100 kW, see Carmel et al (1983), and Kreischer et al (1985) have examined the possibility of constructing gyrotrons of up to 10 MW at 120 GHz. Recently the possibility of producing microwave power using pulsed free electron layers (FEL's) has been described by Orzechowski et al (1986). The proposed MTX experiment at the Lawrence Livermore Laboratory will use a pulsed FEL to heat the ALCATOR-C tokamak, see Nevins et al (1987).

Recent reviews of ECRH theory include Boratici et al (1983), Hsu et al (1983) and Litvak et al (1977).

The resonance condition for ECRH can be written in the form

$$|(\omega - n\omega_{ce})| \leq k_{\parallel} v_{the}. \quad (9)$$

Thus as the tokamak magnetic field varies with major radius R this limits the spatial extent of the resonance region to

$$\Delta R \sim 2R |k_{\parallel} v_{\text{the}} / n w_{\text{ce}}| \quad (10)$$

which, in general, is much smaller than the minor diameter of the plasma. Consequently by tuning the frequency of the incoming wave it is possible to deposit energy in the plasma in a very localised form. As can be seen from the high frequency part of the CMA diagram Figure (1.1) the O-mode propagates until the density rises such that the plasma frequency = wave frequency. If the wave frequency is greater than the cyclotron frequency then the X-mode is reflected before the O-mode when it encounters the low density cyclotron cut-off $R = 0$. If the wave frequency is less than the cyclotron frequency, the X-mode propagates until the high density cut-off is encountered,

$$w = \left| \frac{w_{\text{ce}}}{2} \right| + \left(\frac{w_{\text{ce}}^2}{4} + w_{\text{pe}}^2 \right)^{\frac{1}{2}}. \quad (11)$$

Thus for the X-mode to reach the upper hybrid resonance $w^2 = w_{\text{pe}}^2 + w_{\text{ce}}^2$, it must be injected from the inside of the torus, not attractive from engineering considerations. Both the X-mode and the O-mode undergo cyclotron damping as they approach the electron cyclotron frequency. With the exception of the X-mode at parallel incidence, thermal corrections to the cold plasma wave are small and wave attenuation coefficients are

$$\chi_{\pm} = \frac{\beta_e f_{\pm}(\vartheta, \nu) e^{-\xi_{\pm}^2}}{|w(\xi_{\pm})|^2} \quad (12)$$

where

$$\beta_e = v_{\text{the}}/c \quad \nu = w_{\text{pe}}^2/w^2$$

$$\xi_{\pm} = (w - |w_{\text{ce}}|)/(w n_{\pm} \beta_e \cos \vartheta)$$

where $n_{\pm}$ are the cold plasma refractive indices given by the Appleton-Hartree dispersion relation.

The functions $f_{\pm}(\theta, \nu)$ have been calculated by Litvak et al (1977) and are shown in Figure (1.7). It is clear that 0-mode absorption is greatest at large angles of incidence. The power absorbed by the resonant layer is given by $1 - e^{-\tau}$ where

$$\tau \propto R \frac{n_e K T_e}{B_0} \quad (\text{see Fielding 1982})$$

where

- R - major radius of tokamak
- n_e - electron density
- T_e - electron temperature
- K - Boltzmann's constant
- B_0 - background magnetic field.

Consequently for large tokamaks the plasma is optically thick when $T_e \geq 1$ keV.

Absorption in smaller devices is still significant though due to multiple reflections at the vacuum vessel. Launching X-mode waves from the inside of the torus, to avoid the low density cut-off has been suggested. Maekawa et al (1977) have described how the X-mode undergoes conversion to an electron Bernstein wave at the upper hybrid resonance. The Bernstein waves then propagate almost perpendicular to the magnetic field into the higher density region to be damped at the cyclotron resonance.

ECRH has also been applied at the 2nd harmonic on CLEO at Culham, see Lloyd et al (1985). At second harmonic heating the X-mode is absorbed preferentially over all angles to the 0-mode.

Figure 1.7. Angular Dependence of Absorbtion Coefficients.
(after Litvak et al. (1977))

Well localised power deposition is possible with the X-mode which can be launched from the low field side at the second harmonic.

Another attractive feature of ECRH is the ability to modify the electron temperature profile $T_e(r)$ accurately to stabilise certain modes. Any change in the temperature profile has a corresponding change in the current profile $j_e(r)$ as the plasma resistivity is linked to its temperature by an equation of the form of (7). Internal disruptions giving rise to sawteeth relaxation at the centre of tokamaks are seen as the current density profile becomes more peaked and the condition $q < 1$ is reached, see Riviere (1986). In Ohmic heating regimes a tokamak plasma can display strong $n = 1$, $m = 2$ oscillations owing to the development of a tearing mode at the $q = 2$ surface.

It has been shown in several experiments, see T-10 group (1983) that the amplitude and frequency of the sawtooth oscillation can be modified by choosing the resonance position of the ECRH zone. Heating at centre tends to enhance sawteeth activity where as heating off axis has been shown to suppress this behaviour. In general the change in the electron temperature profile alters the current profile so that q (centre of plasma) > 1 , for stabilisation. Similar techniques have been applied to the CLEO tokamak, Robinson (1985), however offsetting the resonant region too far off axis resulted in the triggering of some soft global disruptions as the $m = 2$ mode became too unstable.

Because the toroidal current in tokamaks is produced by transformer action the physical structure is pulse-loaded. This is undesirable from an engineering point of view and several studies are

being carried out into non-inductive current drive with the aim of maintaining the poloidal magnetic field and toroidal current continuously. Lower hybrid waves are very effective for current drive producing a high energy tail in the parallel electron velocity distribution function. Several experiments have recently combined ECRH with LHRH, see Hoshino et al (1985), and Maekawa et al (1983). Because of the electron tail $T_{\parallel} > T_{\perp}$ and a velocity space instability occurs sporadically seen as sharp increases of voltage limiting the current. The addition of ECRH inhibits the instability by increasing $T_{\perp}$ leading to a more isotropic distribution, see Porkolab et al (1983). ECRH has also been seen to improve the current drive efficiency of LHRH on JFT-2M (Hoshino et al (1985)) and WT-II (Ando et al (1986)) where up to 2 kA currents driven by 50 kW LHRH were increased to 4 kA by the application of 50 kW ECRH.

Current generation by r.f. waves was first proposed by Ohkawa (1970) and later Fisch (1978). Fisch considered injecting waves with net parallel momentum via phased waveguide arrays. This scheme relies upon Landau damping of the high phase velocity r.f. waves travelling only in one toroidal direction parallel to the background magnetic field. The waves have net parallel momentum which, when absorbed by electrons whose velocity is close to the wave parallel phase velocity, exerts a force that drives an electric current. The current is mainly carried by the high-velocity electrons because, being less collisional they maintain their momentum longer. A current drive scheme in which no net toroidal momentum is injected was first discussed by Fisch and Boozer (1980). They suggested that the

collisionality of the plasma be altered in a preferential toroidal direction. If electrons moving in one direction were heated they would become less collisional and a net current would result. They showed that this heating could be in the velocity component perpendicular to the magnetic field and could be achieved by cyclotron damping rather than Landau damping. The relativistic effects of current drive have been explored by Fisch (1981), Cairns et al (1983) and Start et al (1983). A comprehensive review of all types of current drive has been provided by Fisch (1987).

A summary of the advantages of ECRH are:-

1. The high frequency waves penetrate easily into the plasma
2. The penetration is largely independent of plasma edge conditions
3. The power is fed to the plasma through simple wave guides
4. The position of power absorption can be easily controlled allowing for the inhibition of tearing mode instabilities
5. ECRH allows the possibility of non-inductive current drive
6. The theory of ECRH is straightforward and well understood.

1.6 The Diffusion Approach to R.F. Heating

To examine the effect of r.f. heating on the probability function a resonant diffusion approach is often adopted.

We begin with the collisionless Boltzman equation for the electron distribution function $f(\underline{r}, \underline{p}, t)$

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f - e(\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{p}} = 0. \quad (13)$$

The coupling between field and particle dynamics is due to $\underline{E}$ and $\underline{B}$ in the Boltzman equation and to the presence of the current density

$$\underline{j}(\underline{r}, t) = -e \int \underline{v} f(\underline{r}, \underline{p}, t) d^3 p + \underline{j}_i \quad (14)$$

and the charge density

$$\rho(\underline{r}, t) = -e \int f(\underline{r}, \underline{p}, t) d^3 p + \rho_i \quad (15)$$

in Maxwells equations. In the above $\underline{j}_i$ and ρ_i denote a possible current and charge density due to plasma ions. If we take a homogeneous steady plasma equilibrium ($\partial/\partial r = 0$, $\partial/\partial t = 0$) in the absence of wave excitation equation (13) for the equilibrium electron distribution function reduces to

$$(\underline{v} \times \underline{B}_0) \cdot \frac{\partial f_0}{\partial \underline{p}} = 0 \quad (16)$$

leaving

$$f_0(\underline{p}) = f_0(p_\perp, p_\parallel) \quad (17)$$

with $p_{\perp}$, $p_{\parallel}$ components of $\underline{p}$ perpendicular and parallel to the background magnetic field and f_0 independent of the azimuthal angle of $\underline{p}$ about $\underline{B}_0$.

Introducing wave dynamics in a linear form

$$f(\underline{r}, \underline{p}, t) = f_0(p_{\perp}, p_{\parallel}) + f_1(\underline{r}, \underline{p}, t) \quad (18)$$

$$\underline{E}(\underline{r}, t) = \underline{E}_1(\underline{r}, t) \quad (19)$$

$$\underline{B}(\underline{r}, t) = \underline{B}_0 + \underline{B}_1(\underline{r}, t) \quad (20)$$

$$\underline{j}(\underline{r}, t) = \underline{j}_0 + \underline{j}_1(\underline{r}, t) \quad (21)$$

yields in (13)

$$\frac{\partial f_1}{\partial t} + \underline{v} \cdot \underline{\nabla} f_1 - e(\underline{v} \times \underline{B}_0) \cdot \frac{\partial f_1}{\partial \underline{p}} = e(\underline{E}_1 + \underline{v} \times \underline{B}_1) \cdot \frac{\partial f_0}{\partial \underline{p}} \quad (22)$$

and

$$\underline{j}_1 = -e \int f_1 \underline{v} d^3p. \quad (23)$$

Applying Maxwells equations leads us to

$$\underline{n} \times (\underline{n} \times \underline{E}_1) + \underline{\epsilon} \cdot \underline{E}_1 = 0 \quad (24)$$

where $\underline{n}$ is the refractive index and $\underline{\epsilon}$ is the permittivity tensor.

The condition that equation (24) has non-trivial solutions is the dispersion relation,

$$|\underline{\Lambda}| = 0 \quad (25)$$

where

$$\underline{\Lambda} \cdot \underline{E}_1 = \underline{n} \times (\underline{n} \times \underline{E}_1) + \underline{\epsilon} \cdot \underline{E}_1. \quad (26)$$

In the same fashion the quasilinear motion of the electron distribution function f_0 induced by the presence of waves can be calculated. Retaining terms quadratic in wave amplitude yields

$$\frac{\partial f_a}{\partial t} = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \, e \left\langle (\underline{E}_1 + \underline{v} \times \underline{B}_1) \cdot \frac{\partial f_a}{\partial \underline{p}} \right\rangle \quad (27)$$

where $\langle \rangle$ denotes the average over oscillations. Equation (27) can be cast in the form of a diffusion equation

$$\frac{\partial f_a}{\partial t} = \frac{\partial}{\partial \underline{p}} \cdot \left(\underline{D} \cdot \frac{\partial f_a}{\partial \underline{p}} \right) \quad (28)$$

where $\underline{D}$ is the Diffusion Tensor.

For inhomogeneous states, which are normally found in experimental plasmas, the homogeneous case as discussed above is not directly useful, but the general structure of the quasilinear diffusion term is unchanged. Modifications of the diffusion coefficient are due to the space dependence of the plasma equilibrium and to the localization of the interaction between plasma electrons and waves appearing as a finiteness of the beam widths and the spatial dependence of the magnetic field. Consequently modifications to the diffusion coefficient have been described by many authors. Lieberman and Lichtenberg (1972), (1973) considered the diffusion coefficient appropriate to magnetic mirror devices. They used a bounced-averaged Hamiltonian to investigate the resonance breakdown of adiabatic invariance due to secondary island formation and showed that for significantly large electric fields the invariant was destroyed. Grésillon et al (1976) derived an expression for the diffusion coefficient in weakly turbulent plasma by measuring the space-time correlation function which proved accurate when compared with velocity diffusion coefficients found experimentally. The effect of the rotational

transform of the magnetic field in tokamak devices, was considered by Fielding (1980), and the effect on the diffusion coefficient of the limited particle wave interaction time. Fidone (1983) looked at quasilinear saturation effects and the reduction in the homogeneous plasma diffusion coefficient when the ratio of beam area to flux surface area is considered. Cairns and Lashmore-Davies (1986) combined rotational transform and finite beam size effects and Dendy (1985) looked at the possibility of helical current sheets and the corresponding diffusion coefficient. Demio and Engelmann (1986) generalised earlier results, and showed that the same form for the diffusion coefficient could be derived from a kinetic (quasilinear) and test-particle approach. The equivalence of quasilinear theory and Fokker-Planck theory has been studied in detail by Chiu (1985).

1.7 Outline of Thesis

In an attempt to reach higher plasma temperatures the input power of future ECRH experiments is likely to increase. Most recently this has culminated in the use of a free electron laser (FEL) on the Microwave Tokamak Experiment (MTX) at the Lawrence Livermore Laboratory, see Thomassen et al (1986). The proposed FEL will have a peak power of $4-8 \times 10^9$ Watts with short pulses of 5×10^{-8} secs. In this work we shall investigate the nonlinear behaviour of the absorption of the microwave power as relativistic effects become important.

In our study of high power ECRH we will sustain a Fokker-Planck approach, considering the orbits of single particles and combining them to derive forms for the diffusion coefficient, as opposed to using a linearised form of the Boltzmann equation. However it should be noted that, at least in the low field regime, these two approaches are equivalent as shown by Demio and Engelmann (1986).

First of all we shall consider a Lagrangian formalism based on the work of Littlejohn (1983; 1984; 1985) to derive averaged equations of motion for electrons in constant and linearly varying magnetic fields. These equations require less computer processor time to solve and have been incorporated into a Single Particle Experiment on Electron Cyclotron Heating (SPEECH) Code. In Chapter 3 we consider effects of large fields on the velocity diffusion coefficient and derive analytic forms which are later compared with calculations from the SPEECH Code. We then derive analytic forms for the change in perpendicular velocity of the particle for low electric field, high

electric field, and in cases when the background magnetic field is non-uniform based on the work of O'Brien (1987a), (1987b). In Chapter 4 we introduce the SPEECH code and compare our analytic form for $\Delta\mu$ and the probability function $P(\Delta\mu)$ with numerical experiment for a variety of regimes. We include a phase plane analysis of the averaged equations as an aid to understanding the results. Comparison of the analytic forms of the velocity diffusion coefficient with numerical experiment is included in Chapter 5 as well as a discussion of multipass behaviour. The final chapter summarises the results of the previous chapters and gives suggestions for further work.

A Note on Equation Numbering

Equations are numbered in the order they appear in the Chapter they appear. Equations subsequently quoted in later chapters have the number of the chapter in which they originally appeared as a prefix. For example equation (81) in Chapter 2 is numbered as (2.81) when it reappears in Chapter 4.

CHAPTER 2

DERIVATION OF EQUATIONS IN LAGRANGIAN FORM

2.1 Introduction

One of the principle aims of this research is to investigate the effects of the large wave electric fields generated in current and future Electron Cyclotron Heating Experiments. Fields of up to $5 \cdot 10^7 \text{ Vm}^{-1}$ have been suggested for the new MTX experiment at Lawrence Livermore Laboratory using a Free Electron Laser. In particular we shall examine the effect of the relativistic mass change on wave absorption and diffusion of particles in velocity space.

One can gain an insight into the problem by first considering the Lorentz force equations for an electron in a constant magnetic field. If the particle is subjected to a transverse, linearly polarised electromagnetic wave in the 0-mode, the equations are

$$\frac{d\mathbf{p}}{dt} = -e(\mathbf{E}_{\sim} + \mathbf{v} \times \mathbf{B}_{\sim} + \mathbf{v} \times \mathbf{B}_0) \quad (1)$$

$$\text{where} \quad \mathbf{E}_{\sim} = \mathbf{E}_{\sim}^0 e^{i(kx - \omega t)} \quad \mathbf{E}_{\sim}^0 = (0, 0, E_{\sim}^0)$$

$$\mathbf{B}_{\sim} = \mathbf{B}_{\sim}^0 e^{i(kx - \omega t)} \quad \mathbf{B}_{\sim}^0 = (0, B_{\sim}^0, 0)$$

$$\text{and} \quad \mathbf{B}_0 = (0, 0, B_0)$$

e - electronic charge

m_0 - rest mass of electron

$\underline{p} = \gamma m_0 \underline{v}$

$\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$

(see Fig 2.1).

Equations (1) can be rewritten in the form

$$\frac{d}{dt} (\gamma m v_x) = -e(v_y B_0 + v_z B_{\sim} e^{i(kx - \omega t)}) \quad (2)$$

$$\frac{d}{dt} (\gamma m v_y) = e v_x B_0 \quad (3)$$

$$\frac{d}{dt} (\gamma m v_z) = -e(E_{\sim} e^{i(kx - \omega t)} - v_x B_{\sim} e^{i(kx - \omega t)}) \quad (4)$$

Now, if on a short timescale we regard γ constant and define

$v_{\pm} = v_x \pm i v_y$ we obtain

$$\frac{d}{dt} (v_{-}) = -i\Omega v_{-} - \frac{e v_z B_{\sim}}{\gamma m} e^{i(kx - \omega t)} \quad (5)$$

$$\frac{d}{dt} (v_{+}) = i\Omega v_{+} - \frac{e v_z B_{\sim}}{\gamma m} e^{i(kx - \omega t)} \quad (6)$$

with

$$\Omega = \frac{e B_0}{\gamma m_0}.$$

As (5) is linear in v_{-} we can introduce an integrating factor to give

$$\frac{d}{dt} (v_{-} e^{i\Omega t}) = -\frac{e}{m\gamma} \frac{v_z k E_{\sim}}{\omega} e^{i(kx - \omega t)} e^{i\Omega t}. \quad (7)$$

Assuming real v_x and v_y the equation for v_{+} is simply the complex conjugate of (7). Now neglecting the wave field the zero order trajectories are

$$v_{-} = v_{-0} e^{-i(\Omega t + \theta)} \quad (8)$$

Figure 2.1. Omode Wave Polarization.

Figure 2.2. Xmode Wave Polarization.

so
$$v_x = v_{\perp} \cos (\Omega t + \vartheta) \quad (9)$$

$$v_y = v_{\perp} \sin (\Omega t + \vartheta) \quad (10)$$

$$x = \frac{v_{\perp}}{\Omega} \sin (\Omega t + \vartheta) \quad (11)$$

$$y = -\frac{v_{\perp}}{\Omega} \cos (\Omega t + \vartheta). \quad (12)$$

If we now substitute (11) for x on the right hand side of (7) we obtain

$$\frac{d}{dt} (v_{\perp} e^{i\Omega t}) = -\frac{e}{m\gamma} \frac{v_z k E_0}{w} \times e^{i\left(\frac{kv_{\perp}}{\Omega} \sin (\Omega t + \vartheta) - \omega t\right)} e^{i\Omega t}. \quad (13)$$

Now employing the standard expansion

$$e^{iz \sin \vartheta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\vartheta} \quad (14)$$

see Abramowitz and Stegun (1972) where J_n is the Bessel function of the first kind gives

$$\frac{d}{dt} (v_{\perp} e^{-i\vartheta}) = \frac{e}{\gamma m} \frac{v_z k E_0}{w} e^{i(\Omega - \omega)t} \times \sum_{n=-\infty}^{\infty} J_n\left(\frac{kv_{\perp}}{\Omega}\right) e^{in\Omega t}. \quad (15)$$

For heating at the fundamental the slowly varying term is given by the J_0 component of the sum. Further approximating $J_0(kv_{\perp}/\Omega)$ by 1, appropriate for small Larmor radius leaves

$$\frac{d}{dt} (v_{\perp} e^{-i\vartheta}) = -\frac{e}{m\gamma} \frac{v_z k E_0}{w} e^{i(\Omega - \omega)t}. \quad (16)$$

Now on a larger time scale if we take into account the time variation of Ω owing to the relativistic mass shift, or a gradient in the

background field we get

$$\frac{d}{dt} (v_{\perp}) = - \frac{e}{\gamma m} \frac{v_z k E_{\sim}}{w} e^{i \left(\int^t \Omega dt' - \omega t \right)} \quad (17)$$

Clearly if B_0 is a constant and we assume γ also constant then equation (16) shows that a particle, whose gyrofrequency equals the wave frequency, will be driven indefinitely; the interaction being limited only by the time the particle spends in the wave. (Here we have set $k_{\parallel} = 0$ so there is no Doppler term in the resonance condition.) Such an analysis is common - Cairns and Lashmore-Davies (1986); Eldridge (1972). Such an approximation will be appropriate for $\dot{\gamma} = v\dot{v}/c^2\gamma^3$ small and consequently small $E_{\sim}$. However for larger fields one would expect the particle to be driven out of resonance as γ changed.

In the chapters that follow we shall endeavour to quantify this. But first we shall introduce a technique to generate averaged equations for arbitrary wave and background fields.

2.2 Averaged Lagrangian Formulation for Constant Background Field Two Examples

This technique is based on the gyro averaged forms that have been developed by Littlejohn. Here we will consider a simple magnetic field geometry so that the gyro-averaging process is straightforward. The treatment for a more complex geometry is detailed in Appendix 2.

We consider a phase space Lagrangian as used by Littlejohn (1983; 1984; 1985). This is defined in terms of a Hamiltonian $H(\underline{p}, \underline{q}, t)$ by

$$L_o(\underline{q}, \underline{p}, \dot{\underline{q}}, t) = \underline{p} \cdot \dot{\underline{q}} - H(\underline{p}, \underline{q}, t). \quad (18)$$

The standard Euler-Lagrange equations applied to the variational principle

$$\delta \int L \, dt = 0 \quad (19)$$

with $\underline{p}$ and $\underline{q}$ allowed to vary independently give Hamiltons equations in their usual form.

$$\dot{\underline{q}} = \frac{\partial H}{\partial \underline{p}} ; \quad \dot{\underline{p}} = - \frac{\partial H}{\partial \underline{q}} \quad (20)$$

In applications we can transform to any set of six independent variables and the equations of motion are still given by the Euler-Lagrange equations corresponding to (19).

Applying this to an electron with charge $-e$ moving in an electromagnetic field, then the canonical momentum is

$$\underline{p} = \gamma m \underline{v} - e \underline{A} \quad (21a)$$

$$= m \underline{u} - e \underline{A} \quad (21b)$$

where $\underline{u} = \gamma \underline{v}$, γ is the relativistic factor and $\underline{A}$ is the total vector potential of the wave and the background magnetic field.

The Lagrangian L_0 in (18) can be written in terms of $\underline{u}$ and $\underline{x}$ as

$$L_0 = [\underline{m}\underline{u} - e\underline{A}] \cdot \dot{\underline{x}} - \{m^2 c^4 + m^2 c^2 u^2\}^{\frac{1}{2}} + e\Phi \quad (22)$$

where Φ is the scalar potential.

Now suppose $\underline{A}$ is of the form

$$\begin{aligned} \underline{A} = & \underline{A}_0(x, z, t) + \underline{A}_1(x) e^{ik_{\perp} x + ik_{\parallel} z - i\omega t} \\ & + \overline{\underline{A}}_1(x) e^{-ik_{\perp} x - ik_{\parallel} z + i\omega t} \end{aligned} \quad (23)$$

where A_0 is slowly varying in time compared to the cyclotron frequency and in space compared to the Larmor radius. The wave field represented by $\underline{A}_1$ is supposed small enough to be regarded as a small perturbation on the basic cyclotron motion. We can now go to guiding centre coordinates with

$$\underline{x} = \underline{X} + \frac{u_{\perp}}{\Omega} \sin \vartheta \underline{\hat{x}} - \frac{u_{\perp}}{\Omega} \cos \vartheta \underline{\hat{y}} \quad (24)$$

$$\underline{u} = u_{\perp} \cos \vartheta \underline{\hat{x}} + u_{\perp} \sin \vartheta \underline{\hat{y}} + u_{\parallel} \underline{\hat{z}} \quad (25)$$

$$\begin{aligned} \dot{\underline{x}} = & \dot{\underline{X}} + \frac{\dot{u}_{\perp}}{\Omega} \sin \vartheta \underline{\hat{x}} + \frac{u_{\perp} \dot{\vartheta}}{\Omega} \cos \vartheta \underline{\hat{x}} \\ & - \frac{\dot{u}_{\perp}}{\Omega} \cos \vartheta \underline{\hat{y}} + \frac{u_{\perp} \dot{\vartheta}}{\Omega} \sin \vartheta \underline{\hat{y}} \end{aligned} \quad (26)$$

where $(\hat{x}, \hat{y}, \hat{z})$ are the usual cartesian unit vectors, and $\underline{X}$ is the position of the guiding centre with $u_{\perp}/\Omega$ the Larmor radius.

Let us suppose first that $\underline{A}_1 = 0$. Then if we substitute (23), (24), (25) and (26) into (22) we get

$$L_0 = [\mu u_{||} \dot{\underline{z}} - e \underline{A}_0] \cdot \dot{\underline{X}} + \frac{\mu u_{\perp}^2 \dot{\vartheta}}{\Omega} - \{m^2 c^4 + m^2 c^2 u^2\}^{\frac{1}{2}} + e\Phi \quad (27)$$

which is the relativistic analogue of Littlejohn's equation (20) (1983). Littlejohn (1983) shows that this gives the usual particle drifts in a slowly varying field.

As another special case consider the vector potential

$$\underline{A} = B_0 x \underline{\hat{y}} + A_1 \cos(kx - \omega t) \underline{\hat{z}}$$

which corresponds to that used by Suvorov and Tokman (1983) to look at averaged particle behaviour, in an O-mode wave propagating in the $\underline{\hat{x}}$ direction across a steady field in the z direction.

Then

$$\begin{aligned} L_0 = & \left[\mu u_{\perp} \cos \vartheta \dot{\underline{x}} + \mu u_{\perp} \sin \vartheta \dot{\underline{y}} \right. \\ & - e B_0 \left(\frac{u_{\perp}}{\Omega} \sin \vartheta + X \right) \underline{\hat{y}} + \mu u_{||} \dot{\underline{z}} \\ & \left. - e A_1 \cos(kx - \omega t) \underline{\hat{z}} \right] \cdot \left(\dot{\underline{X}} + \frac{\dot{u}_{\perp} \sin \vartheta}{\Omega} \underline{\hat{x}} \right. \\ & \left. + \frac{u_{\perp} \dot{\vartheta} \cos \vartheta}{\Omega} \underline{\hat{x}} - \frac{u_{\perp}}{\Omega} \cos \vartheta \dot{\underline{y}} + \frac{u_{\perp} \dot{\vartheta} \sin \vartheta}{\Omega} \underline{\hat{y}} \right) \\ & - \{m^2 c^4 + m^2 c^2 u^2\}^{\frac{1}{2}}, \text{ putting } \Phi = 0. \end{aligned} \quad (29)$$

Multiplying out and averaging over the ϑ oscillations we get

$$\begin{aligned} L_0 = & \frac{\mu u_{\perp}^2 \dot{\vartheta}}{\Omega} + \mu u_{||} \dot{\underline{z}} - e A_1 \dot{\underline{z}} \cos(kx - \omega t) \\ & - e B_0 X \dot{\underline{Y}} - \frac{1}{2} \frac{e B_0 u_{\perp}^2 \dot{\vartheta}}{\Omega^2} - \{m^2 c^4 + m^2 c^2 u^2\}^{\frac{1}{2}} \end{aligned} \quad (30)$$

$$\begin{aligned}
 &= \frac{mu_{\perp}^2 \dot{\vartheta}}{2\Omega} + mu_{\parallel} \dot{z} - eA_1 \dot{z} \cos(kx - \omega t) \\
 &- eB_0 X \dot{Y} - \{m^2 c^4 + m^2 c^2 u_{\perp}^2\}^{\frac{1}{2}}.
 \end{aligned} \tag{31}$$

In the cosine term we put

$$x = X + \frac{u_{\perp}}{\Omega} \sin \vartheta,$$

or, more simply

$$x = \frac{u_{\perp}}{\Omega} \sin \vartheta$$

as X can be taken to be at the origin. Then as

$$\cos(kx - \omega t) = \frac{1}{2} \sum_n \left\{ J_n \left(\frac{ku_{\perp}}{\Omega} \right) e^{in\vartheta - i\omega t} + J_n \left(\frac{ku_{\perp}}{\Omega} \right) e^{-in\vartheta + i\omega t} \right\} \tag{32}$$

and for a wave at the cyclotron frequency $\dot{\vartheta} \approx \omega$ the slowly varying term is that with $n = 1$.

With the approximation

$$J_1 \left(\frac{ku_{\perp}}{\Omega} \right) \approx \frac{1}{2} \frac{ku_{\perp}}{\Omega} \tag{33}$$

we have finally

$$\begin{aligned}
 L_0 &= \frac{1}{2} \frac{mu_{\perp}^2 \dot{\vartheta}}{\Omega} + mu_{\parallel} \dot{z} - \frac{1}{2} \frac{m\Omega_1 \dot{z} u_{\perp}}{\Omega} \cos(\vartheta - \omega t) \\
 &- eB_0 X \dot{Y} - m \{c^4 + c^2 u_{\perp}^2 + c^2 u_{\parallel}^2\}^{\frac{1}{2}}
 \end{aligned} \tag{34}$$

with

$$\Omega_1 = \frac{ekA_1}{m}. \tag{35}$$

Now we construct the appropriate Euler-Lagrange equations to make $\int L dt$ take a stationary value

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{z}} \right) - \frac{\partial L}{\partial z} = 0 \Rightarrow \frac{d}{dt} \left\{ u_{||} - \frac{1}{2} \frac{\Omega_1 u_{\perp}^2}{\Omega} \cos(\vartheta - \omega t) \right\} = 0. \quad (36)$$

An equation representing the conservation of z-momentum.

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}_{\perp}} \right) - \frac{\partial L}{\partial u_{\perp}} &= 0 \Rightarrow \\ \frac{u_{\perp} \dot{\vartheta}}{\Omega} - \frac{1}{2} \frac{\Omega_1 \dot{z}}{\Omega} \cos(\vartheta - \omega t) - \frac{u_{\perp}}{\gamma} &= 0. \end{aligned} \quad (37)$$

If we put $\psi = \vartheta - \omega t$

$$\dot{\psi} = \frac{\Omega}{\gamma} - \omega + \frac{1}{2} \frac{\Omega_1 \dot{z}}{u_{\perp}} \cos \psi. \quad (38)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vartheta}} \right) - \frac{\partial L}{\partial \vartheta} &= 0 \Rightarrow \\ \frac{d}{dt} \left(\frac{1}{2} \frac{u_{\perp}^2}{\Omega} \right) - \frac{1}{2} \frac{\Omega_1 \dot{z} u_{\perp}}{\Omega} \sin(\vartheta - \omega t) &= 0 \end{aligned} \quad (39)$$

or equivalently

$$\dot{u}_{\perp} = \frac{1}{2} \Omega_1 \dot{z} \sin \psi. \quad (40)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}_{||}} \right) - \frac{\partial L}{\partial u_{||}} = 0 \Rightarrow \dot{z} = \frac{u_{||}}{\gamma} \quad (41)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\gamma}} \right) - \frac{\partial L}{\partial \gamma} = 0 \Rightarrow \dot{X} = 0 \quad (42)$$

and finally

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{X}} \right) - \frac{\partial L}{\partial X} = 0 \Rightarrow \dot{Y} = 0. \quad (43)$$

Equations (42) and (43) show that the guiding centre remains on a fixed field line.

Equations (38) and (40) are those of most interest. Equation (40) determines the change in perpendicular energy of the particle and equation (38) determines the change in ψ , which is the relative phase lag between the wave and the particle. Equations (38) and (40) are similar to equations (4) of Suvorov and Tokman (1983). The latter though would appear to have mixed up sines and cosines, at some point, since differentiating the first of their equations (3), using (4) and substituting into their (2) gives a cos rather than a sine on the right hand side.

2.3 Averaged Lagrangian Formulation for Constant Background Field General Case

We now consider a more general case

$$\begin{aligned} \underline{A} = B_0 x \underline{\hat{y}} + \underline{A}_1 e^{ik_{\perp} x + ik_{\parallel} z - i\omega t} \\ + \overline{\underline{A}}_1 e^{-ik_{\perp} x - ik_{\parallel} z + i\omega t} \end{aligned} \quad (44)$$

Now substituting into

$$L_0 = [m\underline{u} - e\underline{A}] \cdot \dot{\underline{x}} - \{m^2 c^4 + m^2 c^2 u^2\}^{\frac{1}{2}}$$

and going to guiding centre coordinates given by equations (24), (25) and (26) gives

$$\begin{aligned} L_0 = & \left[m(u_{\perp} \cos \vartheta, u_{\perp} \sin \vartheta, u_{\parallel}) \right. \\ & - e \left(A_{1x} e^{ik_{\perp} x + ik_{\parallel} z - i\omega t} + \overline{A}_{1x} e^{-ik_{\perp} x - ik_{\parallel} z + i\omega t}, \right. \\ & B_0 x + A_{1y} e^{ik_{\perp} x + ik_{\parallel} z + i\omega t} + \overline{A}_{1y} e^{-ik_{\perp} x - ik_{\parallel} z + i\omega t} \\ & \left. \left. A_{1z} e^{ik_{\perp} x + ik_{\parallel} z - i\omega t} + \overline{A}_{1z} e^{-ik_{\perp} x - ik_{\parallel} z + i\omega t} \right) \right] \\ & \left(\dot{X} + \frac{\dot{u}_{\perp}}{\Omega} \sin \vartheta + \frac{u_{\perp} \dot{\vartheta}}{\Omega} \cos \vartheta, \right. \\ & \left. \dot{Y} - \frac{\dot{u}_{\perp}}{\Omega} \cos \vartheta + \frac{u_{\perp} \dot{\vartheta}}{\Omega} \sin \vartheta, \dot{Z} \right) \\ & - \{m^2 c^4 + m^2 c^2 u^2\}^{\frac{1}{2}}. \end{aligned} \quad (45)$$

We now construct the gyro-averaged Lagrangian by

$$L_{av} = \frac{1}{2\pi} \int_0^{2\pi} L_0 d\vartheta. \quad (46)$$

In the averaging process we assume $\underline{A}_0$ and $\underline{A}_1$ to be slowly varying and use the expansion

$$e^{ik_{\perp}x + ik_{\parallel}z - i\omega t} = e^{ik_{\perp}X + ik_{\parallel}Z - i\omega t} \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_{\perp}u_{\perp}}{\Omega}\right) e^{in\vartheta}. \quad (47)$$

We shall consider ω close to the gyrofrequency, that is, we shall only consider heating at the fundamental. Consequently only terms going as $e^{\pm i(\vartheta - \omega t)}$ are slowly varying and we shall also neglect terms $O\left(\frac{k_{\perp}u_{\perp}}{\Omega}\right)^2$ and higher.

L_{av} then is

$$\begin{aligned} L_{av} = & \frac{mu_{\perp}^2 \dot{\vartheta}}{2\Omega} + mu_{\parallel} \dot{Z} - eB_0 X \dot{Y} \\ & - \frac{e}{2} \left[A_{1x} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{X} \\ & - \frac{e}{2i} \left[A_{1x} \frac{\dot{u}_{\perp}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \\ & - \frac{e}{2} \left[A_{1x} \frac{u_{\perp} \dot{\vartheta}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \\ & - \frac{e}{2} \left[A_{1y} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{Y} \\ & + \frac{e}{2} \left[A_{1y} \frac{\dot{u}_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \\ & - \frac{e}{2i} \left[A_{1y} \frac{u_{\perp} \dot{\vartheta}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \\ & - \frac{e}{2} \left[A_{1z} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{Z} \\ & - \{m^2 c^4 + m^2 c^2 u^2\}^{\frac{1}{2}}, \end{aligned} \quad (48)$$

where we have used the approximations

$$J_0\left(\frac{k_b u_b}{\Omega}\right) \sim 1 \quad \text{and} \quad J_1\left(\frac{k_b u_b}{\Omega}\right) \sim \frac{k_b u_b}{2\Omega}. \quad (49)$$

We can now extract the equations of motion for a particle in a radiofrequency field of arbitrary configuration. These are obtained directly from the Euler-Lagrange equations

$$1. \quad \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{X}} \right) - \frac{\partial L_{av}}{\partial X} = 0 \quad (50)$$

yields

$$\begin{aligned} \frac{d}{dt} \left(-\frac{e}{2} \left[A_{1x} \frac{k_b u_b}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \right) = \\ -eB_0 \dot{Y} - \frac{e}{2} \left[A_{1x} \frac{k_b^2 u_b}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \dot{X} \\ - \frac{e}{2i} \left[A_{1x} \frac{\dot{u}_b k_b}{\Omega} i e^{i\psi} - \text{complex conjugate} \right] \\ - \frac{e}{2} \left[A_{1x} \frac{u_b k_b \dot{\psi}}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \\ - \frac{e}{2} \left[A_{1y} \frac{k_b^2 u_b}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \dot{Y} \\ + \frac{e}{2} \left[A_{1y} \frac{k_b \dot{u}_b}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \\ - \frac{e}{2i} \left[A_{1y} \frac{u_b k_b \dot{\psi}}{\Omega} i e^{i\psi} - \text{complex conjugate} \right] \\ - \frac{e}{2} \left[A_{1z} \frac{k_b^2 u_b}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \dot{Z} \end{aligned} \quad (51)$$

$$2. \quad \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{Y}} \right) - \frac{\partial L_{av}}{\partial Y} = 0 \quad (52)$$

yields

$$\frac{d}{dt} \left(-eB_0 X - \frac{e}{2} \left[A_{1y} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \right) = 0. \quad (53)$$

$$3. \quad \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{Z}} \right) - \frac{\partial L_{av}}{\partial Z} = 0 \quad (54)$$

yields

$$\begin{aligned} \frac{d}{dt} \left(-\frac{e}{2} \left[A_{1z} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] + mu_{\parallel} \right) = \\ -\frac{e}{2} \left[A_{1x} \frac{k_{\perp} k_{\parallel} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{X} \\ -\frac{e}{2i} \left[A_{1x} \frac{\dot{u}_{\perp} k_{\parallel}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \\ -\frac{e}{2} \left[A_{1x} \frac{u_{\perp} k_{\parallel} \dot{\psi}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \\ -\frac{e}{2} \left[A_{1y} \frac{k_{\perp} k_{\parallel} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{Y} \\ +\frac{e}{2} \left[A_{1y} \frac{\dot{u}_{\perp} k_{\parallel}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \\ -\frac{e}{2i} \left[A_{1y} \frac{u_{\perp} \dot{\psi} k_{\parallel}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \\ -\frac{e}{2} \left[A_{1z} \frac{k_{\perp} k_{\parallel} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{Z}. \quad (55) \end{aligned}$$

$$4. \quad \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{u}_{\perp}} \right) - \frac{\partial L_{av}}{\partial u_{\perp}} = 0 \quad (56)$$

yields

$$\frac{d}{dt} \left(-\frac{e}{2i} \left[A_{1x} \frac{1}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \right) \quad (57)$$

$$\begin{aligned}
 & + \frac{e}{2} \left[A_{1y} \frac{1}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \Bigg) = \\
 & \frac{m u_{\mathbf{b}} \dot{\vartheta}}{\Omega} - \frac{e}{2} \left[A_{1x} \frac{k_{\mathbf{b}}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{\chi} \\
 & - \frac{e}{2} \left[A_{1x} \frac{\dot{\vartheta}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1y} \frac{k_{\mathbf{b}}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{\gamma} \\
 & - \frac{e}{2i} \left[A_{1y} \frac{\dot{\vartheta}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1z} \frac{k_{\mathbf{b}}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{z} \\
 & - \frac{m u_{\mathbf{b}}}{\gamma} .
 \end{aligned}$$

$$5. \quad \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{u}_{\parallel}} \right) - \frac{\partial L_{av}}{\partial u_{\parallel}} = 0 \quad (58)$$

yields

$$m \dot{z} - \frac{m u_{\parallel}}{\gamma} = 0 \quad (59)$$

that is $\dot{z} = u_{\parallel}/\gamma$.

$$6. \quad \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{\vartheta}} \right) - \frac{\partial L_{av}}{\partial \vartheta} = 0 \quad (60)$$

yields

$$\begin{aligned}
 & \frac{d}{dt} \left[\frac{m u_{\mathbf{b}}^2}{2\Omega} - \frac{e}{2i} \left[A_{1y} \frac{u_{\mathbf{b}}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \right. \\
 & \quad \left. - \frac{e}{2} \left[A_{1x} \frac{u_{\mathbf{b}}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] = \right. \\
 & \quad \left. - \frac{e}{2} \left[A_{1x} \frac{k_{\mathbf{b}} u_{\mathbf{b}}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{\chi} \right. \quad (61)
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{e}{2i} \left[A_{1x} \frac{\dot{u}_x}{\Omega} ie^{i\psi} - \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1x} \frac{u_x \dot{\theta}}{\Omega} ie^{i\psi} + \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1y} \frac{k_y u_y}{\Omega} ie^{i\psi} + \text{complex conjugate} \right] \dot{\gamma} \\
 & + \frac{e}{2} \left[A_{1y} \frac{\dot{u}_y}{\Omega} ie^{i\psi} + \text{complex conjugate} \right] \\
 & - \frac{e}{2i} \left[A_{1y} \frac{u_y \dot{\theta}}{\Omega} ie^{i\psi} - \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1z} \frac{k_z u_z}{\Omega} ie^{i\psi} + \text{complex conjugate} \right] \dot{z}.
 \end{aligned}$$

Equations (51) and (53) determine the change in the guiding centre caused by the r.f. wave. The change in parallel momentum can be evaluated from (55). Equation (57) deals with the change of phase of the particle and (61) with the change in magnetic moment, as (61) can be written in the form

$$\frac{d}{dt} (\mu) = \dots \quad (62)$$

2.4 Averaged Equations for O mode and X mode

We can now simplify matters by choosing two particular wave modes. The wave forms we shall consider are the O-mode and the X-mode which propagate transverse to the static field. The O-mode is of linearly polarised form and has electric field component parallel to the background magnetic field. (See Fig 2.1.) The X-mode is elliptically polarised with magnetic component parallel to the background magnetic field (see Fig 2.2). The general theory of these waves has already been discussed in Chapter 1.

O-mode

For the O-mode we shall choose the vector potentials

$$\underline{A}_1 = \left(0, 0, i \frac{A_1 z}{2}\right) \quad (63)$$

and $\underline{A}_0 = (0, B_0 x, 0).$ (64)

These with

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad (65)$$

and $\underline{E} = - \frac{\partial \underline{A}}{\partial t}$ (66)

yield electric and magnetic forms

$$\underline{E}_1 = (0, 0, E_z \cos(kx - \omega t)) \quad (67)$$

$$\underline{B}_1 = (0, B_y \cos(kx - \omega t), 0) \quad (68)$$

$$\underline{B}_0 = (0, 0, B_0). \quad (69)$$

Substituting expressions (63) and (64) into equations (51), (53), (55), (57), (59) and (61) and dropping terms that are second order

in the wave field gives

$$\dot{u}_{\perp} = -\frac{1}{2} \frac{e}{m} \frac{k \dot{z} E_z}{w} \cos \psi \quad (70)$$

$$\dot{u}_{\parallel} = 0 \quad (71)$$

$$\dot{\psi} = \frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{e}{m} \frac{k \dot{z} E_z}{u_{\perp} w} \sin \psi \quad (72)$$

These equations agree with Suvorov and Tokman (1983) (to within a sine or cosine as discussed earlier). Physically the meaning of equations (70), (71) and (72) is straightforward. Equation (71) shows the conservation of z-momentum. This is due to the fact that in the 0-mode the electric field is in the z-direction. Averaging over the cyclotron frequency, which for heating at the fundamental is close to the wave frequency, gives $\langle \dot{E}_z \rangle = 0$. The Lorentz force perpendicular to the background magnetic field is linearly polarised. It can be divided into two circular components one of which is resonant with the electrons gyration. The component of the force tangential to the electron transverse velocity determines the transverse energy variation described by (70). The change in phase of the electron gyration is determined by the difference between the relativistic gyro-frequency and the wave frequency - the first two terms on the right hand side of equation (72), and by the normal part of the same resonant component of the Lorentz force, which is the third term on the right hand side of (72).

In fact $u_{\perp}$ and ψ can be regarded as the polar coordinates of the particle in a frame rotating with angular velocity w

(= wave frequency) (see Fig 2.3). The $1/u_{\parallel}$ dependence of $\dot{\psi}$ is just a geometrical effect. The greater $u_{\parallel}$, the smaller the change in ψ produced by the same change in velocity.

An important result of the equations is that the effective electric field for the 0-mode is

$$E_{\text{eff}} = \frac{k \dot{z} E_z}{w} \quad (73)$$

which depends on the parallel velocity. Particles with low parallel velocity, for example those turning in a mirror field, will experience reduced heating.

X-mode

The electric field of the X-mode is typically elliptically polarised see Boyd and Sanderson (1969) can be decomposed into two counter-rotating components E_+ and E_- , E_- being the one rotating in the same sense as the electrons. In cold plasma theory this E_- component vanishes at the cyclotron frequency. However when finite temperature corrections for plasma waves in a magnetic field are included, an E_- component exists at the cyclotron frequency with $|E_-| \sim v_{th}/c |E_+|$ see Stix (1962).

Substituting

$$\underline{A}_1 = (A_{1x}, A_{1y}, 0) \quad (74)$$

$$\underline{A}_0 = (0, B_0 x, 0) \quad (75)$$

into (51), (53), (55), (57), (59) and (61) as before yields

$$\dot{u}_{\parallel} = -\frac{1}{2} \frac{e}{m} E_- \cos \psi \quad (76)$$

Figure 2.3. Polar Co-ordinates of Particle.

$$\dot{u}_{||} = 0 \quad (77)$$

$$\dot{\psi} = \frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{e}{m} \frac{E_{-}}{u_{||}} \sin \psi \quad (78)$$

which are of the same form as (70), (71) and (72) with $kv_{||}E_z/w$ replaced by E_{-} . In the case of the X-mode the effective electric field is not so dependent on $v_{||}$ and consequently heating for mirrored particles will be enhanced over the O-mode.

It should be noted that the equations (51), (53), (55), (57), (59) and (61) give rise to the possibility of a wave induced drift velocity in a direction perpendicular to the static magnetic field and that of the wave vector k . This drift is of the order of somewhat less than the grad B drift for a typical tokamak and is discussed in detail in Appendix 1.

2.5 Averaged Equations for Linearly Varying $\underline{B}$

We now proceed to a more realistic magnetic field configuration. As a particle moves on a field line in a tokamak it feels a changing magnetic flux density (Fig 2.4). To model this we shall choose a vector potential

$$\underline{A}_0 = (0, B_0 x(1 + z/R), 0) \quad (79)$$

which yields

$$\underline{B} = (-B_0 x/R, 0, B_0(1 + z/R)) \quad (80)$$

where R is the scale length for the variation in the magnetic flux density. The most important component of this field is in the z direction and varies linearly with z (see Fig 2.5). Similar fields have been considered by other authors including Eldridge (1972), O'Brien (1986), Lieberman and Lichtenberg (1973). Such fields provide for the possibility of particles being reflected in a magnetic mirror. The detailed calculations are left to Appendix 2. We shall simply state the governing equations here.

0-mode

$$\dot{u}_\perp = \frac{u_\perp u_{||} \Omega_0}{2\Omega R \gamma} - \frac{1}{2} \frac{e}{m} \frac{k E_z}{w} \frac{u_{||}}{\gamma} \cos \psi \quad (81)$$

$$\dot{u}_{||} = - \frac{u_\perp^2 (\dot{\psi} + w) \Omega_0}{2\Omega^2 R} \quad (82)$$

$$\dot{\psi} = \frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{e}{m} \frac{k E_z u_{||}}{w \gamma u_\perp} \sin \psi \quad (83)$$

where $\Omega_0 = \frac{e B_0}{m}$ and $\Omega = \frac{e B_0}{m} \left(1 + \frac{z}{R}\right)$.

Figure 2.4. Magnetic Field Profile in a Tokamak.

Figure 2.5. Model Magnetic Field Profile.

In the limit of the field gradient going to zero, $R \rightarrow \infty$, and equations (81), (82) and (83) reduce to (70), (71) and (72).

X-mode

$$\dot{u}_\perp = \frac{u_\perp u_{||} \Omega_0}{2\Omega R \gamma} - \frac{1}{2} \frac{e}{m} E_- \cos \psi \quad (84)$$

$$\dot{u}_{||} = - \frac{u_\perp^2 (\dot{\psi} + w) \Omega_0}{2\Omega^2 R} + \frac{1}{2} \frac{e}{m} \frac{E_-}{w} \frac{u_\perp (\dot{\psi} + w) \Omega_0}{\Omega^2 R} \sin \psi \quad (85)$$

$$\dot{\psi} = \frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{e}{m} E_- u_\perp \sin \psi \quad (86)$$

Again equations (84), (85) and (86) reduce to (76), (77) and (78) as $R \rightarrow \infty$.

2.6 Accuracy of Equations

The exact equations for particle motion in an electromagnetic field are the Lorentz force equations discussed in §2.1. For a background field of the form (80) these are:-

0-mode

$$\frac{d}{dt} (\gamma v_x) = - \Omega_0 v_y \left(1 + \frac{z}{R} \right) - \frac{e}{m} v_z \frac{k E_z}{w} \cos(kx - wt) \quad (87)$$

$$\frac{d}{dt} (\gamma v_y) = \Omega_0 \left(\frac{v_z x}{R} + v_x \left(1 + \frac{z}{R} \right) \right) \quad (88)$$

$$\frac{d}{dt} (\gamma v_z) = - \frac{\Omega_0 x v_y}{R} - \frac{e}{m} E_z \cos(kx - wt) \left(1 - \frac{v_x k}{w} \right) \quad (89)$$

and

X-mode

$$\begin{aligned} \frac{d}{dt} (\gamma v_x) = & - \Omega_0 v_y \left(1 + \frac{z}{R} \right) - \frac{e}{m} \left(E_x \cos(kx - wt) \right. \\ & \left. + \frac{v_y k E_y}{w} \sin(kx - wt) \right) \end{aligned} \quad (90)$$

$$\begin{aligned} \frac{d}{dt} (\gamma v_y) = & \Omega_0 v_x \left(1 + \frac{z}{R} \right) + \frac{\Omega_0 v_z x}{R} \\ & - \frac{e}{m} \left(1 - \frac{v_x k}{w} \right) E_y \sin(kx - wt) \end{aligned} \quad (91)$$

and

$$\frac{d}{dt} (\gamma v_z) = - \frac{\Omega_0 v_y x}{R} \quad (92)$$

with

$$E_x = \frac{1}{2} (E_+ + E_-) \quad (93)$$

and

$$E_y = \frac{1}{2i} (E_+ - E_-). \quad (94)$$

We have solved (87)-(92) numerically using standard packages and have compared the results with those from equations (81)-(86) for a variety of regimes (Table 1). The results show good agreement and substantial savings in processor time are made by using the gyroaveraged equations.

Figure 2.6 shows a sample of the results. The case illustrated is for an electron with initial $v_{\perp} = 10^7 \text{ ms}^{-1}$; $v_{\parallel} = 10^7 \text{ ms}^{-1}$ and an 0-mode wave with $E_z = 10^4 \text{ Vm}^{-1}$. The particles perpendicular velocity has increased by 3% but its parallel velocity has simply oscillated and the change averages to zero.

Table 1. Comparison of Exact and Approximate Equations.

Initial Data : $u_{\text{perp}} = 10^7 \text{ ms}^{-1}$ $u_{\text{parallel}} = 10^7 \text{ ms}^{-1}$

$$\Delta t = 10^3 \pi \omega t \quad \Psi_i = 0$$

		UPERP		Ψ	
R (m)	$E_z (\text{Vm}^{-1})$	EXACT	APPROX	EXACT	APPROX
10^3	9.95 E 6	9.95 E 6	1.85 E -3	1.74 E -3
	10^4	9.48 E 6	9.48 E 6	8.38 E -2	9.14 E -2
	10^5	5.22 E 6	5.23 E 6	9.82 E -1	9.94 E -1
	10^6	1.72 E 7	1.73 E 7	1.90 E -1	1.92 E -1
	10^7	3.55 E 7	2.29 E 7	1.20 E 0	3.08 E 0
10^3	10^3	9.95 E 6	9.95 E 6	2.83 E -1	2.90 E -1
	10^4	9.48 E 6	9.48 E 6	3.69 E -1	3.77 E -1
	10^5	5.56 E 6	5.58 E 6	1.31 E 0	1.32 E 0
	10^6	1.81 E 7	1.82 E 7	2.35 E -1	2.37 E -1
	10^7	3.43 E 7	2.16 E 7	2.81 E 0	3.08 E 0
10^2	10^3	9.99 E 6	9.99 E 6	2.81 E 0	2.81 E 0
	10^4	9.78 E 6	9.79 E 6	2.90 E 0	2.90 E 0
	10^5	8.82 E 6	8.83 E 6	3.63 E 0	3.64 E 0
	10^6	2.66 E 7	2.67 E 7	5.48 E -1	5.50 E -1
	10^7	2.75 E 7	1.37 E 7	2.91 E 0	2.79 E 0
10^1	10^3	1.01 E 7	1.01 E 7	2.84 E 0	2.84 E 0
	10^4	1.00 E 7	1.00 E 7	2.86 E 0	2.87 E 0
	10^5	9.49 E 6	9.49 E 6	3.12 E 0	3.13 E 0
	10^6	8.59 E 6	8.59 E 6	4.51 E 0	4.52 E 0
	10^7	6.52 E 7	4.51 E 7	2.55 E 0	2.34 E 0

Figure 2.6 Comparison of Exact and Averaged Equations

```

*****
MODE (1) 17/6/86
*****
PERP = 0.1000E+08
ETA  = 0.3142E+01
      = 0.1000E+08
EC.FD = 0.1000E+05
G.FD  = 0.1000E+01
FREQ  = 0.1757E+12
NO.    = 0.5860E+03
L      = 0.1000E-08
DPNT  = 0.2000E+04
OUT    = 99
*****

```

2.7 Conclusions

We have shown that the averaged Lagrangian formulation based on that described by Littlejohn provides a convenient and systematic way of describing the behaviour of particles averaged over the fast timescale associated with the cyclotron orbit. Comparison of the solutions of the averaged and exact equations show that the former provide a good approximation. The simplification and economy of computational requirements associated with the averaging allow for numerical experiments on particle diffusion discussed in Chapter 4.

CHAPTER 3

Resonant Absorption Theory

3.1 Introduction

Theoretical analyses of the heating mechanism of radio waves at the electron cyclotron frequency have ranged from detailed calculations of nonadiabatic motion of electrons in a mirror trap (Seidl (1964), Kuckes (1968), Jaeger, Lichtenberg and Lieberman (1972)) through the straightforward application of Ohms law by Sprott (1971) to purely stochastic processes by Graw (1969) and Lieberman and Lichtenberg (1973), to a resonance diffusion formalism by Eldridge (1972). This picture is further complicated by their application in tokamak geometry. Recently Dendy (1985) and Cairns and Lashmore Davies (1986) have considered flux surface averaging of the diffusion coefficient taking into account finite beam dimensions and flux surface curvature. Rotational transform effects have also been discussed by a number of authors including Fielding (1980). The general trend in experiments has been away from the magnetic mirror type confinement device which is open ended, discussed by early authors, to the closed confinement device like the tokamak and reverse field pinch. The decline in popularity of the mirror machines was primarily due to unacceptable end losses. Some work on mirror machines is still being carried out primarily at the Lawrence Livermore Laboratory, Berkeley, California.

3.2 Fokker-Planck Equation + Quasilinear Diffusion

We shall begin our analysis with the derivation of the Fokker-Planck equation and the quasilinear diffusion tensor and discuss the approximations made therein. We shall then examine modifications to the quasilinear diffusion coefficient.

We introduce the Fokker-Planck equation in a standard form Cairns (1985), Boyd and Sanderson (1969). The Fokker-Planck equation is a kinetic equation with the collision integral based on a model which describes the behaviour of a collection of particles undergoing a series of small angle scatterings. This is the case for plasmas with many particles in the Debye sphere. We define a function $P(\underline{v}, \Delta \underline{v})$ such that P is the probability that a particle with velocity $\underline{v}$ acquires an increment $\Delta \underline{v}$ in a time Δt . It is generally assumed that this process is a Markov process, that is, P does not depend on the past history of the particles motion.

The distribution at time $t + \Delta t$ may be expressed in terms of the distribution at time t by

$$f(\underline{v}, t + \Delta t) = \int f(\underline{v} - \Delta \underline{v}, t) P(\underline{v} - \Delta \underline{v}, \Delta \underline{v}) d^3 \Delta \underline{v}. \quad (1)$$

We then Taylor expand (1), to give

$$f(\underline{v}, t) + \frac{\partial f(\underline{v}, t)}{\partial t} \Delta t \doteq \int \left\{ f(\underline{v}, t) P(\underline{v}, \Delta \underline{v}) - \frac{\partial (fP)}{\partial v_i} \Delta v_i + \frac{1}{2} \frac{\partial^2 (fP)}{\partial v_i \partial v_j} \Delta v_i \Delta v_j \right\} d^3 \Delta \underline{v} \quad (2)$$

keeping terms first order in Δt and second order in $\Delta \underline{v}$. Here we have assumed f , P are smooth functions of $\underline{v}$ and t and $\Delta \underline{v}$ and Δt to be

small, as we have neglected higher order terms. Such an assumption is usually justified since most plasma collisions are small angle scatterings.

Clearly

$$\int P(\underline{v}, \underline{\Delta v}) d^3 \Delta v = 1 \quad (3)$$

and

$$\int P(\underline{v}, \underline{\Delta v}) \Delta v_i d^3 \Delta v = \langle \Delta v_i \rangle \quad (4)$$

and

$$\int P(\underline{v}, \underline{\Delta v}) \Delta v_i \Delta v_j d^3 \Delta v = \langle \Delta v_i \Delta v_j \rangle \quad (5)$$

where $\langle \rangle$ denote the average value.

Hence equation (2) becomes

$$\frac{\partial f}{\partial t}(\underline{v}, t) \Delta t = - \frac{\partial}{\partial v_i} (\langle \Delta v_i \rangle f) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} (\langle \Delta v_i \Delta v_j \rangle f) \quad (6)$$

which can be rewritten as

$$\frac{\partial f}{\partial t} = - \frac{\partial}{\partial v_i} (A_i f) + \frac{1}{2} \frac{\partial}{\partial v_j} \left(D_{ij} \frac{\partial f}{\partial v_j} \right) \quad (7)$$

where

$$A_i = \frac{\langle \Delta v_i \rangle}{\Delta t} - \frac{1}{2} \frac{\partial}{\partial v_j} \frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t} \quad (8)$$

$$D_{ij} = \frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t} . \quad (9)$$

The first term on the right hand side of equation (8) is associated with the slowing down of particles owing to collisions -

the friction term, the second term is the diffusion term which spreads particles in velocity space. Originally the Fokker-Planck equation was used to describe the effects of particle-particle collisions in a plasma, but one can similarly imagine a radio-frequency heating scheme to involve particle-wave "collisions" which will lead to diffusion of the particles in velocity space. Typically then the Fokker-Planck equation is of the form

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t} \right)_c + \left(\frac{\partial f}{\partial t} \right)_w + \left(\frac{\partial f}{\partial t} \right)_E \quad (10)$$

for a tokamak. The terms on the right hand side of (9) are due to collisions, r.f. wave induced diffusion and the steady toroidal electric field, respectively. Often these quantities will be averaged over the particle bounce time.

In particle-particle collisions the coefficients A_i and D_{ij} are calculated from moments of the initial distribution function, for example

$$\frac{\langle \Delta v_i \rangle}{\Delta t} = - \frac{q_s^2}{4\pi\epsilon_0^2 m_s} \sum_{s'} \frac{q_{s'}^2 \log \Lambda}{\mu_{ss'}} \int \frac{u_i}{u^3} f_{s'}(\underline{v}') d^3 v' \quad (11)$$

for a plasma with particle species, s . Here $\mu_{ss'}$ is the reduced mass corresponding to m_s and $m_{s'}$, and $\underline{u} = \underline{v} - \underline{v}'$. In "particle-wave" collisions one must adopt a different approach. One such approach common to many authors is to use "quasi-linear" theory.

3.3 Quasilinear Theory

The diffusion coefficient owing to a radio frequency wave is most commonly derived using quasilinear theory, O'Brien (1986), Demio and Engelmann (1986). For a magnetised plasma quasilinear theory was first discussed by Bernstein and Engelmann (1966) and Drummond and Pines (1964).

In quasilinear theory the particle distribution function is split into a slowly varying averaged part together with a rapidly varying part produced by the wave.

To simplify matters we will now consider the plasma to be collisionless and to be immersed in a static magnetic field $\underline{B}_0$. The plasma obeys Vlasov's equation

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f - \frac{e}{m} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{v}} = 0. \quad (12)$$

with $E(\underline{x}, t) = \tilde{E}(\underline{x}, t)$, $\underline{B}(\underline{x}, t) = \underline{B}_0(\underline{x}, t) + \tilde{\underline{B}}(\underline{x}, t)$. Denoting by f_0 and $\tilde{f}$ the slowly varying part of the distribution function and the rapidly varying part due to the wave with $\langle \tilde{f} \rangle = \langle \tilde{E} \rangle = \langle \tilde{\underline{B}} \rangle = 0$, then to zero order in the amplitude of the waves and the Larmor radius, Vlasov's equation yields

$$(\underline{v} \times \underline{B}_0) \cdot \frac{\partial f_0}{\partial \underline{v}} = 0. \quad (13)$$

That is $f_0 = f_0(v_{||}, v_{\perp})$. To the next order in the wave amplitudes Vlasov's equation yields

$$\frac{\partial f_0}{\partial t} + \frac{\partial \tilde{f}}{\partial t} + \underline{v} \cdot \underline{\nabla} \tilde{f} - \frac{e}{m} (\tilde{\underline{E}} + \underline{v} \times \tilde{\underline{B}}) \cdot \frac{\partial f_0}{\partial \underline{v}} = \frac{e}{m} (\underline{v} \times \underline{B}_0) \cdot \frac{\partial \tilde{f}}{\partial \underline{v}} \quad (14)$$

and averaging over the short timescale associated with the waves and over the phase ϑ associated with the motion of the electrons around

$\underline{B}_0$ gives the time evolution of f_0 induced by the waves as

$$\frac{\partial f_0}{\partial t} = \frac{e}{m} \int_0^{2\pi} \frac{d\vartheta}{2\pi} \left\langle (\underline{\tilde{E}} + \underline{v} \times \underline{\tilde{B}}) \cdot \frac{\partial \tilde{f}}{\partial \underline{v}} \right\rangle \quad (15)$$

which shows that $\frac{\partial f_0}{\partial t}$ is second order in perturbed quantities.

Substituting (15) back into (14) gives

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial t} + \underline{v} \cdot \underline{\nabla} \tilde{f} - \frac{e}{m} (\underline{\tilde{E}} + \underline{v} \times \underline{\tilde{B}}) \cdot \frac{\partial f_0}{\partial \underline{v}} - \frac{e}{m} (\underline{v} \times \underline{B}_0) \cdot \frac{\partial \tilde{f}}{\partial \underline{v}} \\ = - \frac{e}{m} \int_0^{2\pi} \frac{d\vartheta}{2\pi} \left\langle (\underline{\tilde{E}} + \underline{v} \times \underline{\tilde{B}}) \cdot \frac{\partial \tilde{f}}{\partial \underline{v}} \right\rangle. \end{aligned} \quad (16)$$

The essential approximation in quasilinear theory is to neglect the right hand side of (16), so that $\tilde{f}$ is just described by the usual linearised Vlasov equation, now however f_0 is no longer constant but varies on the slow timescale according to equation (15). Now if we consider an electrostatic wave $\underline{\tilde{B}} = 0$ in an unmagnetised plasma with electric field consisting of a superposition of Fourier components $\underline{E}_k$ corresponding to wavenumber $\underline{k}$ i.e.

$$\underline{E} = \sum_k \underline{E}_k e^{i(\underline{k} \cdot \underline{r} - \omega t)} = \sum_k \underline{E}_k e^{i(\underline{k} \cdot \underline{v} - \omega)t}. \quad (17)$$

Then integrating (14) gives

$$\tilde{f} = i \frac{e}{m} \sum_k \underline{E}_k \cdot \frac{\partial f_0}{\partial \underline{v}} / (\omega - \underline{k} \cdot \underline{v}) \quad (18)$$

which allows equation (15) to be written in the tensor form

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v_i} D_{ij} \frac{\partial f_0}{\partial v_j} \quad (19)$$

with the diffusion coefficient given by

$$D_{ij} = \frac{1}{2} \left(\frac{e}{m} \right)^2 \operatorname{Im} \sum_k \frac{E_{ki} \bar{E}_{kj}}{w - \underline{k} \cdot \underline{v}}. \quad (20)$$

For a magnetised plasma with an electromagnetic wave this takes a more complicated form

$$D_{ij} = \left(\frac{e}{m} \right)^2 a_{k_0} \cdot \left[\int_{-\infty}^t e^{-i \int_t^{t'} \Omega_{k_0} dt''} b_{k_0} - \frac{i}{w} c_{k_0} \right] + \text{complex conjugate} \quad (21)$$

where

$$\Omega_{k_0} = w - k_{||} v_{||} - w_c$$

$$b_{k_0} = \vartheta_{k_0} \left(n \frac{v_{\perp}}{v_{||}} \frac{w - w_c}{w} + \frac{v_{\perp}}{v_{||}} \frac{w_c}{w} \right)$$

$$c_{k_0} = \epsilon_{k_0} \left(n \frac{v}{v_{||}} - \frac{v}{v} \right)$$

$$a_{k_0} = b_{k_0} - \frac{\Omega_{k_0}}{w} c_{k_0}$$

$$\epsilon_{k_0} = J_0 E_{k_0}^- e^{i\psi} + J_2 E_{k_0}^+ e^{-i\psi}$$

$$\vartheta_{k_0} = \epsilon_{k_0} + \frac{v_{||}}{v_{\perp}} J E''_{k_0}$$

with $E_{k_0}^{\pm}$ being the left and right hand polarisations of the electric field and $E''_{k_0} = E_{k_0} \cdot \underline{n}$, the parallel component.

3.4 A Simplified Diffusion Coefficient

We now introduce a simplified diffusion coefficient of the type considered by Cairns and Lashmore-Davies (1986).

Consider a system as illustrated by Fig (3.1). A particle in resonance with the beam when it intersects the flux surface is accelerated according to

$$\dot{v}_z = \frac{e}{m} E_{\text{eff}} \cos \psi. \quad (22)$$

For example for the 0-mode we have equation (2.70)

$$\dot{u}_z = -\frac{1}{2} \frac{e}{m} \frac{kE_z}{w} \dot{z} \cos \psi \quad (23)$$

and equation (2.72)

$$\dot{\psi} = \frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{e}{m} \frac{kE_z}{w u_z} v_{\parallel} \sin \psi. \quad (24)$$

For low powers of heating on resonance it is common to set the right hand side of eq (24) = 0. Hence $\psi = \psi_0$, a constant and we can integrate eq (23) to get

$$\Delta u_z = -\frac{1}{2} \frac{e}{m} \frac{kE_z}{w} \dot{z} \cos \psi_0 \Delta t \quad (25)$$

with $\Delta t = \frac{L}{\dot{z}} \quad L = \text{beam width.} \quad (26)$

Equation (25) describes the kick in v_z given to the particle as it traverses the beam. If we consider the particle to be sufficiently close to the $q=1$ surface so that it continues to return through the beam after a trip round the torus

Figure 3.1. A Schematic Diagram of ECRH.

$$\langle \Delta u_{\mathbf{k}}^2 \rangle = \frac{1}{8} \left(\frac{e}{m} \frac{kL}{w} \right)^2 E_z^2 n \quad (27)$$

or after n passes

$$\frac{\langle \Delta u_{\mathbf{k}}^2 \rangle}{n} = \frac{1}{8} \left(\frac{e}{m} \frac{kL}{w} \right)^2 E_z^2. \quad (28)$$

An expression for particles not on the $q = 1$ surface is given by Dendy (1985).

The diffusion coefficient is then

$$D^{OM} = \frac{1}{2} \frac{\langle \Delta u_{\mathbf{k}}^2 \rangle}{t} = \frac{1}{16} \left(\frac{e}{m} \frac{E_z}{c} \right)^2 L \dot{Z}. \quad (29)$$

However for large E_z it is not appropriate to set the right hand side of equation (24) to zero.

Consider instead

$$\psi = \psi_0 + \delta t \quad (30)$$

with $\delta \ll 2\pi/t$

then substituting equation (30) into equation (23) and integrating gives

$$u_{\mathbf{k}} = -\frac{A}{\delta} \sin(\psi_0 + \delta t) + \text{constant} \quad (31)$$

$$A = \frac{1}{2} \frac{e}{m} \frac{k E_z \dot{Z}}{w} \quad (32)$$

at $t = 0$

$$u_{\mathbf{k}} = u_{\mathbf{k}i}$$

so equation (31) becomes

$$\Delta u_{\mathbf{k}} = \frac{A}{\delta} \left\{ \sin \psi_0 [\cos \delta \Delta t - 1] + \cos \psi_0 \sin \delta \Delta t \right\} \quad (33)$$

on evaluation at $t = \Delta t$.

So

$$(\Delta u_x)^2 = \left(\frac{A}{\delta}\right)^2 \left\{ \sin^2 \psi_0 [\cos \delta \Delta t - 1]^2 + \cos^2 \psi_0 \sin^2 \delta \Delta t \right\} \\ + \text{cross terms that will average out.} \quad (34)$$

Taking an ensemble of particles and averaging over ψ_0 gives

$$\langle (\Delta u_x)^2 \rangle = \frac{1}{2} \left(\frac{e}{m} \frac{k E_z \dot{Z}}{w} \right)^2 \left(\frac{\sin \left(\frac{\delta \Delta t}{2} \right)}{\delta} \right)^2. \quad (35)$$

Giving a revised diffusion coefficient of the form

$$D_{RED}^{OM} = \frac{1}{2} \frac{\langle (\Delta u_x)^2 \rangle}{t} = \frac{1}{4} \frac{\dot{Z}}{L} \left(\frac{e}{m} \frac{k E_z \dot{Z}}{w} \right)^2 \left(\frac{\sin \left(\frac{\delta \Delta t}{2} \right)}{\delta} \right)^2. \quad (36)$$

As one would expect

$$D_{RED}^{OM} \leq D^{OM} \quad (37)$$

and

$$\lim_{\delta \rightarrow 0} D_{RED}^{OM} = D^{OM}. \quad (38)$$

Thus we have derived an expression for the diffusion coefficient in velocity space for an ensemble of electrons subjected to an 0-mode wave. Replacing equation (23) by equation (2.76) and equation (24) by equation (2.78) in the above analysis yields

$$D^{XM} = \frac{1}{16} \left(\frac{e}{m} E_- \right)^2 \frac{L}{\dot{Z}} \quad (39)$$

and

$$D_{\text{RED}}^{\text{XM}} = \frac{1}{4} \left(\frac{e}{m} E_- \right)^2 \frac{\dot{Z}}{L} \left(\frac{\sin \left(\frac{\delta \Delta t}{2} \right)}{\delta} \right)^2. \quad (40)$$

The expressions are valid for

$$\dot{\psi} \lesssim 2\pi \frac{\dot{Z}}{L} \quad (41)$$

and have been compared with numerical experiments in Chapter 4.

3.5 The Inapplicability of the Fokker-Planck Equation

We have already noted that the Fokker-Planck equation is derived by Taylor expanding the probability distribution function

$$f(\underline{v}, t+\Delta t) = \int f(\underline{v}-\Delta\underline{v}, t) P(\underline{v}-\Delta\underline{v}, \Delta\underline{v}) d^3\Delta\underline{v}. \quad (42)$$

This expansion is only valid for

$$|\Delta\underline{v}| \ll \left| f / \frac{\partial f}{\partial \underline{v}} \right| \quad (43)$$

but allows us to proceed with calculations ignorant of the exact forms of $P(\underline{v}-\Delta\underline{v}, \Delta\underline{v})$. We are interested in large electric fields which consequently will often produce large kicks in $\underline{v}$ so we shall proceed to derive forms for $P(\underline{v}-\Delta\underline{v}, \Delta\underline{v})$.

3.6 Derivation of the Probability Function $P(\underline{v}, \Delta \underline{v})$

Following the prescription of O'Brien (1987a) we consider

$$\frac{\partial f}{\partial t} = \frac{1}{\Delta t} \int P(\underline{v}-\Delta \underline{v}, \Delta \underline{v}) (f(\underline{v}-\Delta \underline{v}) - f(\underline{v})) d(\Delta \underline{v}) \quad (44)$$

with $P(\underline{v}, \Delta \underline{v})$ the probability function for a change in velocity from $\underline{v}$ to $\underline{v}+\Delta \underline{v}$ in time Δt .

Then

$$\int P(\underline{v}, \Delta \underline{v}) d(\Delta \underline{v}) = 1 \quad \forall \underline{v}. \quad (45)$$

Now if we change to canonical coordinates

$$\mu = \frac{v_{\perp}^2}{2} \quad \text{and} \quad v_{\parallel}$$

$$\text{then} \quad f \equiv f(\mu, v_{\parallel}) \quad (46)$$

$$\text{and} \quad P(\underline{v}, \Delta \underline{v}) \equiv P(\mu, v_{\parallel}; \Delta \mu, \Delta v_{\parallel}) \quad (47)$$

and assume changes in μ only. We begin by considering B_0 constant.

$$\mu = \frac{u_{\perp}^2}{2} = \frac{1}{2} (\text{Re}(u_x))^2 + \frac{1}{2} (\text{Re}(u_y))^2 \quad (48)$$

$$\text{with} \quad u_{\perp} = \gamma v_{\perp} = \frac{P_{\perp}}{m}.$$

Putting

$$u_{\pm} = u_x \pm i u_y \quad (49)$$

gives

$$\mu = \frac{1}{8} |u_{-}|^2 \quad \text{if} \quad u_{+} = 0. \quad (50)$$

If we choose u_+ to be initially zero it will remain so as only u_- is driven.

Hence

$$\Delta\mu = \frac{1}{8} \Delta\{|u_-|^2\}. \quad (51)$$

From the Lorentz equation we have

$$\frac{du}{dt} = -\frac{e}{m} (\tilde{E} + \underline{v} \times \tilde{B} + \underline{v} \times B_0) \quad (52)$$

giving

$$\frac{du_-}{dt} = -\frac{e}{m} (\tilde{E}_- + i v_- \tilde{B}_z - i v_z \tilde{B}_- + i v_- B_0) \quad (53)$$

with

$$\tilde{E}_\pm = \tilde{E}_x \pm i \tilde{E}_y$$

$$v_\pm = v_x \pm i v_y$$

$$\tilde{B}_\pm = \tilde{B}_x \pm i \tilde{B}_y.$$

Writing $\Omega = \frac{eB_0}{\gamma m}$ gives

$$\frac{du_-}{dt} + i u_- \Omega = -\frac{e}{m} \left(\tilde{E}_- + \frac{i u_- \tilde{B}_z}{\gamma} - \frac{i u_z \tilde{B}_-}{\gamma} \right). \quad (54)$$

Introducing an integrating factor into (54) gives

$$\frac{d}{dt} \left(e^{i \int^t \Omega(\tau) d\tau} u_- \right) = -\frac{e}{m} e^{i \int^t \Omega(\tau) d\tau} \left(\tilde{E}_- + \frac{i u_- \tilde{B}_z}{\gamma} - \frac{i u_z \tilde{B}_-}{\gamma} \right). \quad (55)$$

Taking

$$\tilde{E} \text{ of the form } e^{i(\underline{k} \cdot \underline{r} - \omega t)}$$

with

$$\underline{k} = (k_x, 0, k_z).$$

Then $\underline{k} \times \tilde{E} = \omega \tilde{B}$ and

$$\tilde{B}_- = -\frac{ik_{||}\tilde{E}_-}{w} + \frac{ik_b\tilde{E}_z}{w} \quad (56)$$

and

$$\tilde{B}_z = \frac{k_b}{w} \tilde{E}_y. \quad (57)$$

Hence the effective electric field in the above is

$$\tilde{E}_{eff} = \tilde{E}_- + \frac{ik_b v_-}{w} \tilde{E}_y - \frac{k_{||} v_{||}}{w} \tilde{E}_- + \frac{k_b v_{||}}{w} \tilde{E}_z. \quad (58)$$

For the X-mode with $v \ll c$ this is

$$\tilde{E}_{eff} \sim \tilde{E}_-. \quad (59)$$

For the O-mode with $v \ll c$ this is

$$\tilde{E}_{eff} \sim \frac{k_b v_{||}}{w} \tilde{E}_z \quad (k_{||} = 0). \quad (60)$$

Hence equation (55) becomes

$$\frac{d}{dt} \left(e^{i \int^t \Omega d\tau} u_- \right) = -\frac{e}{m} \tilde{E}_{eff} e^{i \int^t \Omega d\tau}. \quad (61)$$

Now put $\tilde{E}_{eff} = \tilde{E}_0 e^{i(k_b x + k_{||} z - \omega t)}$ to give

$$\left[u_- e^{i \int^t \Omega d\tau} \right]_0^t = -\frac{e}{m} \tilde{E}_0 \int_0^t \exp \left[i(\Omega t' + k_{||} z' + k_b x' - \omega t') \right] dt'. \quad (62)$$

Considering a particle traversing a beam of width L , gives

$$t = L/v_{||}.$$

Now $z' = v_{||} t'$ and $x' = \frac{u_b}{\Omega} \sin(\Omega t' + \psi)$ as before and substituting these into equation (62) gives

$$|u_-(t)| = \left| u_-(0) - \frac{e\tilde{E}_0}{m} \int_0^t \exp \left[i(\Omega + k_{||}v_{||} - \omega)t' + \frac{ik_{\perp}u_{\perp}}{\Omega} \sin(\Omega t' + \psi) \right] dt' \right| \quad (63)$$

Using

$$e^{iz \sin \vartheta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\vartheta} \quad (64)$$

and retaining only the $n=0$ term for resonance with J_0 approximated by 1 gives

$$|u_-(t)| \approx \left| u_-(0) - \frac{e\tilde{E}_0}{m} \int_0^{L/v_{||}} \exp \left[i(\Omega + k_{||}v_{||} - \omega)t' \right] dt' \right|. \quad (65)$$

Initially

$$u_- = 2u_{\perp} e^{i\varphi} \text{ with } \varphi \text{ a phase angle.}$$

Writing

$$K = - \frac{e\tilde{E}_0}{m} \int_0^{L/v_{||}} \exp \left[i(\Omega + k_{||}v_{||} - \omega)t' \right] dt' \quad (66)$$

then

$$u_-(t) \cong u_-(0) + Ke^{i\psi}$$

with ψ another phase factor.

Hence

$$\begin{aligned}
 \Delta\mu &= \frac{1}{8} \Delta\{|u_-|^2\} \\
 &= \frac{1}{8} \left\{ (\bar{u}_-(0) + \bar{K}e^{-i\psi})(u_-(0) + Ke^{i\psi}) - |u_-(0)|^2 \right\} \\
 &= \frac{1}{8} \left\{ |K|^2 + 2|K||u_-(0)|\cos\vartheta \right\}
 \end{aligned}$$

with ϑ a final phase factor.

Hence we have

$$\Delta\mu = \frac{|K|u_b}{2} \cos \vartheta + \frac{1}{8} |K|^2. \quad (67)$$

This expression is appropriate for small kicks in the particles perpendicular energy, that is

$$\frac{1}{8} |K|^2 \ll \frac{|K|u_b}{2}$$

or $|E_{\text{eff}}| \ll 2 \cdot 10^4$

for $v_b \sim v_{||} \sim 10^7 \text{ ms}^{-1}; \quad L \sim 0.1 \text{ m}.$

For the 0-mode this corresponds to

$$|E_z| \ll 7 \cdot 10^5.$$

An expression for $\Delta\mu$ larger fields is derived in Section 3.8 but for arbitrarily large fields $\Delta\mu$ must be calculated numerically. Now we have equation (66)

$$K = - \frac{e\tilde{E}_0}{m} \int_0^{L/v_{||}} \exp \left[i(\Omega + k_{||}v_{||} - \omega)t' \right] dt'.$$

If B_0 is constant then $\Omega \sim \text{constant}$ so

$$K = -\frac{e\tilde{E}_0}{m} \left[\frac{-i}{\Omega + k_{||}v_{||} - w} \exp \left[i(\Omega + k_{||}v_{||} - w)t' \right] \right]_0^{L/v_{||}}$$

so

$$|K| = \frac{e}{m} |\tilde{E}_0| \frac{2}{|\Omega + k_{||}v_{||} - w|} \left| \sin \left(\frac{(\Omega + k_{||}v_{||} - w)L}{2v_{||}} \right) \right|. \quad (68)$$

Putting $\delta = \frac{1}{2}(\Omega + k_{||}v_{||} - w)$ gives

$$|K| = \frac{e}{m} \left| \frac{\tilde{E}_0}{\delta} \sin \left(\frac{\delta L}{v_{||}} \right) \right| \quad (69)$$

with $k_{||} = 0$ and for particles exactly in resonance $\delta = 0$ hence

$$|K| \rightarrow \frac{e}{m} \frac{L}{v_{||}} |\tilde{E}_0|. \quad (70)$$

It is clear from equation (70) that K is in fact the impulse received by the particle from the beam.

Now, as

$$P(\Delta\mu)d(\Delta\mu) = P(\vartheta)d\vartheta \quad (71)$$

then

$$P(\Delta\mu) = P(\vartheta) \left| \frac{d\vartheta}{d(\Delta\mu)} \right|. \quad (72)$$

Now from equation (67) we have

$$\vartheta = \cos^{-1} \left(\frac{2}{|K|u_b} \left(\Delta\mu - \frac{1}{8} |K|^2 \right) \right) \quad (73)$$

so

$$\left| \frac{d\vartheta}{d\Delta\mu} \right| = \frac{1}{\sqrt{1 - \frac{4}{(|K|u_b)^2} \left(\Delta\mu - \frac{1}{8} |K|^2 \right)^2}} \frac{2}{|K|u_b} . \quad (74)$$

Hence

$$P(\Delta\mu) = \frac{2}{|K|u_b\pi} \frac{1}{\sqrt{1 - \frac{4}{(|K|u_b)^2} \left(\Delta\mu - \frac{1}{8} |K|^2 \right)^2}} \quad (75)$$

if we assume $P(\vartheta)$ is constant on $(0, \pi)$.

Hence we have, O'Brien (1987a)

$$P(\mu, \Delta\mu) = \frac{2}{|K|u_b\pi} \frac{1}{\sqrt{1 - \frac{4}{(|K|u_b)^2} \left(\Delta\mu - \frac{1}{8} |K|^2 \right)^2}}$$

$$\text{for } \left| \Delta\mu - \frac{|K|^2}{8} \right| \leq \frac{|K|u_b}{2}$$

$$= 0 \quad \left| \Delta\mu - \frac{|K|^2}{8} \right| > \frac{|K|u_b}{2} \quad (76)$$

and

$$\int_{-\infty}^{\infty} P(\mu, \Delta\mu) d(\Delta\mu) = 1 \text{ as required,}$$

despite the singularity in expression (75) which is integrable.

We can now proceed to calculate the coefficients

$$\langle \Delta\mu \rangle \quad \text{and} \quad \langle (\Delta\mu)^2 \rangle$$

from

$$\langle g \rangle = \int gP(\mu, \Delta\mu) d\Delta\mu.$$

Then

$$\langle \Delta\mu \rangle = \frac{2}{|K|u_b\pi} \int \frac{\Delta\mu}{\sqrt{1 - \frac{4}{(|K|u_b)^2} \left(\Delta\mu - \frac{|K|^2}{8} \right)^2}} d\Delta\mu \quad (77)$$

with the limits of the integral given by

$$-\frac{|K|u_b}{2} \leq \Delta\mu - \frac{|K|^2}{8} \leq \frac{|K|u_b}{2}. \quad (78)$$

Making the substitution

$$x = \frac{2}{|K|u_b} \left(\Delta\mu - \frac{|K|^2}{8} \right) \quad (79)$$

yields

$$\langle \Delta\mu \rangle = \frac{1}{\pi} \int_{-1}^{+1} \frac{\frac{|K|u_b x}{2} + \frac{|K|^2}{8}}{\sqrt{1 - x^2}} dx.$$

The first term in the integrand is an odd function of x and hence has integral zero. The second term evaluates to

$$\langle \Delta\mu \rangle = \frac{1}{\pi} \frac{|K|^2}{8} \left[\sin^{-1} \right]_{-1}^{+1} = \frac{|K|^2}{8}. \quad (80)$$

Similarly

$$\langle (\Delta\mu)^2 \rangle = \frac{1}{\pi} \int_{-1}^{+1} \frac{\left(\frac{|K|u_b}{2} + \frac{|K|^2}{8} \right)^2}{\sqrt{1 - x^2}} dx \quad (81)$$

which evaluates to

$$\langle (\Delta\mu)^2 \rangle = \frac{|K|^2 u_b^2}{8} + \frac{|K|^4}{64} \quad (82)$$

$$= \frac{|K|^2 \mu}{4} + \frac{|K|^4}{64} \quad (83)$$

for $|B_0| = 1$.

This is the correct diffusive form since from Fokker-Planck theory we have

$$\frac{\partial f}{\partial t} = - \frac{\partial}{\partial \mu} \left[\frac{\langle \Delta\mu \rangle}{\Delta t} f \right] + \frac{1}{2} \frac{\partial^2}{\partial \mu^2} \left[\frac{\langle (\Delta\mu)^2 \rangle}{\Delta t} f \right] \quad (84)$$

$$= - \frac{\partial}{\partial \mu} \left(\frac{\langle \Delta\mu \rangle}{\Delta t} \right) f - \frac{\langle \Delta\mu \rangle}{\Delta t} \frac{\partial f}{\partial \mu} + \frac{1}{2} \frac{\partial}{\partial \mu} \left[\frac{\partial}{\partial \mu} \left(\frac{\langle (\Delta\mu)^2 \rangle}{\Delta t} \right) f + \frac{\langle (\Delta\mu)^2 \rangle}{\Delta t} \frac{\partial f}{\partial \mu} \right]. \quad (85)$$

Applying expressions (80) and (83) for $\langle \Delta\mu \rangle$ and $\langle (\Delta\mu)^2 \rangle$ in equation (85) yields

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial \mu} \left[\frac{\langle (\Delta\mu)^2 \rangle}{\Delta t} \frac{\partial f}{\partial \mu} \right] \quad (86)$$

as

$$\frac{\langle \Delta\mu \rangle}{\Delta t} = \frac{1}{2} \frac{\partial}{\partial \mu} \left[\frac{\langle (\Delta\mu)^2 \rangle}{\Delta t} \right] \quad (87)$$

and we can write the Fokker-Planck equation as a diffusion equation. For a fuller discussion see §5.4 Lichtenberg and Lieberman (1983).

3.7 Derivation of K for $B_0 = B_0(z)$

From equation (66) in Section 3.6 we have

$$K = - \frac{e\tilde{E}_0}{m} \int_0^t \exp \left[i(\Omega + k_{||}v_{||} - w)t' \right] dt'.$$

$$K = - \frac{e\tilde{E}_0}{m} T_{\text{eff}}$$

where T_{eff} is the effective time the particle feels the beam.

In Section 3.6 we assumed B_0 and consequently Ω was constant. However if we now allow B_0 to vary with z then as the particle traverses the beam the cyclotron frequency will change in time. If we assume the particle is travelling into a region of higher magnetic field then the cyclotron frequency will increase. If such an increase is large enough, the particle will be reflected. First let us consider transit particles, that is particles which are not reflected back into the beam, following O'Brien (1987b), we have

$$T_{\text{eff}} = |I|$$

where

$$I = \int_0^t e^{i \left(\int_0^{t'} \Omega dt'' - wt' \right)} dt' \text{ putting } k_{||} = 0. \quad (88)$$

Now

$$\Omega = \Omega_0 + \dot{\Omega}t'' \quad \text{so} \quad \int_0^{t'} \Omega dt'' = \Omega t' + \frac{\dot{\Omega}t'^2}{2}.$$

If

$$\Omega = \Omega_0 \left(1 + \frac{Z}{R}\right) \quad \dot{\Omega} = \frac{\Omega_0 v_H}{R}$$

and

$$I \approx \int_0^t e^{i \left(\frac{\dot{\Omega} t'^2}{2} + (\Omega_0 - \omega) t' \right)} dt'. \quad (89)$$

Substituting $a = \Omega_0 - \omega$ and $b = \left| \frac{\dot{\Omega}}{2} \right|$

$$I \approx \int_0^t e^{i(bt'^2 + at')} dt'. \quad (90)$$

Writing $x = \sqrt{b}(t' + a/2b)$ gives

$$|I| = \left| \int_{a/2\sqrt{b}}^{\sqrt{b}(t + a/2b)} e^{ix^2} \frac{dx}{\sqrt{b}} \right| \quad (91)$$

and putting

$$x = \sqrt{\frac{\pi}{2}} y$$

yields

$$\begin{aligned} |I| &= \sqrt{\frac{\pi}{2b}} \left| \int_{\frac{a}{\sqrt{2\pi b}}}^{\sqrt{\frac{2b}{\pi}} \left(t + \frac{a}{2b} \right)} e^{i \frac{\pi}{2} y^2} dy \right| \\ &= \sqrt{\frac{\pi}{2b}} \left| F \left(\sqrt{\frac{2b}{\pi}} \left(t + \frac{a}{2b} \right) \right) - F \left(\frac{a}{\sqrt{2\pi b}} \right) \right|. \end{aligned} \quad (92)$$

where F is the Fresnel integral

$$\int_0^u e^{i \frac{\pi}{2} t^2} dt.$$

We define $F(u) = C(u) + iS(u)$ where

$$C(u) = \int_0^u \cos\left(\frac{\pi}{2} t^2\right) dt \quad (93)$$

$$S(u) = \int_0^u \sin\left(\frac{\pi}{2} t^2\right) dt \quad (94)$$

see Abramowitz and Stegun (1972), §7.3.

Consider now a particle which turns in the beam. At the turning point $\dot{\Omega} = 0$ but $\ddot{\Omega} \neq 0$. Hence

$$\Omega = \Omega_0 + \frac{\ddot{\Omega} t^2}{2} \quad \ddot{\Omega} < 0 \quad (95)$$

and
$$\int \Omega dt = \Omega_0 t + \frac{\ddot{\Omega} t^3}{6} . \quad (96)$$

Substituting equation (96) into equation (88) gives

$$I \approx \int_{-\infty}^{\infty} e^{i \left(\omega t - \Omega_0 t - \frac{\ddot{\Omega} t^3}{6} \right)} dt. \quad (97)$$

Where the limits of the integral have been replaced by $\pm\infty$ on the assumption that the main contributions to the integral come from the resonance $\omega = \Omega$.

Substituting $r = \left(-\frac{\ddot{\Omega}}{6} \right)^{1/3} t$ gives

$$\begin{aligned}
 I &\approx \int_{-\infty}^{\infty} e^{i \left[(w - \Omega_0) \left(-\frac{6}{\Omega} \right)^{1/3} r + r^3 \right]} \left(-\frac{6}{\Omega} \right)^{1/3} dr \\
 &= \left(-\frac{6}{\Omega} \right)^{1/3} \int_{-\infty}^{\infty} e^{i(ar + r^3)} dr \quad a = (w - \Omega_0) \left(-\frac{6}{\Omega} \right)^{1/3} \\
 &= \left(-\frac{6}{\Omega} \right)^{1/3} \left[\int_{-\infty}^{\infty} \cos(ar + r^3) + i \sin(ar + r^3) dr \right] \\
 &= \left(-\frac{6}{\Omega} \right)^{1/3} \int_{-\infty}^{\infty} \cos(ar + r^3) dr \tag{98}
 \end{aligned}$$

as $\sin(ar + r^3)$ is an odd function of r .

Hence

$$\begin{aligned}
 I &\approx 2 \left(-\frac{6}{\Omega} \right)^{1/3} \int_0^{\infty} \cos(ar + r^3) dr \\
 &= 2 \left(-\frac{6}{\Omega} \right)^{1/3} (3)^{-1/3} \pi \text{Ai} \left((3)^{-1/3} \cdot a \right) \\
 &= 2 \left(-\frac{2}{\Omega} \right)^{1/3} \pi \text{Ai} \left((w - \Omega_0) \left(-\frac{2}{\Omega} \right)^{1/3} \right) \tag{99}
 \end{aligned}$$

where $\text{Ai}(z)$ is the first Airy function. See Abramowitz and Stegun (1972), §10.4.

3.8 Derivation of $\Delta\mu$ for Large Electric Fields

So far we have considered the cyclotron frequency to vary as a function of the magnetic field alone, but as the particle is accelerated by the beam the mass and hence the cyclotron frequency will be altered. We shall derive here our equation for $\Delta\mu$ which includes this effect.

We can rewrite equation (61) as

$$u_- e^{i \int_0^t \Omega dt''} = -\frac{e}{m} \tilde{E}_0 \int_0^t e^{i \left(\int_0^{t'} \Omega dt'' - \omega t' \right)} dt' + 2u_+ e^{-i\psi} \quad (100)$$

where $u_-(0) = 2u_+ e^{-i\psi}.$ (101)

Now, considering B_0 constant

$$\Omega = \Omega(t) = \frac{\Omega_0}{\gamma(t)} = \frac{\Omega_0}{\left(1 + \frac{u_+^2 + u_-^2}{c^2} \right)^{\frac{1}{2}}} \quad (102)$$

and $\dot{\Omega} = -\frac{\Omega}{\gamma^2} \frac{u_+ \dot{u}_+}{c^2} \quad \begin{matrix} u_+ = \gamma v_+ \\ u_- = \gamma v_- \end{matrix}$ (103)

assuming u_- constant.

Now let $\Omega = \Omega_1 + \dot{\Omega}t$ with $\dot{\Omega}$ now taken constant.

Then equation (100) becomes

$$u_- = -\frac{e}{m} \tilde{E}_0 e^{-i \left(\frac{(\Omega_1 - \omega)^2}{2\dot{\Omega}} + \Omega_1 t + \frac{\dot{\Omega} t^2}{2} \right)} \times \int_0^t e^{i \frac{\dot{\Omega}}{2} \left(t' + \frac{(\Omega_1 - \omega)}{\dot{\Omega}} \right)^2} dt' + 2u_+ e^{i \left(\psi - \Omega_1 t - \frac{\dot{\Omega} t^2}{2} \right)} \quad (104)$$

So

$$u_- = -\frac{e}{m} \tilde{E}_0 e^{-i\left(\frac{(\Omega_1 - w)^2}{2\dot{\Omega}} + \eta\right)} \sqrt{\frac{2}{|\dot{\Omega}|}} \int_{A'}^{B'} e^{\pm i v^2} dv + 2u_+ e^{i(\psi - \eta)} \quad (105)$$

where

$$A' = \sqrt{\frac{|\dot{\Omega}|}{2}} \left(\frac{\Omega_1 - w}{\dot{\Omega}} \right); \quad B' = \sqrt{\frac{|\dot{\Omega}|}{2}} \left(t + \frac{\Omega_1 - w}{\dot{\Omega}} \right)$$

and
$$\eta = \Omega_1 t + \frac{\dot{\Omega} t^2}{2}$$

and the choice of the $\pm$ sign in the integral in expression (105) depends on the sign of $\dot{\Omega}$. Assuming the particle has initial gyro-frequency $\Omega_1 = w$, equation (105) reduces to

$$u_- = -\frac{e}{m} \tilde{E}_0 e^{-i\eta} \sqrt{\frac{2}{|\dot{\Omega}|}} \int_0^{B''} e^{\pm i v^2} dv + 2u_+ e^{i(\psi - \eta)} \quad (106)$$

with
$$B'' = \sqrt{\frac{|\dot{\Omega}|}{2}} t.$$

Now $\int_0^{B''} e^{\pm i v^2} dv$ can be written in terms of Fresnel integrals.

$$\int_0^{B''} e^{\pm i v^2} dv = \sqrt{\frac{\pi}{2}} \left[C\left(\sqrt{\frac{2}{\pi}} B''\right) \pm i S\left(\sqrt{\frac{2}{\pi}} B''\right) \right] \quad (107)$$

where $C(u)$ and $S(u)$ are as defined by equations (93) and (94) respectively.

Substituting equation (107) into equation (106) gives

$$u_- = -\frac{e}{m} \tilde{E}_0 e^{-i\eta} t \left\{ \frac{1}{Z} (C(z) \pm iS(z)) \right\} + 2u_b e^{i(\psi-\eta)} \quad (108)$$

with

$$Z = \sqrt{\frac{|\dot{\Omega}|}{\pi}} t. \quad (109)$$

Now

$$\begin{aligned} \Delta\mu &= \frac{1}{8} \Delta |u_-|^2 \\ &= \frac{1}{8} \left(K^2 \left(\frac{1}{Z^2} [C^2(z) + S^2(z)] \right) \right. \\ &\quad \left. + \frac{4u_b K}{Z} [C(z) \cos \vartheta \pm S(z) \sin \vartheta] \right) \end{aligned} \quad (110)$$

with

$$K = \frac{e}{m} \tilde{E}_0 t.$$

Thus we have derived a form for $\Delta\mu$ for $\dot{\Omega} \neq 0$ which is valid for

$$\dot{u}_b \ll \frac{v_z}{L} \frac{\gamma^2 c^2}{u_b}. \quad (111)$$

3.9 Conclusions

In this chapter we have examined the theory of electron cyclotron resonance heating from the approach of resonance absorption. We have discussed the Fokker-Planck approach and considered wave-particle "collisions". We have introduced quasilinear theory and illustrated when corrections to the diffusion coefficient are appropriate. For areas of parameter space when the Fokker-Planck equation is no longer valid we have examined forms of the probability function $P(\underline{v}-\underline{\Delta v}, \underline{\Delta v})$ and equations for $\Delta\mu$ which are related to those for $P(\underline{v}-\underline{\Delta v}, \underline{\Delta v})$ through expression (72). We have derived the impulse K for two separate regimes; B_0 constant and $B = B_0(1 + Z/R)$ and developed an equation for $\Delta\mu$ for large electric wave fields.

In chapter 4 we shall compare these analytic results with a particle code and examine their regions of validity.

CHAPTER 4

The SPEECH Code and Analysis of Results

4.1 Introduction to the SPEECH Code

The Single Particle Experiment in Electron Cyclotron Heating (SPEECH) code was written to simulate ECRH in tokamak geometry. The purpose of the code was to investigate the effect of large wave fields on the probability function $P(\underline{v}-\Delta\underline{v}, \Delta\underline{v})$ and consequently to the diffusion coefficient. We consider an experimental set up as shown in Figure (3.1).

An electromagnetic wave is injected perpendicular to the background magnetic field through a wave-guide of width L . The beam is assumed homogeneous and locally coherent, and of uniform intensity across its width. Both ordinary and extraordinary wave modes are included in the code. We have adopted a single particle approach by calculating the effect of the waves on an electrons parallel and perpendicular energy and position as it traverses the beam. This is accomplished by integrating numerically equations (2.81), (2.82) and (2.83) for the O-mode and equations (2.84), (2.85) and (2.86) for the X-mode. We then repeat this process for an ensemble of particles which are regarded as independent. For the simple case of $\underline{B}_0$ constant the time of interaction is simply $L/v_{||}$ and every particle is a transit particle. The phases of the particles are then randomised to simulate the effect of travelling around the torus back to the beam. Initially all particles are given the same parallel and perpendicular velocities

with gyrofrequency equal to the wave frequency, but are evenly distributed in phase between $(0, 2\pi)$. In the more complex case where the background magnetic field is of the form $B_0 = B_0(1 + Z/R)$ then the code checks to see if the particle will be reflected in or near the beam. Consider a particle moving in a region of changing magnetic field. Then from an analysis of the conservation of magnetic moment and energy one can derive an expression for the parallel velocity of the form

$$v_{||}(t) = \left(v_{||0}^2 + v_{\perp 0}^2 \left[1 - \frac{B(t)}{B(0)} \right] \right)^{\frac{1}{2}}. \quad (1)$$

In dt a particle travels a distance dz given by $dz = v_{||}(t)dt$. Then

$$t_2 - t_1 = \int_{z_1}^{z_2} \frac{dz}{v_{||}(z)} \quad \begin{matrix} z_1 = z(t_1) \\ z_2 = z(t_2) \end{matrix} \quad (2)$$

Hence

$$\Delta t = \int_{z_1}^{z_2} \frac{dz}{\left(v_{||1}^2 + v_{\perp 1}^2 \left[1 - \frac{B(z)}{B_1} \right] \right)^{\frac{1}{2}}}. \quad (3)$$

If the field is of the form

$$B(z) = B_0 \left(1 + \frac{Z}{R} \right)$$

then the turning point of the particle is given by

$$Z_{tp} = Z_1 + \left(\frac{v_{||1}}{v_{\perp 1}} \right)^2 (R + Z_1) \quad (4)$$

and

$$\Delta t = \frac{2(R + Z_1)^{\frac{1}{2}}}{v_{\perp 1}} \left[(Z_{tp} - Z_1)^{\frac{1}{2}} - (Z_{tp} - Z_2)^{\frac{1}{2}} \right] \quad (5)$$

for

$$Z_1, Z_2 < Z_{tp}.$$

If the particle turns in the beam the time the particle spends in the beam is given by $2\Delta t$ with Δt evaluated from equation (5) with $Z_2 = Z_{tp}$. If the particle is reflected after exiting the beam we define a correlation region in which the particle retains its phase memory and returns to the beam. Outside this region the particle is assumed scattered and circuits the torus.

It is reasonable to assume particles will retain phase memory as they pass through the beam. Typically the electron-electron collision time is given by

$$\tau_{ee} \sim \frac{2\pi\epsilon_0^2 m_e^{1/2} (2KT)^{3/2}}{ne^4 \ln\Lambda [\varphi(1) - \psi(1)]} \quad (6)$$

and the electron-ion collision time by

$$\tau_{ei} \sim \frac{2\pi\epsilon_0^2 m_e^{1/2} (2KT)^{3/2}}{ne^4 \ln\Lambda} \quad (7)$$

where

ϵ_0 = permittivity of free space

m_e = electron mass

K = Boltzmann's constant

T = temperature

n = particle density

$\Lambda = \lambda_D/b_0$

λ_D = Debye wavelength

b_0 = distance of closest approach

$\varphi(x)$ = error function

$\psi(x) = (\varphi(x) - x\varphi'(x))/2x^2$

For particles of ~ 1 keV; $V_{th} \sim 10^7 \text{ms}^{-1}$ and $\tau_{ee} \sim \tau_{ei} \sim 10^{-5}$ secs. This is two orders of magnitude greater than the time the particle spends in the beam. The input variables for the code are listed in Table 2. Operation of the code was limited to regions of parameter space regarded as most important. For cases with a magnetic field gradient the variable RESPNT determined where exactly across the beam the particles were in resonance with the wave. The variable CORLEN determined how far past the end of the beam turning particles were allowed before re-entering the beam. The power of the beam can be determined from the electric field by considering the Poynting flux. The time averaged value of this is

$$\langle S \rangle = \frac{\epsilon}{2} c |E_0|^2 \quad (8)$$

see Coulson and Boyd (1979).

Some typical values are detailed in Table 3. It should be noted that the wave fields are given and no attempt at self-consistency is considered. This practice is in common with many authors including Sprott (1971), Toyama, Okabayashi and Ishizuka (1968) and Roberts and Buchsbaum (1964).

The SPEECH code was run on the CRAY XMP computer at Harwell through facilities provided by the United Kingdom Atomic Energy Authority at Culham Laboratory. An average run of 1000 particles took 3 seconds of central processing unit time per pass.

We shall begin by discussing the results of the experiment in light of the theory for the probability function $P(\mu, \Delta\mu)$ discussed in Section 3.6.

Table 2. Input Variables of SPEECH Code.

VARIABLE	PHYSICAL QUANTITY	UNITS
UPERP	$\gamma_{v \perp}$	metres (secs) ⁻¹
UZ	$\gamma_{v \parallel}$	metres (secs) ⁻¹
EZ	E_z	Volts (metres) ⁻¹
EMINUS	E_-	Volts (metres) ⁻¹
MAG FD	B_0	Tesla
WVFREQ	ω	(secs) ⁻¹
K	k - wave number	(metres) ⁻¹
R	magnetic field gradient scale length	metres
TOL	tolerance in numerical integration	—————
BEAMW	L - beam width	metres
CORLE	correlation length	metres
RESPNT	point of resonance in beam	—————
BTIME	time spent by particles in beam	secs
PARTIC	number of particles in ensemble	—————

Table 3. Typical Field Equivalent Beam Powers.

gyrotron cross-section

$$\text{Power Flux} = \langle S \rangle = \frac{\epsilon_0 c}{2} |E_0|^2$$

0.0079 m²

Electric Field (V m ⁻¹)	Beam Power
10 ³	10 W
10 ⁴	1 kW
10 ⁵	100 kW
10 ⁶	10 MW
10 ⁷	1 GW

4.2 A Comparison of the First Theoretical Form of the Probability Function $P(\mu, \Delta\mu)$ with Experiment

Figure 4.1 is typical of results from the SPEECH code expressed in graphical form. This is a graph of the distribution function $f(\mu)$ after one pass through the beam. The horizontal axis is μ with μ normalised such that $\mu_{\text{initial}} = 1$. The y axis is $f(\mu)$ and the total area under the graph is one unit. All the particles are taken to have the same initial parallel and perpendicular velocities (in this case 10^7ms^{-1}), differing only in this phase with respect to the wave. Hence the initial probability distribution can be described by the form

$$f(\underline{v}, 0) = \delta(\underline{v} - \underline{v}_i). \quad (9)$$

When we substitute equation (9) into the equation describing the evolution of the probability function

$$f(\underline{v}, \Delta t) = \int f(\underline{v} - \Delta \underline{v}, 0) P(\underline{v} - \Delta \underline{v}, \Delta \underline{v}) d(\Delta \underline{v}) \quad (10)$$

we recover

$$f(\underline{v}, \Delta t) = P(\underline{v}_i, \underline{v} - \underline{v}_i). \quad (11)$$

Hence the distribution function produced by the code after a single transit of the ensemble of particles through the beam is identical to the probability function $P(\underline{v}_i, \underline{v} - \underline{v}_i)$ which is graphed as the continuous line in Figure (4.1). In Figure (4.1) we have neglected the v parallel component and written $P(\mu_i, \Delta\mu)$. The distribution function is of normalised form so that

Figure 4.1 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

$$\int P(\mu_i, \Delta\mu) d\Delta\mu = 1. \quad (12)$$

The dotted line in Figure (4.1) is a graphical expression for the analytic form of $P(\mu, \Delta\mu)$ given by

$$P(\mu_i, \Delta\mu) = \frac{1}{2\pi} \left| \frac{d\Delta\mu}{d\phi} \right|^{-1} \quad (13)$$

$$= \frac{2}{|K| u_{\perp} \pi} \frac{1}{\sqrt{1 - \frac{4}{(|K| u_{\perp})^2} \left(\Delta\mu - \frac{1}{8} |K|^2 \right)^2}} \quad (3.76)$$

with $|K| = \frac{e}{m} \frac{L}{v_{\perp}} |E_{\text{eff}}|$

where we have normalised μ_{initial} to 1.

The case illustrated here does not include any magnetic field gradient as R , the gradient scale length is very large. Consequently all the particles are transit particles and none is trapped. The heating wave is in the 0-mode and has an electric field of 10^3 Volts per metre. The particles that get the strongest kick from the wave have their magnetic moment changed by approximately 0.6% and it should be noted that in this regime as many particles are cooled as are heated. Intuitively those particles exactly in phase with the wave are accelerated in the perpendicular direction whilst those whose phase differs by π from the waves are slowed in the perpendicular direction. The particles parallel energy is unaffected by the 0-mode see Section 2.4.

Figures (4.2), (4.3), (4.4) and (4.5) show the same curves for the 0-mode for electric fields 10^4 , 10^5 , $2 \cdot 10^5$ and $3 \cdot 10^5$ Volts per metre respectively, indicating that expression (3.76) is a very good approximation to the exact form of $P(\mu_1, \Delta\mu)$ for wave fields up to 10^5 Volts per metre. This is of the order expected, as we predicted an expression for $\Delta\mu$ of the form (3.67) on which the expression (3.76) for $P(\Delta\mu)$ is based to be accurate for

$$|E_z| \ll 7 \cdot 10^5 \text{ Volts per metre}$$

in Section 3.6.

Similar good correlation between theory and experiment for the X-mode is shown in Figures (4.6) and (4.7).

It should be noted that as the electric field increases from Figure (4.1) to Figure (4.5) the distribution becomes less symmetrical about $\mu = 1$. This is because of the increase in the quadratic term $|K|^2/8$ in expression (3.67) to the same order as the term linear in $|K|$ in expression (3.67). Consequently in this higher field regime more particles are substantially heated than cooled. For electric fields greater than $3 \cdot 10^5$ Volts per metre the expression for $P(\mu, \Delta\mu)$ given by equation (3.76) becomes progressively less accurate as illustrated by Figures (4.8), (4.9), (4.10), (4.11) and (4.12).

The agreement at the left (cooler) branch of the "U" is much better than for the right (hotter) branch because the approximate form is more accurate for $\dot{\gamma} = v\dot{\gamma}^3/c^2$ small. Thus the larger v becomes, the larger $\dot{\gamma}$ becomes and the more the approximation breaks down. Consequently the cooled branch of the numerical curves agrees

Figure 4.2 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

Figure 4.3 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

Figure 4.4 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

Figure 4.5 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

Figure 4.6 Graph of Numerical and Analytic Forms of the Probability Function Against MU. The Analytic Form is that Derived in §3.6

Figure 4.7 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

Figure 4.8 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

Figure 4.9 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

Figure 4.10 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

Figure 4.11 Graph of Numerical and Analytic Forms of the Probability Function Against MU. The Analytic Form is that Derived in §3.6

Figure 4.12 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

better with the theory than the heated branch. The new spike which becomes dominant near $7 \cdot 10^5$ Volts per metre cannot be explained in terms of the form for $P(\Delta\mu)$ given by expression (3.76) and the more sophisticated form derived in Section 3.8 is required.

4.3 A Comparison of the Second Theoretical Form of the Probability Function $P(\mu, \Delta\mu)$ with Experiment

In Section 3.8 we derived a form for $\Delta\mu$ given by

$$\Delta\mu = \frac{1}{8} \left(\frac{K^2}{z^2} [C^2(z) + S^2(z)] + \frac{4u_z K}{z} (C(z)\cos\vartheta \pm S(z)\sin\vartheta) \right). \quad (3.110)$$

We shall first show that in the small wave field limit expression (3.110) reduces to (3.17).

We can expand $C(z)$ and $S(z)$ in a power series of the form, convergent for all z

$$C(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{2}\right)^{2n} z^{4n+1}}{(2n)!(4n+1)} \quad (14)$$

$$= z - \frac{\pi^2 z^5}{40} + \frac{\pi^4 z^9}{3456} - \frac{\pi^6 z^{13}}{599040} + \dots$$

and

$$S(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{2}\right)^{2n+1} z^{4n+3}}{(2n+1)!(4n+3)}$$

$$= \frac{\pi z^3}{6} - \frac{\pi^3 z^7}{336} + \frac{\pi^5 z^{11}}{42240} - \frac{\pi^7 z^{15}}{9676800} + \dots$$

see Abramowitz and Stegun (1972) §7.3.

Table 4 indicates the largest z allowed for a given accuracy for a given number of terms. If we consider $z \ll 0.27$ then

$$C(z) \sim C_1 \quad \text{and} \quad S(z) \sim 0. \quad (15)$$

Substituting expressions (17) back into equation (3.124) recovers the original form

Table 4. Accuracy of Expansion of
Fresnel Integrals in a Power Series.

$$C(z) = C_1 + C_2 + C_3 + \dots$$

		Accuracy	
		1 %	10 %
Number of Terms	C_1	0.01	0.10
	$C_1 + C_2$	0.53	0.83
	$C_1 + C_2 + C_3$	0.89	1.15
	$C_1 + C_2 + C_3 + C_4$	1.15	1.37

$$S(z) = S_1 + S_2 + S_3 + \dots$$

		Accuracy	
		1 %	10 %
Number of Terms	S_1	0.27	0.58
	$S_1 + S_2$	0.73	1.01
	$S_1 + S_2 + S_3$	1.03	1.27
	$S_1 + S_2 + S_3 + S_4$	1.26	1.47

For example $C(z) = z - \frac{\pi^2 z^5}{40}$ is accurate to 1 % for $z < 0.53$.

$$\Delta\mu = \frac{|K|u_b}{z} \cos \vartheta + \frac{|K|^2}{8} \quad (3.67)$$

This expression is accurate to 1% for $z < 0.01$ and to 10% for $z < 0.1$.

Now we might expect visible differences between the analytic form (3.67) and the experimental form to be noticeable when the analytic form is in error by 10% or at $z = 0.1$, or

$$\sqrt{\frac{|\dot{\Omega}|}{\pi}} t = 0.1 \text{ from (3.109).}$$

Now substituting

$$\dot{\Omega} = -\frac{\Omega}{\gamma^2} \frac{u_b \dot{u}_b}{c^2} \quad (3.103) \text{ for } \dot{\Omega}$$

and an expression of the form (3.23) for $\dot{u}_b$ neglecting the $\cos \psi$ dependence gives

$$\sqrt{\frac{|\dot{\Omega}|}{\pi}} t \sim \left(\frac{1}{2\pi} \frac{\Omega}{\gamma^2} \frac{u_b}{c^3} \frac{e}{m} v_{||} E_z t^2 \right)^{\frac{1}{2}} \sim 0.1 \quad (16)$$

which corresponds to

$$E_z \sim 1.9 \cdot 10^5 \text{ Volts per metre}$$

for $u_b = u_{||} \sim 10^7 \text{ ms}^{-1}$; $L = 0.1 \text{ m}$ and $\Omega = 1.757 \cdot 10^{11} \text{ s}^{-1}$. Hence for $E_z \gtrsim 2 \cdot 10^5 \text{ Volts per metre}$ we expect some deviation of the simple theory with observations as shown in Figure (4.4).

As we continue to increase E_z in Figures (4.5)-(4.14) the form

$$\Delta\mu = \frac{|K|u_b}{2} \cos \vartheta + \frac{|K|^2}{8} \quad (3.67)$$

becomes less and less accurate. Although the analytic form is a good approximation for those particles that are cooled significant

Figure 4.13 Graph of Numerical and Analytic Forms of the Probability Function Against MU. The Analytic Form is that Derived in §3.6

Figure 4.14 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.6

breakdown occurs for those that are heated. This is because of the relativistic change in the gyrofrequency and because $\dot{\gamma} \propto v$, as explained in Section 4.2.

Another feature of Figures (4.5)-(4.14) is the emergence of a new spike. This phenomenon cannot be explained by consideration of expression (3.76) but requires expression (3.110). As stated earlier spikes in the $P(\Delta\mu)$ diagrams occur when $d\Delta\mu/d\vartheta = 0$. Although we can differentiate expression (3.110) this is not a very illuminating process as $\Delta\mu$ is a complex function of ϑ . However it is sufficient to plot (3.110) as we shall discuss later.

Let us now compare the more sophisticated form for $\Delta\mu$ given by (3.110) with experiment. As can be seen from Figures (4.15) to (4.18) the expression (3.110) agrees with experiment to about $E_z \sim 4 \cdot 10^5$ Volts per metre. Figures (4.19) to (4.24) depict expression (3.110) for fields up to $9 \cdot 10^5$ Volts per metre and although the precision deteriorates as E_z approaches 10^6 Volts per metre it does give the correct form of the distribution. Expression (3.110) eventually becomes a poor approximation because the assumption in its derivation, that $|\dot{\Omega}| = \text{constant}$ eventually breaksdown at high fields.

Thus we have compared analytic forms for $\Delta\mu$ and hence $P(\Delta\mu)$ with experiment and have shown good agreement for electric fields up to $4 \cdot 10^5$ Volts per metre, and of the correct form for fields up to $9 \cdot 10^5$ Volts per metre. It is also illuminating to compare (μ, psi) scatter plots. For low fields $\leq 10^5$ Volts per metre we have noted that (3.67)

Figure 4.15 Graph of Numerical and Analytic Forms of the Probability Function Against MU. The Analytic Form is that Derived in §3.8

Figure 4.16 Graph of Numerical and Analytic Forms of the Probability Function Against MU. The Analytic Form is that Derived in §3.8

Figure 4.17 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.8

Figure 4.18 Graph of Numerical and Analytic Forms of the Probability Function Against MU. The Analytic Form is that Derived in §3.8

Figure 4.19 Graph of Numerical and Analytic Forms of the Probability Function Against MU. The Analytic Form is that Derived in §3.8

Figure 4.20 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.8

Figure 4.21 Graph of Numerical and Analytic Forms of the Probability Function Against MU. The Analytic Form is that Derived in §3.8

Figure 4.22 Graph of Numerical and Analytic Forms of the Probability Function Against MU. The Analytic Form is that Derived in §3.8

Figure 4.23 Graph of Numerical and Analytic Forms of the Probability Function Against MU. The Analytic Form is that Derived in §3.8

Figure 4.24 Graph of Numerical and Analytic Forms of the
Probability Function Against MU. The Analytic
Form is that Derived in §3.8

$$\Delta\mu = \frac{|K|u_{\frac{1}{2}}\cos\vartheta}{2} + \frac{|K|^2}{8} .$$

Hence $\mu_f = 1 + \Delta\mu/\mu_i$ will have a cosine dependence on ϑ . If we replace ϑ by $\psi + \pi$ then (3.67) becomes

$$\Delta\mu = \frac{|K|^2}{8} - \frac{|K|u_{\frac{1}{2}}}{2} \cos\psi . \quad (17)$$

This plot is depicted in Figure (4.25), together with the $\Delta\mu$'s for the SPEECH code. In this case the two are indistinguishable. The spikes in the previous curve (4.3) correspond to points of zero gradient in curve (4.25) since

$$P(\Delta\mu) = \frac{1}{2\pi} \left/ \left| \frac{d\Delta\mu}{d\vartheta} \right| \right|$$

and points of zero gradient in curve (4.25) occur when $d\Delta\mu/d\vartheta = 0$.

If we continue to increase the field using now our expression (3.110) to calculate $\Delta\mu$ then our theory and experimental curves slowly diverge, see Figures (4.26) to (4.35), with the analytic curve being the upper one in each case and the numerical one the lower. As we increase the fields the curves are gradually flattened at $\psi = \pi/2$. This corresponds to the new spike which developed in curves (4.10) to (4.14). In order to increase our understanding of this mechanism we shall next consider a phase plane analysis.

Figure 4.25 Graph of MU (final) as a Function of Initial Phase

The numerical and analytic forms are identical here

Figure 4.26 Graph of MU (final) as a Function of Initial Phase

The analytic form is the upper and the numerical form the lower

OMODE, PSI/MU PLOT

Figure 4.27 Graph of MU (final) as a Function of Initial Phase

The analytic form is the upper and
the numerical form the lower

MU

OMODE, PSI/MU PLOT

UPERP = 0.1000E 08
 UZ = 0.1000E 08
 EZ = 0.4000E 06
 MAG FD = 0.1000E 01
 WVFREQ = 0.1757E 12
 K = 0.5860E 03
 R = 0.1000E 31
 TOL = 0.1000E -07
 BEAMV = 0.1000E 00
 COR LE = 0.0000E 00
 RESPNT = 0.0000E 00
 PARTIC = 1000

Figure 4.28

Graph of MU (final) as a Function of Initial Phase

The analytic form is the upper and the numerical form the lower

OMODE,PSI/MU PLOT

Figure 4.29 Graph of MU (final) as a Function of Initial Phase

The analytic form is the upper and
the numerical form the lower

OMODE,PSI/MU PLOT

Figure 4.30 Graph of MU (final) as a Function of Initial Phase

The analytic form is the upper and the numerical form the lower

MU

OMODE,PSI/MU PLOT

Figure 4.31 Graph of MU (final) as a Function of Initial Phase

The analytic form is the upper and the numerical form the lower

OMODE,PSI/MU PLOT

Figure 4.32 Graph of MU (final) as a Function of Initial Phase

The analytic form is the upper and the numerical form the lower

OMODE,PSI/MU PLOT

Figure 4.33 Graph of MU (final) as a Function of Initial Phase

The analytic form is the upper and the numerical form the lower

OMODE, PSI/MU PLOT

Figure 4.34 Graph of MU (final) as a Function of Initial Phase

The analytic form is the upper and the numerical form the lower

OMODE,PSI/MU PLOT

Figure 4.35 Graph of MU (final) as a Function of Initial Phase

The analytic form is the upper and the numerical form the lower

4.4 A Phase Plane Analysis of the Averaged Equations for an Electron in an O-mode Field

The guiding centre equations for an electron in the 0-mode are

$$\dot{u}_b = -\frac{1}{2} A \cos \psi \quad (3.23)$$

$$\dot{\psi} = \frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{A}{u_b} \sin \psi \quad (3.24)$$

where
$$A = \frac{1}{2} \frac{e}{m} E_z \frac{v_{th} k}{w} .$$

Curves in the (u_b, ψ) phase plane are integrals of

$$\frac{d\psi}{du_b} = \frac{\frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{A}{u_b} \sin \psi}{-\frac{1}{2} A \cos \psi} . \quad (18)$$

Critical points occur for these curves where

$$\cos \psi = 0 \quad \text{that is} \quad \psi = (n + \frac{1}{2})\pi \quad n = 0, 1, 2, \dots$$

and

$$\frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{A}{u_b} \sin \psi = 0, \quad (19)$$

that is

$$\frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{A}{u_b} = 0$$

or

$$\frac{\Omega}{\gamma} - w - \frac{1}{2} \frac{A}{u_b} = 0.$$

Depending on whether $\sin \psi = \pm 1$.

If $\sin \psi = +1$ we have

$$\frac{\Omega}{\gamma} - w = -\frac{1}{2} \frac{A}{u_b} \quad (20)$$

so $w > \frac{\Omega}{\gamma}$.

If $u_b^{(0)}$ is the value of u_b satisfying $\frac{\Omega}{\gamma} = w$ then

$u_b > u_b^{(0)}$ for this case.

If $\sin \psi = -1$ we have

$$\frac{\Omega}{\gamma} - w = \frac{1}{2} \frac{A}{u_b} \quad (21)$$

so $w < \frac{\Omega}{\gamma}$ and $u_b < u_b^{(0)}$.

The contours will be horizontal for

$$\frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{A}{u_b} \sin \psi = 0$$

and vertical on

$$\cos \psi = 0.$$

Expressions (20) and (21) have been graphed in Figures (4.36), (4.37), (4.38) and (4.39) for $E_z = 10^3, 10^4, 10^5$ and 10^6 Volts per metre respectively. The continuous line is $\frac{\Omega}{\gamma} - w$ as a function of u_b and the dashed lines represent $\pm \frac{1}{2} \frac{A}{u_b}$, with $u_b^{(0)} = 10^7 \text{ ms}^{-1}$. The curves show that equations (20) and (21) have one root each which lie close to $u_b^{(0)}$ for electric wave fields of the order of $\lesssim 10^4$ Volts per metre. Strictly speaking equation (21) has another root at small u_b . For fields of order 10^5 Volts per metre the two roots of equation (21) coalesce and then vanish while the root of (20) moves slowly to higher u_b .

Figures (4.40), (4.41) and (4.42) are phase plane diagrams of equation (20) for differing values of A and hence E_z . The x-axis

Figure 4.36 Graph of $\frac{\Omega}{\gamma} - w$ and $\frac{A}{u_B}$ for $E_z = 10^3 \text{ Vm}^{-1}$

Figure 4.37 Graph of $\frac{\Omega}{\gamma} - w$ and $\frac{A}{u_B}$ for $E_z = 10^4 \text{ Vm}^{-1}$

Figure 4.38 Graph of $\frac{\Omega}{\gamma} - w$ and $\frac{A}{u_B}$ for $E_z = 10^5 \text{ Vm}^{-1}$

Figure 4.39 Graph of $\frac{\Omega}{\gamma} - w$ and $\frac{A}{u_B}$ for $E_z = 10^6 \text{ Vm}^{-1}$

Figure 4.40 Phase plane Diagram for (ψ, u_B) with $E_z = 10^4 \text{ Vm}^{-1}$

Figure 4.41 Phase plane Diagram for (ψ, u_B) with $E_z = 10^5 \text{ Vm}^{-1}$

Figure 4.42 Phase Plane Diagram for (ψ, u_H) with $E_z = 10^6 \text{ Vm}^{-1}$

is u_b in arbitrary units with $u_b^{(0)} = 0.7$. The y-axis is ψ varying from $-\frac{\pi}{2}$ to $\frac{3\pi}{2}$. The diagrams are periodic (with period 2π) outside this region. Diagrams (4.40), (4.41) and (4.42) correspond to diagrams (4.37), (4.38) and (4.39) respectively, in the size of the electric fields applied. The root of equation (20), u_{bR} say, corresponds to a critical point at $\left(u_{bR+}, \frac{\pi}{2}\right)$ which occurs at the centre of the set of circular contours in each case. For low E_z equation (21) has two roots. One at $u_b \sim 0$ and one at $u_{bR-} \sim u_b^{(0)}$. The critical points thus occur at $\left(0, -\frac{\pi}{2}\right)$ and $\left(0, \frac{3\pi}{2}\right)$ which are unimportant and $\left(u_{bR-}, -\frac{\pi}{2}\right)$ and $\left(u_{bR+}, \frac{3\pi}{2}\right)$ which occur at the X-points of the separatrix in Figure (4.40). For $E_z \sim 10^5$ Volts per metre equation (20) still has one root which still corresponds to the centre of the circular contours in Figure (4.41). The two roots of equation (21) are now quite close and correspond to the centre of the triangular region and the X-point in Figure (4.41). For electric wave fields substantially higher than 10^5 Volts per metre equation (21) has no roots and hence there are no critical points for $\psi = -\frac{\pi}{2}$, $\frac{3\pi}{2}$ and so on, corresponding to Figure (4.42). The spike in Figure (4.13) occurs for particles with $\psi_{\text{initial}} \sim \frac{\pi}{2}$ as can be seen from Figure (4.33). These particles lie close to the critical point at $u_b = 0.8$, $\psi = \frac{\pi}{2}$ in Figure (4.41). These particles lie on closed contours and a large change in ψ produces a small change in $u_b \Rightarrow \frac{du}{d\psi}$ small.

Figure (4.42) shows the possibility of strong heating off resonance. If the bulk of particles are in the low velocity (crosshatched) region then they could all be heated to substantial

velocities.

In the previous sections we have considered two analytic expressions for $\Delta\mu$ and compared them to numerical results showing good agreement for electric fields up to $5 \cdot 10^5$ Volts per metre. We have discussed the form of $P(\Delta\mu)$ and looked at a phase plane analysis of the 0-mode equations with the background magnetic field a constant. We shall now consider the X-mode equations

$$\dot{u}_b = -\frac{1}{2} \frac{e}{m} E_- \cos \psi \quad (2.76)$$

and

$$\dot{\psi} = \frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{e}{m} \frac{E_-}{u_b} \sin \psi \quad (2.78)$$

which are of the same form as the 0-mode equations with E_- replacing $kv_{||}E_z/w$. Hence all the previous analysis carries over to the X-mode in a simple fashion. The primary difference between the X-mode and the 0-mode is the $v_{||}$ dependence of the effective electric field in the 0-mode which does not exist in the X-mode.

Now

$$\Delta u_b \propto E_{\text{eff}} \Delta t = E_{\text{eff}} \frac{L}{v_{||}}$$

where L is the beam width, from (3.23).

For the 0-mode $E_{\text{eff}} = kv_{||}E_z/w$ so Δu_b is independent of $u_{||}$ whereas for the X-mode the size of the kick received will be $\propto 1/u_{||}$. This is demonstrated in Figures (4.43) to (4.47) for the 0-mode. Where despite spending times in the beam varying by two orders of magnitude the resultant distribution is the same. In the corresponding

Figure 4.43 Diagram of Probability Function as a Function of MU

Figure 4.44 Diagram of Probability Function as a Function of MU

Figure 4.45 Diagram of Probability Function as a Function of MU

Figure 4.46 Diagram of Probability Function as a Function of MU

Figure 4.47 Diagram of Probability Function as a Function of MU

runs for the X-mode the distribution gets progressively narrower as the particles spend less time in the beam - see Figures (4.48) to (4.52).

So far in this chapter we have only considered B_0 constant. In the next section we shall consider the implications of a background magnetic field with a gradient.

Figure 4.48 Diagram of Probability Function as a Function of MU

Figure 4.49 Diagram of Probability Function as a Function of MU

Figure 4.50 Diagram of Probability Function as a Function of MU

Figure 4.51 Diagram of Probability Function as a Function of MU

Figure 4.52 Diagram of Probability Function as a Function of MU

4.5 An Analysis of Results for $B = B(z)$

As was noted in the previous chapter the introduction of a background magnetic field with a gradient has two effects.

1. It limits the wave particle resonance. This is because the resonance condition

$$\Omega - w = 0$$

becomes

$$\Omega(|B(z)|) - w = 0 \quad (22)$$

and hence the particle will only be in resonance with the wave over a small region of the beam.

2. It alters the time the particle spends in the beam.

We have already noted

$$v_{||}(t) = \left(v_{||0}^2 + v_{\perp 0}^2 \left[1 - \frac{B(t)}{B(0)} \right] \right)^{\frac{1}{2}}. \quad (1)$$

We shall consider two cases. If the magnetic field $B(z)$ satisfies

$$|B(z_0)| = \frac{E_0}{\mu_0} \quad (23)$$

where

$$E_0 = \frac{1}{2} m v_{\perp i}^2 + \frac{1}{2} m v_{|| i}^2$$

$$\mu_0 = \frac{\frac{1}{2} m v_{\perp i}^2}{|B(0)|}$$

for some z_0 which lies within the beam then the particle will be reflected, assuming the energy gain (or loss) is negligible.

If

$$|B(z)| < \frac{E_0}{\mu_0} \quad \forall z \text{ in the beam}$$

then the particle will transit the beam. In Section 3.7 we devised an expression for K , the impulse given to the particle, in this regime.

$$K = - \frac{eE_0}{m} T_{\text{eff}}$$

with

$$T_{\text{eff}} = \sqrt{\frac{\pi}{2b}} \left| F\left(\sqrt{\frac{2b}{\pi}} \left(t + \frac{a}{2b}\right)\right) - F\left(\frac{a}{\sqrt{2\pi b}}\right) \right| \quad (3.92)$$

with $a = \Omega_0 - w$ and $b = \left|\frac{\dot{\Omega}}{2}\right|$.

Thus

$$\Delta\mu = \frac{|K|u_b}{2} \cos \vartheta + \frac{1}{8} |K|^2 \quad (3.67)$$

with K given by the above.

The results of the code for two runs are given in Figures (4.53) and (4.54). Typically the scale length of the magnetic field gradient will be about 100m. If we consider a $q=1$ surface in JET then a particle will travel from the outside of the torus to the inside in πr metres where r is the major radius of the plasma $\sim 3m$. In this time the magnetic flux density will change by about 0.1 Tesla hence a gradient scale length (R) of $\sim 100m$. Figure (4.53) has $R = 50m$ and Figure (4.54) has $R = 150m$. Both show excellent agreement between numerical experiment and theory.

Next we consider particles which are reflected. We derived the expression

$$T_{\text{eff}} = \left| 2 \left(-\frac{2}{\dot{\Omega}}\right)^{1/3} \pi \text{Ai}\left((w - \Omega_0) \left(-\frac{2}{\dot{\Omega}}\right)^{1/3}\right) \right| \quad (3.99)$$

Figure 4.53 Diagram of Probability Function as a Function of MU

Transit 1

0 MODE = 0.1000E 08

UPERP = 0.1000E 08

UZ = 0.1000E 08

EZ = 0.1000E 04

MAG FD = 0.9990E 00

WVFREQ = 0.1757E 12

K = 0.5860E 03

R = 0.5000E 02

TOL = 0.1000E-07

BEAMW = 0.1000E 00

COR LE = 0.0000E 00

RESPNT = 0.5000E 00

BTIME = 0.9930E-08

PARTIC = 1000

Figure 4.54 Diagram of Probability Function as a Function of MU

where A_1 is the first Airy function, in Section 3.7. Figures (4.55) to (4.59) are the $P(\Delta\mu)$ curves for a group of particles which are reflected at the halfway point of the beam. The resonance point has been varied across the beam in Figures (4.55-59). In Figure (4.55) the width of the theory curve (dotted) is larger than the exact curve (continuous) because the particles spend most of their time away from the resonant region. In Figure (4.56) theory and experiment agree well. However in Figure (4.57) the particles again get a smaller kick than expected. This is because although the effective electric field for the 0-mode given by $E_{\text{eff}} \sim kv_{\parallel}/E_z$ is zero as $v_{\parallel} \sim 0$. In Figures (4.58) and (4.59) the kicks are smaller than expected because the particles never reach the resonant regions. Curves for the X-mode present the same trends except for Figure (4.60) which shows good agreement with theory - because the effective field for the X-mode is not strongly $v_{\parallel}$ dependent. Similar behaviour is shown in Figures (4.61) to (4.67) which contain the $P(\Delta\mu)$ curves for varying electric field strengths. The important points shown are that for comparable fields $E_z \sim E_+$ the diffusion is greater in the case of the X-mode than for the 0-mode owing to the $v_{\parallel}$ dependence for the 0-mode effective field as mentioned above.

Another case of interest is shown in Figures (4.68) and (4.69). The particles here would turn at the edge of the correlation region if unaffected by the wave. However some have been cooled and others heated by resonant interaction. The result has been that the cooled particles are no longer reflected but continue away from the beam

Figure 4.55 Diagram of Probability Function as a Function of MU

Figure 4.56 Diagram of Probability Function as a Function of MU

Figure 4.57 Diagram of Probability Function as a Function of MU

Figure 4.58 Diagram of Probability Function as a Function of MU

Figure 4.59 Diagram of Probability Function as a Function of MU

Figure 4.60 Diagram of Probability Function as a Function of MU

Figure 4.61 Diagram of Probability Function as a Function of MU

Figure 4.62 Diagram of Probability Function as a Function of MU

Figure 4.63 Diagram of Probability Function as a Function of MU

Figure 4.64 Diagram of Probability Function as a Function of MU

Figure 4.65 Diagram of Probability Function as a Function of MU

Figure 4.66 Diagram of Probability Function as a Function of MU

Figure 4.67 Diagram of Probability Function as a Function of MU

Figure 4.68 Diagram of Probability Function as a Function of MU

Figure 4.69 Diagram of Probability Function as a Function of MU

whilst the heated particles are reflected and pass through the beam a second time. This is clearly shown in Figure (4.68). The right hand branch of the "U" has passed through the beam again and hence the $P(\Delta\mu)$ mapping has been applied a second time giving the two extra spikes shown.

4.6 Conclusions

In this chapter we have described the SPEECH code and compared the probability curves for $P(\Delta\mu)$ with the analytic theories described in Chapter 3. We have found good agreement between the two for electric wave fields up to 5×10^5 Volts per metre. We have discussed the heating mechanism through a phase-plane analysis of the governing guiding centre equations and shown experimental data for a variety of regimes including both passing and trapped particles. The trapped particles showed enhanced diffusion under the influence of the X-mode waves over the O-mode case as expected. In the next chapter we will discuss the modifications to the diffusion coefficient required in high field regimes and we shall consider multiple-pass diffusion.

CHAPTER 5

Modifications to the Diffusion Coefficient and Multipass Behaviour

5.1 Modifications to the Diffusion Coefficient

In Chapter 3 we considered forms of the diffusion coefficient introduced by writing the Fokker-Planck equation in the form

$$\left(\frac{\partial f}{\partial t}\right)_{\text{wave}} = \frac{\partial}{\partial v} \left(D \frac{\partial f}{\partial v} \right) \quad (1)$$

where

$$D = \frac{1}{2} \frac{\langle \Delta v^2 \rangle}{\Delta t}$$

to describe the effect of an r.f. wave on the distribution function f .

Consequently we can derive numerical values for the diffusion coefficient by considering the variance in perpendicular velocity of a group of particles as they pass through the beam.

For the 0-mode we had

$$D^{\text{OM}} = \frac{1}{2} \frac{\langle \Delta u_{\perp}^2 \rangle}{\Delta t} = \frac{1}{16} \left(\frac{e}{m} \frac{E_z}{c} \right)^2 L v_{\parallel} \quad (3.29)$$

and

$$D_{\text{RED}}^{\text{OM}} = \frac{1}{2} \frac{\langle \Delta u_{\perp}^2 \rangle}{\Delta t} = \frac{1}{4} \left(\frac{e}{m} \frac{v_{\parallel}}{c} E_z \right)^2 \frac{v_{\parallel}}{L} \left(\frac{\sin\left(\frac{\delta \Delta t}{2}\right)}{\delta} \right)^2 \quad (3.36)$$

and for the X-mode we had

$$D^{XM} = \frac{1}{2} \frac{\langle \Delta u_{\perp}^2 \rangle}{\Delta t} = \frac{1}{16} \left(\frac{e}{m} E_{-} \right)^2 \frac{L}{v_{\parallel}} \quad (3.39)$$

and

$$D_{RED}^{XM} = \frac{1}{2} \frac{\langle \Delta u_{\perp}^2 \rangle}{\Delta t} = \frac{1}{4} \left(\frac{e}{m} E_{-} \right)^2 \frac{v_{\parallel}}{L} \left(\frac{\sin\left(\frac{\delta \Delta t}{2}\right)}{\delta} \right)^2 \quad (3.40)$$

with $\delta = \dot{\psi}_{\text{initial}}$.

Table 5 shows a comparison of expressions (3.29), (3.36), (3.39) and (3.40) with those derived from the SPEECH code. The runs tabulated were for various electric field strengths with a constant background magnetic field and were for one pass. The analytic forms show good agreement for electric fields $\leq 10^5$ Volts per metre. Above this value $\dot{\psi}$ is no longer constant across the beam and the approximation

$$\dot{\psi} \lesssim 2\pi\dot{z}/L \quad (3.41)$$

is invalid. However we have shown that for large electric fields of the order of 5×10^5 Volts per metre and larger one can expect a substantial decrease in velocity diffusion over linear theory owing to particles being heated out of resonance.

Table 5. Comparison of Analytic Forms of the Diffusion Coefficient with Numerical Experiment.

OMODE			
E_z	D^{OM}	D_{RED}^{OM}	$\frac{1}{2} \frac{\langle \Delta v_L^2 \rangle}{t}$
$(V m^{-1})$	$(m^2 s^{-3})$	$(m^2 s^{-3})$	$(m^2 s^{-3})$
10^3	$2.15 \cdot 10^{16}$	$2.15 \cdot 10^{16}$	$2.13 \cdot 10^{16}$
$5 \cdot 10^3$	$5.40 \cdot 10^{17}$	$5.40 \cdot 10^{17}$	$5.35 \cdot 10^{17}$
10^4	$2.15 \cdot 10^{18}$	$2.15 \cdot 10^{18}$	$2.15 \cdot 10^{18}$
$5 \cdot 10^4$	$5.40 \cdot 10^{19}$	$5.35 \cdot 10^{19}$	$5.30 \cdot 10^{19}$
10^5	$2.15 \cdot 10^{20}$	$2.09 \cdot 10^{20}$	$2.09 \cdot 10^{20}$
$5 \cdot 10^5$	$5.40 \cdot 10^{21}$	$2.48 \cdot 10^{21}$	$1.74 \cdot 10^{21}$

XMODE			
E_+	D^{XM}	D_{RED}^{XM}	$\frac{1}{2} \frac{\langle \Delta v_L^2 \rangle}{t}$
$(V m^{-1})$	$(m^2 s^{-3})$	$(m^2 s^{-3})$	$(m^2 s^{-3})$
10^3	$1.74 \cdot 10^{16}$	$1.74 \cdot 10^{16}$	$1.74 \cdot 10^{16}$
$5 \cdot 10^3$	$4.36 \cdot 10^{17}$	$4.36 \cdot 10^{17}$	$4.36 \cdot 10^{17}$
10^4	$1.74 \cdot 10^{18}$	$1.74 \cdot 10^{18}$	$1.75 \cdot 10^{18}$
$5 \cdot 10^4$	$4.36 \cdot 10^{19}$	$4.26 \cdot 10^{19}$	$4.33 \cdot 10^{19}$
10^5	$1.74 \cdot 10^{20}$	$1.59 \cdot 10^{20}$	$1.70 \cdot 10^{20}$
$5 \cdot 10^5$	$4.36 \cdot 10^{21}$	$1.45 \cdot 10^{21}$	$2.40 \cdot 10^{21}$

5.2 Multi-Pass Diffusion

If we now consider an ensemble of particles making many circuits of the torus and passing through the beam a second time the distribution function will be given by

$$f(\mu, \Delta t) = \iint f(\mu - \Delta\mu_2 - \Delta\mu_1, 0) P(\mu - \Delta\mu_2 - \Delta\mu_1, \Delta\mu_1) P(\mu - \Delta\mu_2, \Delta\mu_2) d\Delta\mu_1 d\Delta\mu_2 \quad (2)$$

where $P(\mu, \Delta\mu)$ will be given by (3.76).

Similarly after n passes

$$f(\mu, n(\Delta t + T)) = \int \dots \int f\left(\mu - \sum_{i=1}^n \Delta\mu_i, 0\right) \prod_{j=1}^n \left[P\left(\mu - \sum_{k=j}^n \Delta\mu_k, \Delta\mu_j\right) d\Delta\mu_j \right] \quad (3)$$

where Δt is the time the particle takes to pass through the beam and T is the time the particle takes to circuit the tokamak. Consequently we can write

$$\left(\frac{\partial f}{\partial t}\right)_{\text{wave}} = \frac{1}{n(\Delta t + T)} \int \dots \int f\left(\mu - \sum_{i=1}^n \Delta\mu_i, 0\right) \prod_{j=1}^n \left[P\left(\mu - \sum_{k=j}^n \Delta\mu_k, \Delta\mu_j\right) d\Delta\mu_j \right] \quad (4)$$

instead of the simple diffusive form

$$\left(\frac{\partial f}{\partial t}\right)_{\text{wave}} = \frac{\partial}{\partial v} \left(D \frac{\partial f}{\partial v} \right).$$

Here we have assumed that the distribution function remains unchanged outside the beam.

Although such an expression as (3) is not amenable to straightforward analysis it is simply repeated operations by the probability function $P(\mu, \Delta\mu)$ on the initial distribution function. The development of the distribution function from the SPEECH code is shown for two cases in Figures (5.1)-(5.10) and (5.11)-(5.20). Figures (5.1)-(5.10) show the development of the distribution function subjected to an 0-mode field of $4 \cdot 10^3$ Volts per metre. Figure (5.1) shows the standard "U" form. Figure (5.2) contains five spikes. This can be explained intuitively by considering the two spikes in Figure (5.1) as delta-functions and applying the P operator to them. This produces four more spikes totalling six in all but the two at the centre have merged leaving five only. Subsequent transits spread the profile in velocity space. In this low field regime the profile is symmetrical about $\mu = 1$ the initial value. This is because if we consider

$$P(\Delta\mu) = \frac{2}{|K|u_E\pi} \frac{1}{\sqrt{1 - \frac{4}{(|K|u_E)^2} \left(\Delta\mu - \frac{1}{8} |K|^2 \right)^2}} \quad (3.76)$$

for small fields and hence small K, $P(\Delta\mu)$ is symmetrical about $\Delta\mu = 0$, that is as many particles are cooled as are heated. However increasing the electric field increases K so that $P(\Delta\mu)$ is no longer symmetric

Figure 5.1 Probability of MU as a Function of
 MU for $E_z = 4 \cdot 10^3 \text{ Vm}^{-1}$ - Pass 1

Figure 5.2 Probability of MU as a Function of MU for $E_z = 4 \cdot 10^3 \text{ Vm}^{-1}$ - Pass 2

Figure 5.3 Probability of MU as a Function of
MU for $E_z = 4 \cdot 10^3 \text{ Vm}^{-1}$ - Pass 3

Transit 3

0 MODE = 0.1000E 08
 UPERP = 0.1000E 08
 UZ = 0.4000E 04
 EZ = 0.1000E 01
 MAG FD = 0.1757E 12
 WVFREQ = 0.5860E 03
 K = 0.1000E 31
 R = 0.1000E-05
 TOL = 0.1000E 00
 BEAMW = 0.0000E 00
 COR LE = 0.1001E-07
 BTIME = 0.0000E 00
 RESPNT = 1000
 PARTIC =

Figure 5.4 Probability of MU as a Function of MU for $E_z = 4 \cdot 10^3 \text{ Vm}^{-1}$ - Pass 4

Figure 5.5 Probability of MU as a Function of
MU for $E_z = 4 \cdot 10^3 \text{ Vm}^{-1}$ - Pass 5

Figure 5.6 Probability of MU as a Function of
MU for $E_z = 4 \cdot 10^3 \text{ Vm}^{-1}$ - Pass 6

Figure 5.7 Probability of MU as a Function of
MU for $E_z = 4 \cdot 10^3 \text{ Vm}^{-1}$ - Pass 7

Figure 5.8 Probability of MU as a Function of
MU for $E_z = 4 \cdot 10^3 \text{ Vm}^{-1}$ - Pass 8

Figure 5.9 Probability of MU as a Function of
MU for $E_z = 4 \cdot 10^3 \text{ Vm}^{-1}$ - Pass 9

Transit 9

0 MODE	=	0.1000E 08
UPERP	=	0.1000E 08
UZ	=	0.4000E 04
EZ	=	0.1000E 01
MAG FD	=	0.1757E 12
WVFREQ	=	0.5860E 03
K	=	0.1000E 31
R	=	0.1000E-05
TOL	=	0.1000E 00
BEAMW	=	0.0000E 00
COR LE	=	0.1001E-07
BTIME	=	0.0000E 00
RESPNT	=	1000
PARTIC	=

Figure 5.10 Probability of MU as a Function of
MU for $E_z = 4 \cdot 10^3 \text{ Vm}^{-1}$ - Pass 10

Transit 10

0 MODE	=	0.1000E 08
UPERP	=	0.1000E 08
UZ	=	0.4000E 04
MAG FD	=	0.1000E 01
WVFREQ	=	0.1757E 12
K	=	0.5860E 03
R	=	0.1000E 31
TOL	=	0.1000E-05
BEAMW	=	0.1000E 00
COR LE	=	0.0000E 00
BTIME	=	0.1001E-07
RESPNT	=	0.0000E 00
PARTIC	=	1000

Figure 5.11 Probability of MU as a Function of
MU for $E_z = 10^6 \text{ Vm}^{-1}$ - Pass 1

Figure 5.12 Probability of MU as a Function of
MU for $E_z = 10^6 \text{ Vm}^{-1}$ - Pass 2

Figure 5.13 Probability of MU as a Function of
 MU for $E_z = 10^6 \text{ Vm}^{-1}$ - Pass 3

Figure 5.14 Probability of MU as a Function of MU for $E_z = 10^6 \text{ Vm}^{-1}$ - Pass 4

Figure 5.15 Probability of MU as a Function of
MU for $E_z = 10^6 \text{ Vm}^{-1}$ - Pass 5

Figure 5.16 Probability of MU as a Function of
MU for $E_z = 10^6 \text{ Vm}^{-1}$ - Pass 6

Figure 5.17 Probability of MU as a Function of
 MU for $E_z = 10^6 \text{ Vm}^{-1}$ - Pass 7

Figure 5.18 Probability of MU as a Function of
MU for $E_z = 10^6 \text{ Vm}^{-1}$ - Pass 8

Figure 5.19 Probability of MU as a Function of
MU for $E_z = 10^6 \text{ Vm}^{-1}$ - Pass 9

Figure 5.20 Probability of MU as a Function of
MU for $E_z = 10^6 \text{ Vm}^{-1}$ - Pass 10

about $\Delta\mu = 0$. This is clear in Figures (5.11)-(5.20) for $E_z = 10^6 \text{Vm}^{-1}$. The distribution in spreading out gets "piled-up" at the $\mu = 0$ boundary producing a very asymmetric distribution. In Figures (5.21)-(5.24) we have graphed the variance in μ against the number of passes, for 0-mode fields of 10^4 , $7 \cdot 10^4$, 10^5 , 10^6 Volts per metre. Figure (5.21) shows a linear increase in the variance of μ with the number of passes appropriate for low fields but Figures (5.22) and (5.23) show the variance saturating after several passes. Figure (5.24) shows strong linear behaviour and a saturation in diffusion. Behaviour in the X-mode shows a similar saturation as shown in Figures (5.25) and (5.26).

Hence we have shown that for electric wave fields above 10^5 Volts per metre the diffusion coefficient becomes substantially reduced as relativistic effects become important. We have considered a reduced form of the diffusion coefficient which is appropriate for fields up to 10^6 Volts per metre but for higher fields the diffusion formalism breaks down. In this region one must adopt the integro-differential approach of

$$\left(\frac{\partial f}{\partial t}\right)_{\text{wave}} = \frac{1}{n(\Delta t + T)} \int \dots \int f\left(\mu - \sum_{i=1}^n \Delta\mu_i, 0\right) \prod_{j=1}^n \left[P\left(\mu - \sum_{k=j}^n \Delta\mu_k, \Delta\mu_j\right) d\Delta\mu_j \right] \quad (4)$$

An alternative approach is to use the probability function for a single pass and combine it into a Fokker-Planck code which includes a collisional term, and a term for steady toroidal electric field. Such an approach has recently been adopted by O'Brien (1987c).

GRAPH OF VARIANCE IN MAGNETIC MOMENT AGAINST TRANSIT NO.

Figure 5.21 Graph of Variance in Magnetic Moment
Against Transit Number for an O-mode
Wave

GRAPH OF VARIANCE IN MAGNETIC
MOMENT AGAINST TRANSIT NO.

Figure 5.22 Graph of Variance in Magnetic Moment
Against Transit Number for an O-mode
Wave

GRAPH OF VARIANCE IN MAGNETIC
MOMENT AGAINST TRANSIT NO.

Figure 5.23 Graph of Variance in Magnetic Moment
Against Transit Number for an O-mode
Wave

GRAPH OF VARIANCE IN MAGNETIC
MOMENT AGAINST TRANSIT NO.

Figure 5.24 Graph of Variance in Magnetic Moment
Against Transit Number for an O-mode
Wave

GRAPH OF VARIANCE IN MAGNETIC
MOMENT AGAINST TRANSIT NO.

Figure 5.25 Graph of Variance in Magnetic Moment
Against Transit Number for an X-mode
Wave

GRAPH OF VARIANCE IN MAGNETIC MOMENT AGAINST TRANSIT NO.

Figure 5.26 Graph of Variance in Magnetic Moment
Against Transit Number for an X-mode
Wave

CHAPTER 6

Summary and Conclusions

In this thesis we have been interested in the mathematics of high power electron cyclotron resonance heating. In attempts to seek even higher plasma temperatures larger power ECRH experiments are being planned - most recently the MTX experiment at the Lawrence Livermore Laboratory has introduced a free electron laser to generate electric wave fields of 10^7 - 10^8 Vm⁻¹, replacing the gyrotrons commonly used in other experiments. In the past the effect of waves on particles has generally been described by quasilinear theory whereby the Vlasov equation is linearized in a perturbation of the electric field. Averaging over the fast timescale associated with the wave allows the perturbed part of the distribution function to be described by the usual linearised Vlasov equation except that the equilibrium part of the distribution function is no longer constant but varies on a slow timescale, in a diffusive manner

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} \left(D \frac{\partial f_0}{\partial v} \right)$$

where the diffusion coefficient is given in terms of the wave fields. For high wave fields this linearization process is likely to become less accurate and so we adopt a Fokker-Planck formalism to derive a modified diffusion coefficient, and ultimately dispense with the Fokker-Planck formalism altogether.

We begin in Chapter 1 with a discussion of the background and history of fusion and a short summary of cold plasma theory, warm plasma theory and ECRH and current drive theory. We examine the reflection and absorption of the X-mode and O-mode in tokamak experiments and consider non-inductive current drive by injecting waves with parallel momentum along the toroidal field and the proposition of Fisch to drive currents by creating an asymmetry in the plasma resistivity.

In Chapter 2 we derive averaged equations for particles subjected to arbitrarily polarised R.F. waves, and in particular X-mode and O-mode waves, from a Lagrangian formalism based on the work of Littlejohn (1983; 1984; 1985) who shows how the guiding centre motion of a particle can be derived from a phase space Lagrangian defined by

$$L(\underline{q}, \underline{p}, \dot{\underline{q}}, t) = \underline{p} \cdot \dot{\underline{q}} - H(\underline{p}, \underline{q}, t) \quad (1)$$

where H is the Hamiltonian. The standard Euler-Lagrange equations applied to the variational principle

$$\delta \int L dt = 0 \quad (2)$$

with $\underline{p}$ and $\underline{q}$ allowed to vary independently give Hamilton's equations. Transforming to any set of six independent variables, the equations of motion are still given by the variational principle. We consider two background magnetic field regimes, one constant and one linearly varying in space. At the end of this chapter we compare these averaged equations with integrated forms of exact Lorentz force

equations. The averaged equations exhibit good accuracy and a substantial saving in computer processor time.

In Chapter 3 we discuss the theory of resonant absorption. Starting with an equation of the form

$$f(\underline{v}, t + \Delta t) = \int f(\underline{v} - \Delta \underline{v}, t) P(\underline{v} - \Delta \underline{v}, \Delta \underline{v}) d\underline{v} \quad (3)$$

to describe the evolution of the distribution function $f(\underline{v}, t)$ when radio frequency heating is applied. $P(\underline{v} - \Delta \underline{v}, \Delta \underline{v})$ is the probability function that a particle with velocity $\underline{v}$ receives a kick $\Delta \underline{v}$ from the r.f. beam. The Fokker-Planck approximation is then to expand the integrand on the right hand side of (3) in a Taylor expansion to first order in Δt and second order in $\Delta \underline{v}$. This allows us to write the derivative of the distribution function with respect to time in the form of a diffusion equation

$$\frac{\partial f}{\partial t} = - \frac{\partial}{\partial v_i} (A_i f) + \frac{1}{2} \frac{\partial}{\partial v_i} \left(D_{ij} \frac{\partial f}{\partial v_j} \right) \quad (4)$$

where

$$A_i = \frac{1}{\Delta t} \int P \Delta v_i d^3 \Delta v$$

and
$$D_{ij} = \frac{1}{\Delta t} \int P \Delta v_i \Delta v_j d^3 \Delta v.$$

The first term in equation (4) represents the effects of particle-particle collisions tending to slow down the particle, hence this is called the frictional term. The second term introduces the effect of the wave and describes its diffusion effect on the particle

in velocity space, and hence is called the diffusive term.

There are two common methods for calculating the diffusion coefficient D . The first is the method of simply looking at the dynamical equation for Δv for a single particle or "test particle" and averaging over an ensemble of such particles (often itself called a "Fokker-Planck" approach). Another possibility is to employ quasilinear theory. In quasilinear theory one can derive a diffusion equation by linearising the Vlasov equation $f \approx f_{eq} + f_{wave}$ giving the diffusion coefficient in terms of the wave electric fields. However both methods are likely to become less accurate as the electric field increases in size, as the first method relies on small kicks in Δv to allow the Taylor expansion to be accurate and the second method relies on the perturbed part of the distribution function being small compared to the equilibrium distribution function.

In the first part of Chapter 3 we look at modifying the Diffusion Coefficient to incorporate relativistic particle dynamics by considering a test particle approach. We then depart from Fokker-Planck theory somewhat by considering the probability function $P(\underline{v}-\underline{\Delta v}, \Delta v)$ directly. Following O'Brien (1987 a,b) we derive forms for P in low electric fields (with and without magnetic field gradients) and then look at the form of P for large electric fields.

In Chapter 4 we introduce the Single Particle Experiment in Electron Cyclotron Heating (SPEECH) code. This code simulates ECRH by integrating the averaged equations for an ensemble of particles for a variety of field regimes. We express the output from the code graphically and have compared the theoretical predictions for

$P(\underline{v}-\Delta\underline{v},\Delta\underline{v})$ (or more particularly $P(\underline{\mu},\Delta\underline{\mu})$) with those from the code. Good agreement is found for electric wave fields below about 10^6 Vm^{-1} but for higher fields the analytic approximations break-down. For low applied wave fields the probability function takes the form of a U within a certain range of $\underline{\mu}$. As the wave field is increased a new spike emerges in its profile. This is analysed with the help of phase-plane diagrams.

In Chapter 5 we consider the modified diffusion coefficients and compare them with output from the code. We also consider the behaviour of the distribution function over several passes and show that although it develops in a symmetrical fashion for low wave fields it exhibits strong asymmetry at high powers of ECRH.

Suggestions for Further Work

In the SPEECH code only the fundamental X-mode and O-mode propagating perpendicularly to the background magnetic field were considered. The possibility therefore occurs to extending to include $k_{\parallel}$ in the code and possibly an analysis of second harmonic X-mode heating as this is popular in current experiments. The beam profile chosen was a top hat and a more realistic approximation would be a Gaussian profile. Some work on beams with Gaussian profiles has recently been done by Nevins et al (1987).

APPENDIX 1

Spatial Diffusion Induced by the R.F. Wave

Equation (2.61) yields

$$\begin{aligned}
 \frac{m u_{\perp} \dot{u}_{\perp}}{\Omega} &= \frac{e}{2} \left[A_{1x} \frac{u_{\perp}}{\Omega} (k_{\parallel} \dot{Z} - w) e^{i\psi} + \text{complex conjugate} \right] \\
 &- \frac{e}{2i} \left[A_{1y} \frac{u_{\perp}}{\Omega} (k_{\perp} \dot{X} + k_{\parallel} \dot{Z} - w) e^{i\psi} - \text{complex conjugate} \right] = \\
 &- \frac{e}{2} \left[A_{1y} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{Y} \\
 &- \frac{e}{2} \left[A_{1z} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{Z}. \tag{A1.1}
 \end{aligned}$$

Multiplying by $k_{\perp}$ and subtracting expression (2.51) yields

$$\begin{aligned}
 \frac{m k_{\perp} u_{\perp} \dot{u}_{\perp}}{\Omega} &= e B_0 \dot{Y} + \frac{e}{2} \left[A_{1y} \frac{k_{\perp} u_{\perp}}{\Omega} (k_{\perp} \dot{X} + k_{\parallel} \dot{Z} + \dot{\vartheta} - w) e^{i\psi} \right. \\
 &\quad \left. + \text{complex conjugate} \right] \\
 &+ \frac{e}{2i} \left[A_{1y} \frac{k_{\perp} \dot{u}_{\perp}}{\Omega} e^{i\psi} - \text{complex conjugate} \right]. \tag{A1.2}
 \end{aligned}$$

Hence

$$\dot{Y} = \left[\frac{m k_{\perp} u_{\perp} \dot{u}_{\perp}}{\Omega} - \frac{d}{dt} \left\{ \frac{e}{2i} \left[A_{1y} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \right\} \right] / e B_0. \tag{A1.3}$$

Thus for the X-mode with

$$\begin{aligned}
 A_{1y} &= \frac{E_y}{2w i} \\
 \dot{Y} &= \frac{k_{\perp} u_{\perp} \dot{u}_{\perp}}{\Omega^2} + \frac{d}{dt} \left\{ \frac{1}{2} \frac{E_y}{w} \frac{k_{\perp} u_{\perp}}{B_0 \Omega} \cos \psi \right\} \tag{A1.4}
 \end{aligned}$$

and for the 0-mode with

$$\begin{aligned} A_{1y} &= 0 \\ \dot{Y} &= \frac{k_{\perp} u_{\perp} \dot{u}_{\perp}}{\Omega^2} \end{aligned} \quad (A1.5)$$

or

$$\frac{\dot{Y}}{\frac{d}{dt} \left(\frac{1}{2} m u_{\perp}^2 \right)} = \frac{k_{\perp}}{m \Omega^2} . \quad (A1.6)$$

If a power P is being absorbed by the electrons there will be a current

$$J = \frac{ek}{m \Omega^2} P \quad (A1.7)$$

in a direction perpendicular to the field. This can be interpreted as an external force drift produced by the loss of wave momentum. Clearly this current is proportional to the wavelength of the r.f. beam and proportional to (magnetic flux density)⁻². Hence one would expect the largest current on the outside of the plasma.

Let us compare this wave driven drift with the gradient drift.

The gradient drift velocity is given by

$$v_g = \frac{1}{2} \frac{e}{m} v_{\perp}^2 \frac{\nabla B_0}{\Omega^2}$$

see Cairns (1985)

$$= \frac{1}{2} \frac{e}{m} v_{\perp}^2 \frac{B_0}{R \Omega^2} \quad \text{where } R = \text{gradient scale} \sim \text{tokamak major radius.} \quad (A1.8)$$

So

$$\frac{\dot{Y}}{v_g} = \frac{B_0}{B_0} k R \frac{v_{\parallel}}{v_{\perp}} .$$

Now if $f = 100 \text{ GHz} \Rightarrow k \sim 2 \cdot 10^3 \text{ m}^{-1}$ so for $R \sim 1 \text{ m}$ $kR \approx 2 \cdot 10^3$.

The incident wave power flux is

$$\frac{1}{2\mu_0} B_{\sim}^2 c$$

from the Poynting flux.

If $w = 10^7 \text{ Watts per (metre)}^2$ which corresponds to 100 kw from a 100cm^2 cross section wave guide.

Then

$$B_{\sim}^2 \approx \frac{10^7 \times 8\pi \times 10^{-7}}{3 \times 10^8} \approx 10^{-7}.$$

So

$$B_{\sim} \approx 3.3 \cdot 10^{-2} \text{ Tesla.}$$

Thus if

$$B_0 \approx 1 \text{ T} \quad \text{then} \quad \frac{\dot{Y}}{v_g} \approx 1.$$

Hence when the particle is in the beam the wave driven drift and grad B drift are of the same order. Out of the beam $k = 0$ and the wave driven drift vanishes. Consequently these currents should be strongly localised near the heating regions.

Looking at individual particles $\frac{d}{dt} (\frac{1}{2} m u_{\perp}^2)$ is positive and negative depending on the relative phase of the wave and the particle. Similarly $\dot{Y}$ is positive and negative, and the diffusion in velocity space which leads to heating is accompanied by a diffusion of particles in physical space.

APPENDIX 2

Derivation of Averaged Equations for a Linearly Varying Magnetic Field

We shall consider the vector potential

$$\underline{A}_0 = \left(0, B_0 x \left(1 + \frac{z}{R} \right), 0 \right) \quad (A2.1)$$

which yields

$$\underline{B}_0 = \left(-\frac{B_0 x}{R}, 0, B_0 \left(1 + \frac{z}{R} \right) \right). \quad (A2.2)$$

The x-component of $\underline{B}_0$ will, in general, be much smaller than the z-component. We shall use this to write $\Omega = \Omega(z)$ a function of z only. To zero order in the perturbed quantities

$$\left. \begin{aligned} x &= X + \frac{u_b}{\Omega} \sin \vartheta \\ z &= Z. \end{aligned} \right\} \quad (A2.3)$$

and

Hence

$$\underline{A}_0 = \underline{A} \text{ constant field} + \left(0, B_0 \left(X + \frac{u_b}{\Omega} \sin \vartheta \right) \frac{Z}{R}, 0 \right) \quad (A2.4)$$

where

$$\underline{A} \text{ constant field} = (0, B_0 x, 0)$$

from (2.44).

Consequently we get an extra term in the Lagrangian given by (2.45).

We had

$$- eA_{oy}\dot{y} = - eB_o \left(X + \frac{u_b}{\Omega} \sin \vartheta \right) \left(\dot{Y} - \frac{\dot{u}_b}{\Omega} \cos \vartheta + \frac{u_b \dot{\vartheta}}{\Omega} \sin \vartheta \right) \quad (A2.5)$$

which averaged to

$$- eB_o X \dot{Y} - \frac{eB_o u_b^2 \dot{\vartheta}}{2\Omega^2} = - eB_o X \dot{Y} - \frac{\mu u_b^2 \dot{\vartheta}}{2\Omega} \quad (A2.6)$$

Now we get

$$\begin{aligned} - eA_{oy}\dot{y} = & - eB_o \left(X + \frac{u_b}{\Omega} \sin \vartheta \right) \left(\dot{Y} - \frac{\dot{u}_b}{\Omega} \cos \vartheta + \frac{u_b \dot{\vartheta}}{\Omega} \sin \vartheta + \frac{u_b \dot{\Omega}}{\Omega^2} \cos \vartheta \right) \\ & - eB_o \left(X + \frac{u_b}{\Omega} \sin \vartheta \right) \frac{Z}{R} \left(\dot{Y} - \frac{\dot{u}_b}{\Omega} \cos \vartheta + \frac{u_b \dot{\vartheta}}{\Omega} \sin \vartheta + \frac{u_b \dot{\Omega}}{\Omega^2} \cos \vartheta \right) \end{aligned} \quad (A2.7)$$

This leads to an averaged Lagrangian

$$\begin{aligned} L_{av} = & \frac{\mu u_b^2 \dot{\vartheta}}{2\Omega} + \mu u_{||} \dot{z} - eB_o X \dot{Y} \left(1 + \frac{Z}{R} \right) \\ & - \frac{e}{2} \left[A_{1x} \frac{k_b u_b}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{X} \\ & - \frac{e}{2i} \left[A_{1x} \left(\frac{\dot{u}_b}{\Omega} - \frac{u_b \dot{\Omega}}{\Omega^2} \right) e^{i\psi} - \text{complex conjugate} \right] \\ & - \frac{e}{2} \left[A_{1x} \frac{u_b \dot{\vartheta}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \\ & - \frac{e}{2} \left[A_{1y} \frac{k_b u_b}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{Y} \\ & + \frac{e}{2} \left[A_{1y} \left(\frac{\dot{u}_b}{\Omega} - \frac{u_b \dot{\Omega}}{\Omega^2} \right) e^{i\psi} + \text{complex conjugate} \right] \\ & - \frac{e}{2i} \left[A_{1y} \frac{u_b \dot{\vartheta}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \end{aligned}$$

$$\begin{aligned}
 & - \frac{e}{2} \left[A_{1z} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{z} \\
 & - \{m^2 c^4 + m^2 c^2 u^2\}^{\frac{1}{2}}
 \end{aligned} \tag{A2.8}$$

where ϑ -gyro angle of particle

$$u_{\perp} = \gamma v_{\perp} \quad u_{\parallel} = \gamma v_{\parallel} \quad \Omega = \frac{eB}{m} \cong \frac{eB_0}{m} \left(1 + \frac{Z}{R} \right)$$

$$\psi = k_{\perp} X + k_{\parallel} Z + \vartheta - \omega t$$

$$\dot{\Omega} = \frac{eB_0 \dot{Z}}{Rm}; \quad \frac{\dot{\Omega}}{\Omega} = \frac{\dot{Z}}{R \left(1 + \frac{Z}{R} \right)}$$

$$\frac{\partial \Omega}{\partial Z} = \frac{\dot{\Omega}}{\dot{Z}} = \frac{eB_0}{Rm}.$$

Lagrange's equations applied to Lav yield

$$1. \quad \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{X}} \right) - \frac{\partial L_{av}}{\partial X} = 0$$

$$\begin{aligned}
 & \frac{d}{dt} \left(- \frac{e}{2} \left[A_{1x} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \right) = \\
 & - eB_0 \dot{\gamma} \left(1 + \frac{Z}{R} \right) - \frac{e}{2} \left[A_{1x} \frac{k_{\perp}^2 u_{\perp}}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \dot{X} \\
 & - \frac{e}{2i} \left[A_{1x} k_{\perp} \left(\frac{\dot{u}_{\perp}}{\Omega} - \frac{u_{\perp} \dot{\Omega}}{\Omega^2} \right) i e^{i\psi} - \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1x} \frac{k_{\perp} u_{\perp} \dot{\vartheta}}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1y} \frac{k_{\perp}^2 u_{\perp}}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \dot{Y} \\
 & + \frac{e}{2} \left[A_{1y} k_{\perp} \left(\frac{\dot{u}_{\perp}}{\Omega} - \frac{u_{\perp} \dot{\Omega}}{\Omega^2} \right) i e^{i\psi} + \text{complex conjugate} \right]
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{e}{2i} \left[A_{1y} \frac{k_{\perp} u_{\perp} \dot{\psi}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1z} \frac{k_{\perp}^2 u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{Z}.
 \end{aligned} \tag{A2.9}$$

This is the equation for $\dot{Y}$.

$$\begin{aligned}
 2. \quad & \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{Y}} \right) - \frac{\partial L_{av}}{\partial Y} = 0 \\
 & \frac{d}{dt} \left(-eB_0 X \left(1 + \frac{Z}{R} \right) - \frac{e}{2} \left[A_{1y} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \right) = 0.
 \end{aligned} \tag{A2.10}$$

$$\begin{aligned}
 3. \quad & \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{Z}} \right) - \frac{\partial L_{av}}{\partial Z} = 0 \\
 & \frac{d}{dt} \left(m u_{\parallel} + \frac{e}{2i} \left[A_{1x} \frac{u_{\perp} e B_0}{R m \Omega^2} e^{i\psi} - \text{complex conjugate} \right] \right. \\
 & \quad - \frac{e}{2} \left[A_{1y} \frac{u_{\perp} e B_0}{R m \Omega^2} e^{i\psi} + \text{complex conjugate} \right] \\
 & \quad \left. - \frac{e}{2} \left[A_{1z} \frac{k_{\perp} u_{\perp}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \right) = \\
 & \quad - \frac{m u_{\perp} \dot{\psi}}{2 \Omega^2} \frac{e B_0}{R m} - \frac{e B_0 X \dot{Y}}{R} \\
 & \quad + \frac{e}{2} \left[A_{1x} \frac{k_{\perp} u_{\perp}}{\Omega^2} \frac{e B_0}{R m} e^{i\psi} + \text{complex conjugate} \right] \dot{X} \\
 & \quad + \frac{e}{2i} \left[A_{1x} \left(\frac{\dot{u}_{\perp}}{\Omega^2} \frac{e B_0}{R m} - \frac{2 u_{\perp} \dot{\Omega}}{\Omega^3} \frac{e B_0}{R m} \right) e^{i\psi} - \text{complex conjugate} \right] \\
 & \quad - \frac{e}{2} \left[A_{1x} \frac{u_{\perp} \dot{\psi}}{\Omega^2} \frac{e B_0}{R m} e^{i\psi} + \text{complex conjugate} \right] \\
 & \quad + \frac{e}{2} \left[A_{1y} \frac{k_{\perp} u_{\perp}}{\Omega^2} \frac{e B_0}{R m} e^{i\psi} + \text{complex conjugate} \right] \dot{Y}
 \end{aligned}$$

$$\begin{aligned}
& - \frac{e}{2} \left[A_{1y} \left(\frac{\dot{u}_b}{\Omega^2} - \frac{2u_b \dot{\Omega}}{\Omega^3} \right) \frac{eB_0}{Rm} e^{i\psi} + \text{complex conjugate} \right] \\
& + \frac{e}{2i} \left[A_{1y} \frac{u_b \dot{\Omega}}{\Omega^2} \frac{eB_0}{Rm} - \text{complex conjugate} \right] \\
& + \frac{e}{2} \left[A_{1z} \frac{k_{||} u_b}{\Omega^2} \frac{eB_0}{Rm} + \text{complex conjugate} \right] \dot{z} \\
& - \frac{e}{2} \left[A_{1x} \frac{k_{||} k_{\perp} u_b i e^{i\psi}}{\Omega} + \text{complex conjugate} \right] \dot{x} \\
& - \frac{e}{2i} \left[A_{1x} k_{||} \left(\frac{\dot{u}_b}{\Omega} - \frac{u_b \dot{\Omega}}{\Omega^2} \right) i e^{i\psi} - \text{complex conjugate} \right] \\
& - \frac{e}{2} \left[A_{1x} \frac{k_{||} u_b \dot{\Omega}}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \\
& - \frac{e}{2} \left[A_{1y} \frac{k_{||} k_{\perp} u_b}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \dot{y} \\
& + \frac{e}{2} \left[A_{1y} k_{||} \left(\frac{\dot{u}_b}{\Omega} - \frac{u_b \dot{\Omega}}{\Omega^2} \right) i e^{i\psi} + \text{complex conjugate} \right] \\
& - \frac{e}{2i} \left[A_{1y} \frac{k_{||} u_b \dot{\Omega}}{\Omega} i e^{i\psi} - \text{complex conjugate} \right] \\
& - \frac{e}{2} \left[A_{1z} \frac{k_{||} k_{\perp} u_b}{\Omega} i e^{i\psi} + \text{complex conjugate} \right] \dot{z}. \tag{A2.11}
\end{aligned}$$

This is the Z-momentum equation.

$$4. \quad \frac{d}{dt} \frac{\partial L_{av}}{\partial \dot{\Omega}} - \frac{\partial L_{av}}{\partial \Omega} = 0$$

$$\begin{aligned}
& \frac{d}{dt} \left(\frac{m u_b^2}{2\Omega} - \frac{e}{2} \left[A_{1x} \frac{u_b}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \right. \\
& \left. - \frac{e}{2i} \left[A_{1y} \frac{u_b}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \right) =
\end{aligned}$$

$$\begin{aligned}
 & - \frac{e}{2} \left[A_{1x} \frac{k_b u_b}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{x} \\
 & - \frac{e}{2i} \left[A_{1x} \left(\frac{\dot{u}_b}{\Omega} - \frac{u_b \dot{\Omega}}{\Omega^2} \right) e^{i\psi} - \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1x} \frac{u_b \dot{\Omega}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1y} \frac{k_b u_b}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{y} \\
 & + \frac{e}{2} \left[A_{1y} \left(\frac{\dot{u}_b}{\Omega} - \frac{u_b \dot{\Omega}}{\Omega^2} \right) e^{i\psi} + \text{complex conjugate} \right] \\
 & - \frac{e}{2i} \left[A_{1y} \frac{u_b \dot{\Omega}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1z} \frac{k_b u_b}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{z}. \tag{A2.12}
 \end{aligned}$$

This equation gives the change in magnetic moment owing to the r.f. wave.

$$5. \quad \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{u}_u} \right) - \frac{\partial L_{av}}{\partial u_u} = 0$$

$$m\dot{z} - \frac{mu_u}{\gamma} \Rightarrow \dot{z} = \frac{u_u}{\gamma} \text{ as before.}$$

$$6. \quad \frac{d}{dt} \left(\frac{\partial L_{av}}{\partial \dot{u}_b} \right) - \frac{\partial L_{av}}{\partial u_b} = 0$$

$$\begin{aligned}
 & \frac{d}{dt} \left(- \frac{e}{2i} \left[A_{1x} \frac{1}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \right. \\
 & \left. + \frac{e}{2} \left[A_{1y} \frac{1}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \right) =
 \end{aligned}$$

$$\begin{aligned}
 & \frac{mu_b \dot{\psi}}{\Omega} - \frac{e}{2} \left[A_{1x} \frac{k_b}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{x} \\
 & + \frac{e}{2i} \left[A_{1x} \frac{\dot{\Omega}}{\Omega^2} e^{i\psi} - \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1x} \frac{\dot{\psi}}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1y} \frac{k_b}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{y} \\
 & - \frac{e}{2} \left[A_{1y} \frac{\dot{\Omega}}{\Omega^2} e^{i\psi} + \text{complex conjugate} \right] \\
 & - \frac{e}{2i} \left[A_{1y} \frac{\dot{\psi}}{\Omega} e^{i\psi} - \text{complex conjugate} \right] \\
 & - \frac{e}{2} \left[A_{1z} \frac{k_b}{\Omega} e^{i\psi} + \text{complex conjugate} \right] \dot{z} \\
 & - \frac{mu_b}{\gamma} .
 \end{aligned} \tag{A2.13}$$

Zero Field Equations

Putting $A_{1x} = A_{1y} = A_{1z} = 0$ in equation (A2.11) yields

$$\dot{u}_{||} = - \frac{\Omega_1 \rho u_b^2}{2\gamma\Omega R} \tag{A2.14}$$

which gives the parallel velocity as a function of the magnetic field and perpendicular velocity. Similarly equation (A2.12) yields

$$\frac{d}{dt} \left(\frac{mu_b^2}{2\Omega} \right) = 0 \tag{A2.15}$$

the invariance of the magnetic moment.

0-mode Equations

If we choose $A_{1x} = A_{1y} = 0$ and $A_{1z} = E_z/2\omega i$ for the wave vector potential and set $k_{||} = 0$ then equation (A2.12) reduces to

$$\dot{u} = \frac{u_b u_{||} \Omega_0}{2\Omega R \gamma} - \frac{1}{2} \frac{e}{m} \left[\frac{E_z}{c} \frac{u_{||}}{\gamma} \cos \psi \right]. \quad (A2.16)$$

The first term in equation (A2.16) expresses the change in u_b owing to the gradient in B and the second term introduces the effect of the radio frequency wave.

(A2.13) yields

$$\dot{\psi} = \frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{e}{m} \left[\frac{E_z}{c} \frac{u_{||}}{\gamma u_b} \sin \psi \right] \quad (A2.17)$$

as before.

The expression (A2.11) gives

$$\dot{u}_{||} = - \frac{u_b^2 (\dot{\psi} + w) \Omega_0}{2\Omega^2 R}. \quad (A2.18)$$

In all the above $\Omega_0 = \frac{eB_0}{m}$ and $\Omega = \frac{eB_0}{m} \left(1 + \frac{Z}{R} \right)$.

X-mode Equations

For the X-mode we choose $A_{1z} = 0$, $A_{1-} = A_{1x} - iA_{1y}$ and $k_{||} = 0$. Equations (A2.11), (A2.12) and (A2.13) yield

$$\dot{u}_{||} = - \frac{u_b^2 (\dot{\psi} + w) \Omega_0}{2\Omega^2 R} + \frac{1}{2} \frac{e}{m} \frac{E_-}{w} \frac{u^2 (\dot{\psi} + w)}{\Omega^2} \frac{\Omega_0}{R} \sin \psi \quad (A2.19)$$

$$\dot{u}_b = \frac{u_b u_{||} \Omega_0}{2\Omega R \gamma} - \frac{1}{2} \frac{e}{m} E_- (\dot{\psi} + w) \cos \psi \quad (A2.20)$$

$$\dot{\psi} = \frac{\Omega}{\gamma} - w + \frac{1}{2} \frac{e}{m} \frac{E_-}{w} \frac{(\dot{\psi} + w)}{u_b} \sin \psi. \quad (A2.21)$$

REFERENCES

- Abramowitz, M and Stegun, I A (1972), "Handbook of Mathematical Functions", Dover Publ. Inc., New York.
- Adams, J B (1959), European Fusion Research : Report of CERN Study Group on Fusion Problems; CERN 59-16 PS/J/2.
- Ando, A et al (1986), Nuclear Fusion, 26, 107.
- Andronov, A A (1978), Infrared Physics, 18, 385.
- Appert, A et al (1986), Plasma Physics and Controlled Fusion, 28, 133.
- Artsimovich, L A (1964), "Controlled Thermo-Nuclear Reactions", pp 286 et seq., Oliver and Boyd, Edinburgh.
- Aston, F W (1919), Phil. Mag., 38, 707.
- Aston, F W (1920), Phil. Mag., 39, 449 and 611.
- Atkinson, R de'E and Houtermans, F (1929), Z. Phys., 54, 656.
- Baldwin, D E (1977), Review of Modern Physics, 49, 317.
- Behn, R et al (1984), Plasma Physics and Controlled Fusion, 26, 173.
- Bernstein, I B and Englemann, F (1966), Physics of Fluids, 9, 5, 937.
- Bickerton, R J and the JET Team (1986), Plasma Physics and Controlled Fusion, 28, 12B, 1943.
- Bornatici, M et al (1983), Nuclear Fusion, 23, 1153.
- Boyd, T J M and Sanderson, J J (1969), "Plasma Dynamics", Nelson, London.
- Bretscher, E, French, A P and Seidl, F G P (1948), Phys. Rec. 73, 815.

- Cairns, R A et al (1983), Physics of Fluids, 26, 12, 3245.
- Cairns, R A (1985), "Plasma Physics", Blackie, Glasgow and London.
- Cairns, R A and Lashmore-Davies, C N (1986), Plasma Physics and Controlled Fusion, 28, 10, 1539.
- Carmel, Y et al (1983), Physics Review Letters, 50, 112.
- Chen, F F (1984), "Introduction to Plasma Physics and Controlled Fusion", 2nd edn., Plenum Press, London.
- Chiu, S C (1985), GA PROJECT 3411, GA-A17944.
- Clemmow, P C and Dougherty, J P (1969), "Electrodynamics of Particles and Plasmas", Addison-Wesley, London.
- Coulson, C A and Boyd, T J M (1979), "Electricity", 2nd ed., Longman, London.
- Demio, L and Engelmann, F (1986), Plasma Physics and Controlled Fusion, 28, 12A, 1851.
- Dendy, R O (1985), Plasma Physics and Controlled Fusion, 27, 1243.
- Drummond, W E and Pines, D (1964), Annals of Physics (New York), 28, 478.
- Eddington, A S (1920), Observatory, 43, 353; Reports of the British Association (1920), 34.
- Eldridge, O (1972), Physics of Fluids, 15, 4, 676.
- Eubank, H (1982), Proceedings of the 3rd Varenna-Grenoble International Symposium 1982.
- Fidone, I et al (1983), Physics of Fluids, 26, 3292.
- Fielding, P J (1980), Proceedings of the Joint Workshop on Electron Cyclotron Emission and ECRH, Oxford, CLM-ECR.
- Fielding, P J (1982), "Plasma Physics and Nuclear Fusion Research", Academic Press, London.

- Fisch, N J (1978), Physics Review Letters, 41, 13, 873.
- Fisch, N J (1981), Physics Review A, 24, 6, 3245.
- Fisch, N J (1987), Review of Modern Physics, 59, 1, 175.
- Fisch, N J and Boozer, A H (1980), Physics Review Letters, 45, 9, 720.
- Flynn, R W (1971), Physics of Fluids, 14, 5, 956.
- Gill, R D et al (1986), 13th European Conference on Controlled Fusion and Plasma Heating, 10C, 1, 21.
- Gormezano, G (1982), Proceedings of the 3rd Joint Varenna-Grenoble International Symposium, 2, 439.
- Grawe, H (1969), Plasma Physics, 11, 151.
- Grésillon, D et al (1976), Physics Review Letters, 36, 19, 1138.
- Hoshino, K et al (1985), 12th European Conference on Controlled Fusion and Plasma Physics (Budapest), 2, EPS 184.
- Hsu, J Y et al (1983), Physics of Fluids, 26, 3300.
- Jacquiot, J et al (1985), Plasma Physics and Controlled Fusion, 27, 12A, 1379.
- Jaeger, F, Lichtenberg, A J and Lieberman, M A (1972), Plasma Physics, 14, 1073.
- Jory, H et al (1980), I.E.D.M. Tech. Dig., 304.
- Kennel, C F and Engelmann, F (1966), Physics of Fluids, 9, 12, 2377.
- Kreischer, K E et al (1985), Plasma Physics and Controlled Fusion, 27, 12A, 1449.
- Kuckes, A F (1968), Plasma Physics, 10, 367.
- Landau, L D (1946), Journal of Physics (USSR), 10, 25.
- Lichtenberg, A J and Lieberman, M A (1983), "Regular and Stochastic Motion", Springer-Verlag, New York.

- Lieberman, M A and Lichtenberg, A J (1973), Plasma Physics, 15, 125.
- Littlejohn, R G (1983), Journal of Plasma Physics, 29, 111.
- Littlejohn, R G (1984), Physics of Fluids, 27, 976.
- Littlejohn, R G (1985), Physics of Fluids, 28, 2015.
- Litvak, A G et al (1977), Nuclear Fusion 17, 659.
- Lloyd, B et al (1985), Course and Workshops on Applications of R.F. waves to Tokamak Plasmas (Varennna), International School of Plasma Physics, Milan.
- Maekawa, T et al (1977), Physics Review Letters, 40, 1379.
- Maekawa, T et al (1983), Nuclear Fusion, 23, 242.
- Miyamoto, K (1978), Nuclear Fusion, 18, 243.
- Nevins, W M et al (1987), To be published in Physics Review Letters.
- O'Brien, M R (1987a), Private Communications 1.
- O'Brien, M R (1987b), Private Communications 2.
- O'Brien, M R (1987c), Private Communications 3.
- O'Brien, M R, Cox, M and Start, D F H (1986), Nuclear Fusion, 26, 12, 1625.
- Ohkawa, T (1970), Nuclear Fusion, 10, 185.
- Olipphant, M L, Hartek, P and Lord Rutherford (1934), Proceedings of the Royal Society, 144, 692.
- Orzechowski, T J et al (1986), Physics Review Letters, 57, 2172.
- Porkolab, M et al (1983), Plasma Physics and Controlled Nuclear Fusion (IAEA Baltimore), 1, Nuclear Fusion Supplement 227.
- Porkolab, M et al (1984), Proceedings of the 4th Joint Varennna-Grenoble International Symposium, 1, 259.
- Riviere, A C (1986), Plasma Physics and Controlled Fusion, 28, 9A, 1263.

- Roberts, C S and Buchsbaum, S J (1964), Physical Review, 135, 2A, 381.
- Robinson, D C (1985), Proceedings of Conference on Physical Processes in Toroidal Confinement. Varenna 1985.
- Rose, D J and Clark, M (1961), "Plasmas and Controlled Fusion", MIT and Wiley, New York, Chapter 16.
- Seidl, M (1964), Plasma Physics, 6, 597.
- Sprott, J C (1971), Physics of Fluids, 14, 1795.
- Start, D F H et al (1983), Plasma Physics, 25, 1431.
- Stix, T H (1962), "The Theory of Plasma Waves", McGraw-Hill, New York.
- Stix, T H (1965), Physics Review Letters, 15, 878.
- Stix, T H (1975), Nuclear Fusion, 15, 737.
- Suvorov, E V and Tokman, M D (1983), Plasma Physics, 25, 7, 723.
- T-10 Group (1983), 11th European Conference on Controlled Fusion and Plasma Physics (Aachen), 1, 289, EPS.
- Thomassen, K I et al (1986), "Free Electron Laser Experiments in Alcator-C", LLL-PROP-00202, U.S. Government P.O., 1986/8687-002/44101.
- Thomson, G P and Blackman, M (1946), British Patent 817, 681 : application date 8 May 1946.
- Thonemann, P C (1958), Proceedings of the 2nd Conference on Peaceful Uses of Atomic Energy, 31, 34, U.N. Geneva.
- Toyama, H, Okabayashi, M and Ishizuka, H (1968), Plasma Physics, 10, 319.
- Whang, K W and Morales, G J (1983), Nuclear Fusion, 23, 4, 481.