

Two Fraïssé-style theorems for homomorphism-homogeneous relational structures

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Abstract

In this paper, we state and prove two Fraïssé-style results that cover existence and uniqueness properties for twelve of the eighteen different notions of homomorphism-homogeneity as introduced by Lockett and Truss, and provide forward directions and implications for the remaining six cases. Following these results, we completely determine the extent to which the countable homogeneous undirected graphs (as classified by Lachlan and Woodrow) are homomorphism-homogeneous; we also provide some insight into the directed graph case.

Keywords: homomorphism-homogeneous, relational structures, Fraïssé theory, infinite graph theory.

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1 Introduction

A relational first-order structure \mathcal{M} is *homogeneous* (or *ultrahomogeneous*, depending on source) if every isomorphism between finite substructures of \mathcal{M} extends to an automorphism of \mathcal{M} . Examples of homogeneous structures include the countable dense linear order without endpoints $(\mathbb{Q}, <)$, the random graph R and the universal tournament T [26]. The celebrated theorem of Fraïssé [16] states that a structure \mathcal{M} is homogeneous if and only if the class \mathcal{C} of finite substructures of \mathcal{M} (often called the *age* of \mathcal{M}) satisfies four distinct conditions; furthermore, \mathcal{M} is unique up to isomorphism if this is the case. A large body of literature, in a range of subjects across mathematics, is devoted to the study of homogeneous structures. Particular areas of interest include classification results [36] [23] [10], combinatorial aspects [6] and model-theoretic properties, both as objects in their own right and via a link to \aleph_0 -categorical structures [17]. Through the link to \aleph_0 -categorical structures and the famous Ryll-Nardzewski theorem (also Engeler, Svenonius [17]) there is a well-established connection between automorphisms of countable homogeneous structures and interesting infinite permutation groups [3]. Many properties of automorphism groups of homogeneous structures have been studied, such as topological dynamics [20] and applications to constraint satisfaction problems on infinite domains [5]. Furthermore, the groups themselves have lent themselves to research into properties such as: simplicity [27]; generation [15]; reconstruction of the structure from the automorphism group [2]; the existence of generic elements [37]; and the small index property [18] [21]. An excellent overview of this vast subject is [26].

The idea of homogeneity has been extended to other types of finite partial maps between relational structures. This is the concept of *homomorphism-homogeneity*; originally developed by Cameron and Nešetřil in 2006 [9] and Mašulović in 2007 [29]. This definition was taken to its logical conclusion in the papers of Lockett and Truss [24] [25], in which they detailed eighteen

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different notions of homomorphism-homogeneity based on both the finite partial map and the type of endomorphism (see [Definition 2.1](#)). An example of this is MB-homogeneity; a structure \mathcal{M} is MB-homogeneous if every monomorphism between finite substructures of \mathcal{M} extends to a bijective endomorphism of \mathcal{M} . The development of this subject has been rapid, culminating in detailed accounts of homomorphism-homogeneous graphs [\[35\]](#) [\[12\]](#), finite tournaments with loops [\[19\]](#), and posets [\[25\]](#). Aside from these combinatorial studies, finding analogues of results about automorphism groups for endomorphism monoids is a motivating factor in this subject; examples of these include the development of *oligomorphic transformation monoids* [\[30\]](#) [\[12\]](#) and the idea of *generic endomorphisms* [\[24\]](#).

Throughout this article, let σ be a countable relational signature. Suppose that \mathcal{C} is a class of finite σ -structures. The proof of one direction of Fraïssé’s theorem uses conditions on \mathcal{C} to inductively construct a homogeneous structure \mathcal{M} whose age is \mathcal{C} . One of these conditions is the *joint embedding property* (JEP); this property (along with two others) ensures that we can construct a countable σ -structure \mathcal{M} with age \mathcal{C} . The second is the *amalgamation property* (AP); this ensures that \mathcal{M} is homogeneous. This final claim is verified by showing that \mathcal{M} has the *extension property*, a necessary and sufficient condition for a countable σ -structure \mathcal{M} to be homogeneous. Fraïssé’s theorem also states that any two homogeneous σ -structures with the same age are isomorphic; this is shown using a back-and-forth argument between two similarly constructed structures that builds the desired isomorphism. The forth part of the argument ensures that the extended map is totally defined; the back part ensures that the map is eventually surjective.

Cameron and Nešetřil [\[9\]](#) proved an analogue of Fraïssé’s theorem for *MM-homogeneity*, where every monomorphism between finite substructures of some structure \mathcal{M} extends to a monomorphism of \mathcal{M} (see [Definition 2.1](#)). This proof necessitated modification of the amalgamation property to ensure MM-homogeneity; resulting in the *mono-amalgamation property* (MAP). In a slight departure from the technique used to prove Fraïssé’s theorem, the proof of the analogous theorem for MM-homogeneity in [\[9\]](#) utilised a forth alone argument; this is because the extended map need not be surjective. On the uniqueness side, the same article also showed that two MM-structures with the same age may be non-isomorphic; instead detailing that two MM-homogeneous structures were unique up to a weaker notion called *mono-equivalence*. The proof of this again used a forth alone argument. Finally, work has been done on the case of *HH-homogeneity*, where every finite partial homomorphisms of \mathcal{M} extends to an endomorphism of \mathcal{M} . The notion of a *homo-amalgamation property* (HAP) first appeared in the preprint of Pech and Pech [\[33\]](#), in which they used this to prove a version of Fraïssé’s theorem for HH-homogeneity by using a forth alone argument. In addition, they also showed that two HH-homogeneous structures with the same age are homomorphism-equivalent. These results were later published in their following paper [\[34\]](#). Further insights were made by Dolinka [\[14\]](#), who used the HAP (and the equivalent one-point homomorphism extension property (1PHEP)) to determine which structures were both homogeneous and HH-homogeneous.

In the case of MB-homogeneity, a forth alone approach does not suffice. As the extended map must be surjective, we are required to use a back-and-forth argument. The fact that monomorphisms are not invertible in general necessitates the use of a second amalgamation property alongside the MAP of [\[9\]](#); this was defined by Coleman, Evans and Gray [\[12\]](#) using *antimonomorphisms* in the *bi-amalgamation property* (BAP). In a similar situation to [\[9\]](#), two MB-homogeneous structures with the same age may not be isomorphic but instead are unique up to *bi-equivalence*; the proof of this also requires a back-and-forth argument.

In light of these previous generalisations of Fraïssé’s theorem and the multitude of types of homomorphism-homogeneity (see [Definition 2.1](#)), the natural aim would be to find an “umbrella” version of Fraïssé’s theorem; one that encapsulates all possible notions of homomorphism-homogeneity. This result would supply Fraïssé’s theorem, and the versions of [\[9\]](#), [\[34\]](#) and [\[12\]](#), as corollaries. Such a theorem could help to determine the extent to which a structure is

homomorphism-homogeneous based on structural properties. In turn, this would provide a rich source of oligomorphic transformation monoids [12, Theorem 1.7].

However, a compromise must be reached between idealism and practicality for two reasons. First, as discussed above, differing approaches are required if the extended map is surjective; see the contrast between analogues of Fraïssé’s theorem for MM-homogeneity [9] and MB-homogeneity [12] for a case in point. In the forth alone case, we can utilise a single modified amalgamation property in order to construct the structure and extend the map. The issue is that monomorphisms and homomorphisms are not “invertible” in general. This is particularly problematic in the homomorphism case; what could you use to accurately describe the ‘back’ condition for homomorphisms, given that the underlying function may not be invertible? As evidenced in [12], we need *two* modified amalgamation properties in the back-and-forth case; one for the forth part to ensure the extended map is totally defined, and one for the back part to ensure the resulting map is surjective. Second, some kinds of homomorphism-homogeneity are easier to deal with than others. There is a distinct dichotomy in the set of notions of homomorphism-homogeneity, split between those whose extended maps are not necessarily the same “type” as the partial map (such as MH-homogeneity, in that a homomorphism is not necessarily an monomorphism), and those whose extended maps are definitely of the same type as the partial map (such as MM, or MI-homogeneity). The former case causes issues in inductively constructing a structure due to the lack of certainty about the extended map; this is discussed in further detail in Section 2.

The first of these reasons therefore necessitate *two* similar but markedly different theorems (Theorem 1.1 and Theorem 1.2) based on whether or not the proof uses forth alone or a back-and-forth argument; these form the central theorems of the paper. The second only allows the two theorems to cover twelve of the eighteen different notions of homomorphism-homogeneity. In the statement of the theorems below, what constitutes the “relevant” amalgamation property and notion of equivalence will be explained in Sections 2 and 4.

Theorem 1.1. *Let $XY \in \{II, MI, MM, HI, HM, HH\}$.*

- (1) *If \mathcal{M} is an XY -homogeneous σ -structure, then $\text{Age}(\mathcal{M})$ has the relevant amalgamation property.*
- (2) *If \mathcal{C} is a class of finite σ -structures with countably many isomorphism types, is closed under isomorphisms and substructures, has the JEP and the relevant amalgamation property, then there exists an XY -homogeneous σ -structure \mathcal{M} with age \mathcal{C} .*
- (3) *Any two XY -homogeneous σ -structures with the same age are equivalent up to a relevant notion of equivalence.*

Theorem 1.2. *Let $XZ \in \{IA, MA, MB, HA, HB, HE\}$.*

- (1) *If \mathcal{M} is an XZ -homogeneous σ -structure, then $\text{Age}(\mathcal{M})$ has the two relevant amalgamation properties.*
- (2) *If \mathcal{C} is a class of finite σ -structures with countably many isomorphism types, is closed under isomorphisms and substructures, has the JEP and the two relevant amalgamation properties, then there exists an XZ -homogeneous σ -structure \mathcal{M} with age \mathcal{C} .*
- (3) *Any two XZ -homogeneous σ -structures with the same age are equivalent up to a relevant notion of equivalence.*

While not the ideal “umbrella” theorem, these two results are still useful in determining the extent to which a structure is homomorphism-homogeneous; thus providing interesting examples of oligomorphic transformation monoids. To that end, this article is dedicated to the proof of

these two theorems; as well as determining a complete picture of homomorphism-homogeneity for some well-known structures.

Section 2 begins by defining the eighteen different notions of homomorphism-homogeneity, and contains the proof of **Theorem 1.1** split into three propositions (2.3, 2.4, 2.6) that correspond to the three points of the theorem. **Section 3** utilises the idea of the converse of a function to introduce the concept of an *antihomomorphism* between two σ -structures; essentially, this is the preimage of a homomorphism that preserves non-relations. This machinery underpins the ‘back’ condition that is used to extend a map between finite substructures of a σ -structure \mathcal{M} to a surjective endomorphism of \mathcal{M} . Following this, **Section 4** is dedicated to the proof of **Theorem 1.2**. In **Section 5** we introduce the idea of a *maximal homomorphism-homogeneity class* (mhh-class), and determine mhh-classes for every countable homogeneous undirected graph in the classification of Lachlan and Woodrow [23], as well as turning our attention to the directed graph case.

Throughout the article, we will write \underline{x} to mean an n -tuple of some set; this non-standard notation is motivated by the use of barred notation to mean converses of functions, which appear more regularly. The notation $^{-1}$ is reserved exclusively for the inverse of a function. For some countable indexing set I , we define $\sigma = \{R_i : i \in I\}$ to be a relational signature. Usually, \mathcal{M} will denote a countable σ -structure on domain M . If \mathcal{A} and \mathcal{B} are two σ -structures, then a *homomorphism* $f : \mathcal{A} \rightarrow \mathcal{B}$ is a function $f : A \rightarrow B$ such that if $\underline{a} \in R_i^{\mathcal{A}}$ then $\underline{a}f \in R_i^{\mathcal{B}}$; that is, f preserves relations in \mathcal{A} . We say that $f : \mathcal{A} \rightarrow \mathcal{B}$ is an *embedding* if f is a homomorphism and $\underline{a} \notin R_i^{\mathcal{A}}$ implies that $\underline{a}f \notin R_i^{\mathcal{B}}$; so f preserves relations and non-relations of \mathcal{A} . The *age* $\text{Age}(\mathcal{M})$ of \mathcal{M} is the class of all finite structures that can be embedded in a σ -structure \mathcal{M} . For more on the introductory concepts of model theory, [17] is a good place to start.

2 Proof of **Theorem 1.1**

We recall the eighteen different notions of homomorphism-homogeneity as developed in the two papers of Lockett and Truss [24] [25]. Following their lead, we denote each type of endomorphism by a symbol: H for endomorphism, E for epimorphism (surjective endomorphisms), M for monomorphism (injective endomorphism), B for bismorphism (bijective endomorphism), I for embedding and A for automorphism. We cannot assert that a finite partial map is surjective; there is no well defined notion of a finite partial epimorphism, for instance. Therefore, there are only three types of finite partial map of a structure: H for homomorphism, M for monomorphism, and I for embedding. Without loss of generality, maps between finite substructures can be taken to be surjective.

Definition 2.1. Let \mathcal{M} be a first-order structure, and take $P \in \{H, M, I\}$ and $Q \in \{H, E, M, B, I, A\}$. Say that \mathcal{M} is *XY-homogeneous* if every finite partial map of type P of \mathcal{M} extends to a map of type Q of \mathcal{M} . We denote the collection of all notions of homomorphism-homogeneity by \mathfrak{H} . Furthermore, we denote the *class of all PQ-homogeneous structures* by PQ, and say that \mathbb{H} is the set of all classes of PQ-homogeneous structures.

For example, a structure \mathcal{M} is HE-homogeneous if every finite partial homomorphism (H) of \mathcal{M} extends to an epimorphism (E) of \mathcal{M} . Regular homogeneity (as in [26], for instance) corresponds to IA-homogeneity using this notation. All possible types of homomorphism-homogeneity given in **Definition 2.1** are outlined in **Table 1**.

It is important to make the distinction between a notion of homomorphism-homogeneity and the associated class of homomorphism-homogeneous structures. For example, II-homogeneity and IA-homogeneity represent two different notions of homomorphism-homogeneity under consideration. As outlined in **Table 1**, II-homogeneity is where every finite partial isomorphism extends to an *embedding*; IA-homogeneity is where every finite partial isomorphism extends to an automorphism. However, the classes II and IA of homomorphism-homogeneous structures

	isomorphism (I)	monomorphism (M)	homomorphism (H)
End(\mathcal{M}) (H)	IH	MH	HH
Epi(\mathcal{M}) (E)	IE	ME	HE
Mon(\mathcal{M}) (M)	IM	MM	HM
Bi(\mathcal{M}) (B)	IB	MB	HB
Emb(\mathcal{M}) (I)	II	MI	HI
Aut(\mathcal{M}) (A)	IA	MA	HA

Table 1: Table of PQ-homogeneity: \mathcal{M} is PQ-homogeneous if a finite partial map of type X (column) extends to a map of type Y (row) in the associated monoid. The collection of all notions of homomorphism-homogeneity is denoted by \mathfrak{H} .

coincide. For countable structures, it was shown by Lockett and Truss [25] that a structure is II (MI, HI)-homogeneous if and only if it is IA (MA, HA)-homogeneous; that is, $II = IA$, $MI = MA$ and $HI = HA$. This difference between notions of homomorphism-homogeneity and classes of homomorphism-homogeneous structures is apparent in Section 4, where we re-prove this result of [25] from a Fraïssé-theoretic perspective.

It follows that some notions of homomorphism-homogeneity are stronger than others. For instance, as every bimorphism is a monomorphism, it follows that every MB-homogeneous structure is also MM-homogeneous. Similarly, as an isomorphism is both a monomorphism and a homomorphism, it follows that if a structure \mathcal{M} is PQ-homogeneous then it is also IQ-homogeneous. This natural concept inversely corresponds to a natural containment order on the set \mathfrak{H} of homomorphism-homogeneity classes; see Figure 1 for a diagram of this order. Notice that the stronger the notion of homomorphism-homogeneity, the smaller the class of countable structures that satisfy that notion. This difference is explained in more detail in Section 5.

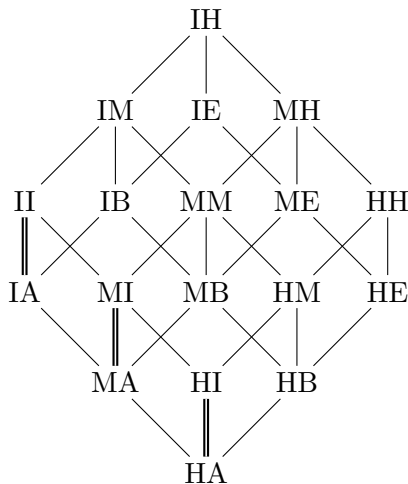


Figure 1: The set \mathfrak{H} of homomorphism-homogeneity classes for countable first-order structures, partially ordered by inclusion. Lines indicate inclusion, double lines indicate equality.

As discussed in the introduction, it is necessary to partition \mathfrak{H} into two pieces based on whether or not the extended map is surjective. This represents the division between cases where a forth alone argument will suffice and the other when we require a back-and-forth construction. Furthermore, there are some elements of \mathfrak{H} that are weaker notions of homogeneity than others.

These are of the form PQ where a map being of type Q does not necessarily imply that it is a map of type P; for instance, a homomorphism is not necessarily a monomorphism. These phenomena motivate the division of \mathfrak{H} into the following subsets:

- forth alone $\mathfrak{F} = \{XY \in \mathfrak{H} : X, Y \in \{H, M, I\}\}$;
- back-and-forth $\mathfrak{B} = \{XZ \in \mathfrak{H} : X \in \{H, M, I\}, Z \in \{E, B, A\}\}$;
- no implication $\mathfrak{N} = \{IH, IE, IM, IB, MH, ME\}$;
- implication $\mathfrak{J} = \mathfrak{H} \setminus \mathfrak{N}$.

This partitions \mathfrak{H} into four parts based on the intersections of $\mathfrak{B}, \mathfrak{F}$ with $\mathfrak{N}, \mathfrak{J}$ (see [Figure 2](#), where the boxes represent intersections).

	\mathfrak{F}	\mathfrak{B}
\mathfrak{N}	IH IM MH	IE IB ME
\mathfrak{J}	HH HM HI MM MI II	HE HB HA MB MA IA

Figure 2: Diagram illustrating the subsets $\mathfrak{F}, \mathfrak{B}, \mathfrak{N}$, and \mathfrak{J} of \mathfrak{H} .

We move on to establish the machinery required for the proof of [Theorem 1.1](#). This result deals with types of homomorphism-homogeneity in \mathfrak{F} (see [Figure 2](#)); those that only require a forth construction to prove. Consequently, we have that $X, Y \in \{H, M, I\}$ throughout this section. When we say a *map of type X*, we are referring to this instance; so if α is a map of type H, it is a homomorphism. Notice that $I \subseteq M \subseteq H$.

A critical step in Fraïssé's proof is the establishment that the inductively constructed structure \mathcal{M} is homogeneous; namely by showing that \mathcal{M} satisfies the *extension property*, a necessary and sufficient condition for homogeneity. Since different kinds of homomorphism-homogeneity rely on extending different kinds of maps, this property needs to be generalised and then shown to be an equivalent condition to the relevant notion of homomorphism-homogeneity. To that end, we define the *XY-extension property* (XYEP), where $X, Y \in \{H, M, I\}$:

(XYEP) A structure \mathcal{M} with age \mathcal{C} has the XYEP if for all $A \subseteq B \in \mathcal{C}$ and maps $f : A \rightarrow \mathcal{M}$ of type X, there exists a map $g : B \rightarrow \mathcal{M}$ of type Y extending f .

For example, the standard extension property in Fraïssé's theorem is the IIAP, and the *mono-extension property* of [\[9\]](#) is the MMAP.

Ideally, we would like to say a structure \mathcal{M} is XY-homogeneous if and only if \mathcal{M} has the XYEP, generalising the observation of Fraïssé. However, complications occur in the proof of the converse direction of this statement; this is due to the inductive construction of the extended map. For example, suppose that \mathcal{M} has the IMEP and that $f : A \rightarrow B$ is an isomorphism. Extending this using the IMEP gives a monomorphism $g : A' \rightarrow B'$ where $A \subseteq A'$ and $B \subseteq B'$. However, g is a monomorphism between finite substructures; and so in general, we cannot extend g to another monomorphism h between finite substructures. The only way we can

continue extending is if the map of type Y is also of type X. This behaviour is the motivating factor in splitting \mathfrak{H} into \mathfrak{J} and \mathfrak{N} (see page 6). In light of this, we show that the XYEP is a necessary condition for XY-homogeneity in general, and that it is also sufficient when the extended map of type Y is also a map of type X.

Proposition 2.2. *Let \mathcal{M} be a countable σ -structure with age \mathcal{C} , and let $XY \in \mathfrak{F}$.*

(1) *If \mathcal{M} is XY-homogeneous, then \mathcal{M} has the XYEP.*

(2) *Suppose also that $XY \in \mathfrak{J}$. If \mathcal{M} has the XYEP, then \mathcal{M} is XY-homogeneous.*

Proof. (1) Let $A \subseteq B \in \mathcal{C}$ and $f : A \rightarrow \mathcal{M}$ be a map of type X. As $\text{Age}(\mathcal{M}) = \mathcal{C}$, there exists an isomorphism $\theta : B \rightarrow B\theta \subseteq \mathcal{M}$. Therefore, $\theta^{-1}f : A\theta \rightarrow Af$ is a map of type X between finite substructures of \mathcal{M} . As \mathcal{M} is XY-homogeneous, extend $\theta^{-1}f$ to a map $\alpha : \mathcal{M} \rightarrow \mathcal{M}$ of type Y. Hence, $\theta\alpha : B \rightarrow \mathcal{M}$ is a map of type Y. It remains to show that $\theta\alpha$ extends f ; for any $a \in A$ it follows that

$$af = a\theta\theta^{-1}f = a\theta\alpha$$

as α extends $\theta^{-1}f$. Therefore \mathcal{M} has the XYEP.

(2) Suppose that $f : A \rightarrow B$ is a map of type X between finite substructures of \mathcal{M} . We use a forth argument to extend f to a map α of type Y. As \mathcal{M} is countable, we enumerate the points of $M = \{m_0, m_1, \dots\}$. Set $A = A_0, B = B_0$ and $f = f_0$ and assume that we have extended f to a map $f_k : A_k \rightarrow B_k$, where $A_i \subseteq A_{i+1}$ and $B_i \subseteq B_{i+1}$ for all $0 \leq i \leq k-1$. At most, we can assume that f_k is a map of type Y. Select $m_i \in \mathcal{M} \setminus A_k$, where i is the least natural number such that $m_i \notin A_k$. We can see that $A_k \cup \{m_i\} \subseteq \mathcal{M}$ belongs to \mathcal{C} . As $XY \in \mathfrak{J}$, the map f_k of type Y is also of type X; so use the XYEP to find a map $f_{k+1} : A_k \cup \{m_i\} \rightarrow \mathcal{M}$ of type Y extending f_k . Repeating this process infinitely many times, ensuring that each m_i appears at some stage, extends f to a map $\alpha : \mathcal{M} \rightarrow \mathcal{M}$ of type Y; so \mathcal{M} is XY-homogeneous. \square

We now move to the proof of [Theorem 1.1](#). Our eventual aim is to construct a countable structure \mathcal{M} with age \mathcal{C} , where \mathcal{M} is XY-homogeneous. Recall (from [\[26\]](#)) that a class \mathcal{C} of finite σ -structures has the *joint embedding property* (JEP) if for all $A, B \in \mathcal{C}$ there exists a $D \in \mathcal{C}$ such that D jointly embeds A and B . This property, along with \mathcal{C} being closed under substructures and isomorphisms, and having countably many isomorphism types, is required to construct a countable structure \mathcal{M} with age \mathcal{C} ; it has nothing to do with the homogeneity of the structure \mathcal{M} . In Fraïssé's theorem, it is the amalgamation property that is central to ensuring that the constructed structure is homogeneous; following the lead of [\[9\]](#), this must be generalised in order to ensure XY-homogeneity. So to construct a countable, XY-homogeneous structure \mathcal{M} with age \mathcal{C} , the class of finite structures \mathcal{C} must have the JEP and some generalised amalgamation property.

Since different types of homogeneity require different amalgamation properties, it then makes sense to define an ‘‘umbrella’’ condition; one that encompasses every required amalgamation property. This is the *XY-amalgamation property* (XYAP), where $X, Y \in \{\text{H}, \text{M}, \text{I}\}$:

(XYAP) Let \mathcal{C} be a class of finite σ -structures. Then \mathcal{C} has the XYAP if for all $A, B_1, B_2 \in \mathcal{C}$, map $f_1 : A \rightarrow B_1$ of type X and embedding $f_2 : A \rightarrow B_2$, there exists a $D \in \mathcal{C}$, embedding $g_1 : B_1 \rightarrow D$ and map $g_2 : B_2 \rightarrow D$ of type Y such that $f_1g_1 = f_2g_2$ (see [Figure 3](#)).

Based on choices for X and Y, the XYAP yields nine different amalgamation properties; one for each notion of XY-homogeneity in \mathfrak{F} . For instance, the IIAP is the standard amalgamation property, the MMAP is the MAP in [\[9\]](#) and the HHAP is the HAP from [\[14\]](#). These are the relevant amalgamation properties as alluded to in [Theorem 1.1](#). Our next result demonstrates [Theorem 1.1](#) (1).

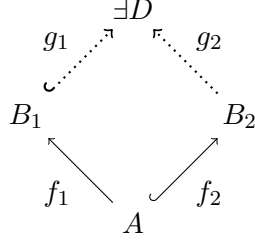


Figure 3: Diagram of the XY-amalgamation property (XYAP).

Proposition 2.3 ([Theorem 1.1](#) (1)). *Suppose that $XY \in \mathfrak{F}$. If a countable σ -structure \mathcal{M} is XY-homogeneous, then $\text{Age}(\mathcal{M})$ has the XYAP.*

Proof. Suppose that $A, B_1, B_2 \in \text{Age}(\mathcal{M})$, $f_1 : A \rightarrow B_1$ is a map of type X and $f_2 : A \rightarrow B_2$ is an embedding. Without loss of generality, suppose that f_2 is the inclusion map and that $A, B_1, B_2 \subseteq \mathcal{M}$. Using XY-homogeneity of \mathcal{M} , extend $f_1 : A \rightarrow B_1$ to a map $\alpha : \mathcal{M} \rightarrow \mathcal{M}$ of type Y. Set $D = B_1 \cup B_2 \alpha$ and induce the structure on D with relations from \mathcal{M} . Finally, take $g_1 : B_1 \rightarrow D$ to be the inclusion map and define g_2 to be the map $g_2 = \alpha|_{B_2} : B_2 \rightarrow D$ of type Y. We can see that $f_1 g_1 = f_2 g_2$ and so these choices verify the XYAP for $\text{Age}(\mathcal{M})$. \square

Now, we proceed with the proof of [Theorem 1.1](#) (2). As in [9], different stages of the inductive construction are achieved at even and odd steps.

Proposition 2.4 ([Theorem 1.1](#) (2)). *Suppose that $XY \in \mathfrak{F} \cap \mathfrak{J}$. Let \mathcal{C} be a class of finite σ -structures that is closed under isomorphism and substructures, has countably many isomorphism types, and has the JEP and XYAP. Then there exists a countable XY-homogeneous σ -structure \mathcal{M} with age \mathcal{C} .*

Proof. Along the lines of similar proofs of Fraïssé's theorem (see [3], [9]), the idea is to construct \mathcal{M} over countably many stages, assuming that M_k has been constructed at some stage $k \in \mathbb{N}$, with M_0 being some member of \mathcal{C} . As the number of isomorphism types in \mathcal{C} is countable, we can choose a countable set S of pairs (A, B) with $A \subseteq B \in \mathcal{C}$ such that every pair $A' \subseteq B' \in \mathcal{C}$ is represented by some pair $(A, B) \in S$. Choose a bijection $\beta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that $\beta(i, j) \geq i$.

Assume first that k is even. As \mathcal{C} has countably many isomorphism types, we can enumerate the isomorphism types of \mathcal{C} by $\{T_0, T_1, T_2, \dots\}$. Use the JEP to find a structure D that contains both M_k and a copy of $C \cong T_{k/2} \in \mathcal{C}$; define M_{k+1} to be this structure D . Now suppose that k is odd. Let $L_k = (A_{kj}, B_{kj}, f_{kj})_{j \in \mathbb{N}}$ be the list of all triples (A, B, f) such that $(A, B) \in S$ and $f : A \rightarrow M_k$ is a map of type X. This list is countable as S is and there are finitely many maps of type X from A into M_k . Let $k = \beta(i, j)$. Then as $\beta(i, j) \geq i$, the map $f_{ij} : A_{ij} \rightarrow M_i \subseteq M_k$ exists. Therefore, we can use the XYAP to define M_{k+1} such that $M_k \subseteq M_{k+1}$ and the map $f_{ij} : A_{ij} \rightarrow M_k$ of type X extends to some map $g_{ij} : B_{ij} \rightarrow M_{k+1}$ of type Y. This ensures that every possible XY-amalgamation occurs.

Define $\mathcal{M} = \bigcup_{k \in \mathbb{N}} M_k$. Our construction ensures that every isomorphism type of \mathcal{C} appears at an even stage, so every structure in \mathcal{C} embeds into \mathcal{M} . Conversely, we have that $M_k \in \mathcal{C}$ for all $k \in \mathbb{N}$. As \mathcal{C} is closed under substructures, every structure that embeds into \mathcal{M} is in \mathcal{C} , showing that $\text{Age}(\mathcal{M}) = \mathcal{C}$.

It remains to show that \mathcal{M} is XY-homogeneous. As $XY \in \mathfrak{F} \cap \mathfrak{J}$, it is enough to show that \mathcal{M} has the XYEP by [Proposition 2.2](#). So assume that $A \subseteq B \in \mathcal{C}$ and that $f : A \rightarrow \mathcal{M}$ is a map of type X. As Af is finite, it follows that there exists $j \in \mathbb{N}$ such that $Af \subseteq M_j$. Furthermore, there exists a triple $(A_{j\ell}, B_{j\ell}, f_{j\ell}) \in L_j$ such that there exists an isomorphism $\theta : B \rightarrow B_{j\ell}$ with $A\theta|_A = A_{j\ell}$ and $f = \theta|_A f_{j\ell}$. Define $n = \beta(j, \ell)$; as $n \geq j$, it follows that $Af = A_{j\ell} f_{j\ell} \subseteq M_j \subseteq M_n$. Here, M_{n+1} is constructed by XY-amalgamating M_n and $B_{j\ell}$ over $A_{j\ell}$; this provides an extension $g_{j\ell} : B_{j\ell} \rightarrow M_{n+1}$ of type Y to the map $f_{j\ell}$ of type X. It follows

that $g = \theta g_{j\ell} : B \rightarrow \mathcal{M}$ is a map of type Y extending the map f of type X (see Figure 4 for a diagram). Therefore, \mathcal{M} has the XYEP, proving that it is XY-homogeneous. \square

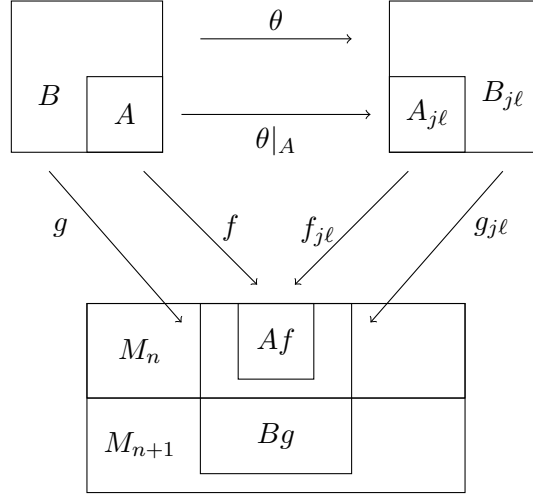


Figure 4: Diagram of maps in the proof of Proposition 2.4. The map $g = \theta g_{j\ell} : B \rightarrow \mathcal{M}$ of type Y extends the map f of type X.

All that remains to show is part (3) of Theorem 1.1. It was previously mentioned in [9] that two MM-homogeneous structures with the same age need not be isomorphic, but are instead *mono-equivalent*. This inspires another collective definition; and is the relevant notion of equivalence as mentioned in the statement of Theorem 1.1.

Definition 2.5. Let \mathcal{M}, \mathcal{N} be σ -structures and suppose that $Y \in \{H, M, I\}$. Say that \mathcal{M} and \mathcal{N} are *Y-equivalent* if $\text{Age}(\mathcal{M}) = \text{Age}(\mathcal{N})$ and every embedding $f : A \rightarrow \mathcal{N}$ from a finite substructure A of \mathcal{M} can be extended to a map $g : \mathcal{M} \rightarrow \mathcal{N}$ of type Y, and vice versa.

Note that if two structures \mathcal{M}, \mathcal{N} are M-equivalent, then they are mono-equivalent in the sense of [9]. If two structures \mathcal{M}, \mathcal{N} are I-equivalent, then they are mutually embeddable.

Proposition 2.6 (Theorem 1.1 (3)). *Let \mathcal{M}, \mathcal{N} be countable σ -structures, and suppose that $XY \in \mathfrak{F} \cap \mathfrak{J}$.*

(1) *Suppose that \mathcal{M}, \mathcal{N} are Y-equivalent. Then \mathcal{M} is XY-homogeneous if and only if \mathcal{N} is.*

(2) *If \mathcal{M}, \mathcal{N} are XY-homogeneous and $\text{Age}(\mathcal{M}) = \text{Age}(\mathcal{N})$ then \mathcal{M}, \mathcal{N} are Y-equivalent.*

Proof. (1) By Proposition 2.2 (2), it suffices to show that \mathcal{N} has the XYEP. Suppose then that $A \subseteq B \in \text{Age}(\mathcal{N})$ and there exists a map $f : A \rightarrow A' \subseteq \mathcal{N}$ of type X. Note that A need not be isomorphic to A' . As $\text{Age}(\mathcal{M}) = \text{Age}(\mathcal{N})$, there exists a copy A'' of A' in \mathcal{M} ; fix an isomorphism $e : A' \rightarrow A''$ between the two. Therefore, $e^{-1} : A'' \rightarrow A'$ is an isomorphism from a finite substructure of \mathcal{M} into \mathcal{N} ; as the two are Y-equivalent, we extend this to a map $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ of type Y. Now, define a map $h = fe : A \rightarrow A''$; this is a map of type X from A into \mathcal{M} . Since \mathcal{M} is XY-homogeneous, it has the XYEP by Proposition 2.2 (1) and so we extend h to a map $h' : B \rightarrow \mathcal{M}$ of type Y. Now, the map $h'\alpha : B \rightarrow \mathcal{N}$ is a map of type Y; we need to show it extends f . Using the facts that α extends e^{-1} and h' extends $h = fe$, for all $a \in A$:

$$af = afee^{-1} = ah'\alpha.$$

Therefore \mathcal{N} has the XYEP. A diagram of this process can be found in Figure 5.

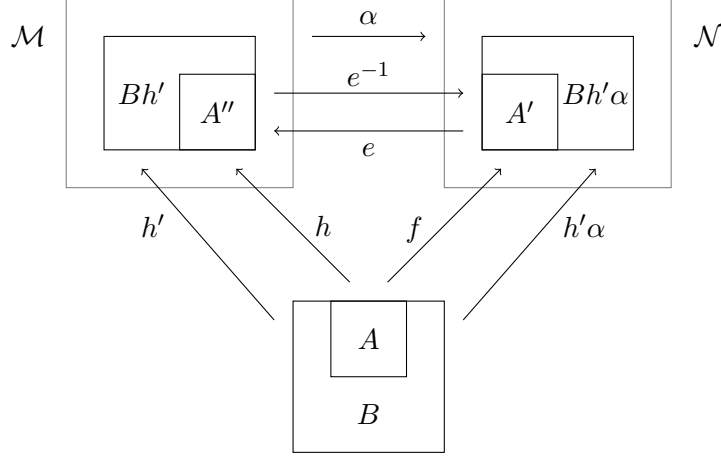


Figure 5: Diagram of maps in the proof of [Proposition 2.6](#). The map $h'\alpha$ is a map of type Y extending the map f of type X, proving that \mathcal{N} has the XYEP.

(2) Let $A \subseteq \mathcal{M}, B \subseteq \mathcal{N}$ and suppose that $f : A \rightarrow B$ is an embedding; trivially, f is also a map of type X. We extend f to a map $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ of type Y via an inductive argument. As \mathcal{M} is countable, we enumerate its elements $M = \{m_0, m_1, \dots\}$. Set $A = A_0, B = B_0$ and $f = f_0$, and suppose that $f_k : A_k \rightarrow B_k$ is a map of type Y where $A_i \subseteq A_{i+1}$ and $B_i \subseteq B_{i+1}$ for all $0 \leq i \leq k-1$. As $XY \in \mathfrak{F} \cap \mathfrak{J}$, f_k is also a map of type X. Select a point $m_i \in \mathcal{M} \setminus A_k$, where i is the least natural number such that $m_i \notin A_k$. We see that $A_k \cup \{m_i\}$ is a substructure of \mathcal{M} and is therefore an element of $\text{Age}(\mathcal{N})$ by assumption. As \mathcal{N} is XY-homogeneous, by [Proposition 2.2](#) (1) it has the XYEP. Using this, extend $f_k : A_k \rightarrow \mathcal{N}$ to a map $f_{k+1} : A_k \cup \{m_i\} \rightarrow \mathcal{N}$ of type Y. As $XY \in \mathfrak{F} \cap \mathfrak{J}$, we can repeat this process infinitely many times; by ensuring that every $m_i \in \mathcal{M}$ is included at some stage, we can extend the map f to a map $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ of type Y as required. We can use a similar argument to construct a map $\beta : \mathcal{N} \rightarrow \mathcal{M}$ of type Y; therefore \mathcal{M} and \mathcal{N} are Y-equivalent. \square

3 Multifunctions and antihomomorphisms

As mentioned in the introduction, homomorphisms are not “invertible” in general. For instance, there could be a homomorphism between two relational σ -structures $\alpha : \mathcal{A} \rightarrow \mathcal{B}$ sending a non-relation of \mathcal{A} to a relation in \mathcal{B} ; that is, such that $\underline{a} \notin R_i^{\mathcal{A}}$ but $\underline{a}\alpha \in R_i^{\mathcal{B}}$. Furthermore, there is no guarantee that the homomorphism is even injective; so α could send two points in \mathcal{A} to the same point in \mathcal{B} . In establishing a suitable ‘back’ amalgamation property for our Fraïssé-style theorem, both of these considerations must be taken into account. This is achieved by the use of the converse of a function.

For a relation $\rho \subseteq X \times Y$, define the *converse* of ρ to be the set

$$\rho^c = \{(y, x) : (x, y) \in \rho\} \subseteq Y \times X$$

We say that a relation $\bar{f} \subseteq Y \times X$ is a *partial multifunction* if $(y, x), (z, x) \in \bar{f}$ implies that $y = z$; and that \bar{f} is a *multifunction* if, in addition, for all $y \in Y$ there exists $x \in X$ such that $(y, x) \in \bar{f}$. It is easy to see that \bar{f} is a partial multifunction if and only if it is the converse f^c of a partial function f , and that \bar{f} is a multifunction if and only if the partial function f is surjective. A multifunction $\bar{f} \subseteq Y \times X$ is *surjective* if for all $x \in X$ there exists $y \in Y$ such that $(y, x) \in \bar{f}$. Consequently, \bar{f} is a surjective multifunction if and only if it is the converse of a surjective function f . It is clear that a (partial) multifunction \bar{f} is a (partial) function if and

only if it is the converse f^c of a (partial) injective function f . We adopt this barred notation throughout the rest of this article; if $f \subseteq X \times Y$ is a function, denote the partial multifunction given by the converse f^c of f by $\bar{f} \subseteq Y \times X$, and vice versa. Note that $\bar{\bar{f}} = f$ for any function f .

Example 3.1. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$ be two sets, and suppose that $f = \{(1, b), (2, b), (3, a), (4, c)\}$ is a function. Then the converse \bar{f} of f is a partial multifunction given by $\bar{f} = \{(b, 1), (b, 2), (a, 3), (c, 4)\}$ (see [Figure 6](#)). By restricting the codomain Y of f to its image $\text{im } f$, the resulting function $g : X \rightarrow \text{im } f$ that behaves like f is surjective. In this case, the converse $\bar{g} : \text{im } f \rightarrow X$ of g is a surjective and totally defined multifunction (see shaded portion of [Figure 6](#)). This technique will be used frequently in [Section 4](#).

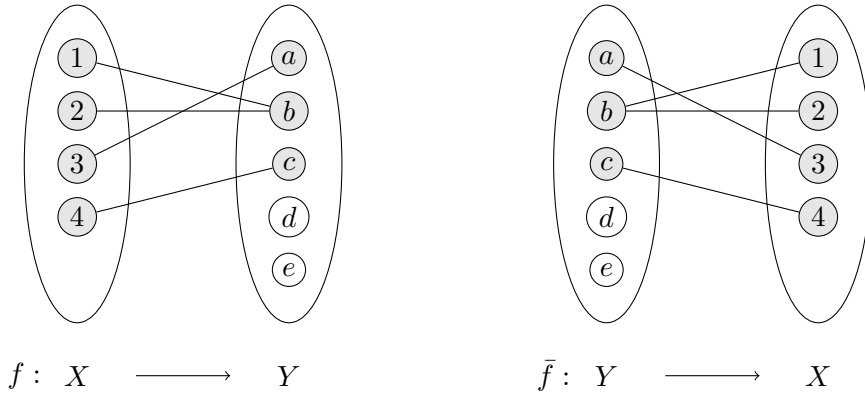


Figure 6: Diagram of a function f and its converse, the partial multifunction \bar{f} , from [Example 3.1](#). The surjective function g and its converse, the surjective multifunction \bar{g} , are shaded.

If $\bar{f} \subseteq Y \times X$ is a multifunction, we will abuse notation and write $\bar{f} : Y \rightarrow X$ where the context is clear. If $y \in Y$, define the set $y\bar{f} = \{x \in X : (y, x) \in \bar{f}\}$. By contrast with a function, notice that $y\bar{f}$ is a set and not a single point; in fact, the set could be infinite (see [Example 3.2](#)). For a tuple $\underline{y} = (y_1, \dots, y_n) \in Y^n$, define $\underline{y}\bar{f}$ to be the following set of tuples

$$\underline{y}\bar{f} = \{(x_1, \dots, x_n) : x_i \in y_i\bar{f} \text{ for all } 1 \leq i \leq n\}.$$

For a subset W of Y , we write

$$W\bar{f} = \{x \in X : (w, x) \in \bar{f} \text{ for some } w \in W\} = \bigcup_{w \in W} w\bar{f}.$$

Abusing terminology, we say that $Y\bar{f}$ is the *image* of \bar{f} .

Example 3.2. Recall that the sign function $s : \mathbb{R} \rightarrow \{-1, 0, 1\}$ is the surjective function defined by

$$s(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Let $\bar{s} : \{-1, 0, 1\} \rightarrow \mathbb{R}$ be the corresponding surjective multifunction. For example, $1\bar{s} = \mathbb{R}^+$ where \mathbb{R}^+ is the set of positive real numbers, $(0, -1)\bar{s} = \{(0, x) \in \mathbb{R}^2 : x \in \mathbb{R}^-\}$ where \mathbb{R}^- is the set of negative real numbers, and $\{-1, 1\}\bar{s} = 1\bar{s} \cup -1\bar{s} = \mathbb{R} \setminus \{0\}$.

Remark. We note the equivalence between the image of the converse of a function f and the preimage of f . The reason for the expression of these sets in terms of converses of functions, and not preimages, is for ease of use and notation.

For a multifunction $\bar{f} : Y \rightarrow X$ and a subset $W \subseteq Y$, we say that the multifunction $\bar{f}|_W : W \rightarrow X$ that acts like \bar{f} on W is the *restriction of \bar{f} to W* . If $Y \subseteq B$ and $X \subseteq A$ are sets, and $\bar{f} : Y \rightarrow X$ and $\bar{g} : B \rightarrow A$ are two multifunctions, then we say that \bar{g} *extends \bar{f}* if $y\bar{f} = y\bar{g}$ for all $y \in Y$.

Throughout, we would like to be able to compose functions with multifunctions and vice versa; we achieve this by composing them as relations.

Lemma 3.3. (1) *Suppose that $\bar{g} : C \rightarrow B$ and $\bar{f} : B \rightarrow A$ are multifunctions. Then $\bar{g} \circ \bar{f} = \bar{g}\bar{f} \subseteq C \times A$ is a multifunction.*

(2) *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions, and suppose that $fg : A \rightarrow C$ is their composition. Then the converse map $\overline{fg} : C \rightarrow A$ is equal to $\bar{g}\bar{f} : C \rightarrow A$, where \bar{g} and \bar{f} are composed as relations.* \square

Remark. We previously noted that a function g is also a multifunction if and only if it is injective; so by this lemma, the composition of a multifunction \bar{f} with an injective function g (or vice versa) is again a multifunction. Furthermore, the assumption that the function g is injective in this case is necessary for the composition $\bar{f} \circ g$ to be a multifunction.

Now, we extend the theory of multifunctions into the setting of relational first-order structures.

Definition 3.4. Suppose that \mathcal{A}, \mathcal{B} are two σ -structures and that $\bar{f} : \mathcal{B} \rightarrow \mathcal{A}$ is a multifunction. We say that \bar{f} is an *antihomomorphism* if $\neg R_i^{\mathcal{B}}(\underline{b})$ in \mathcal{B} implies that $\neg R_i^{\mathcal{A}}(\underline{a})$ in \mathcal{A} for all $\underline{a} \in \underline{b}\bar{f}$ and $R_i \in \sigma$.

Remark. This definition is equivalent to saying that $\bar{f} : \mathcal{B} \rightarrow \mathcal{A}$ is an antihomomorphism if for all $R_i \in \sigma$ and for all $\underline{a} \in \underline{b}\bar{f}$, then $R_i^{\mathcal{A}}(\underline{a})$ implies that $R_i^{\mathcal{B}}(\underline{b})$.

Informally, an antihomomorphism is a multifunction that preserves non-relations. The motivation behind this definition is explained by the following alternative characterisation of antihomomorphisms.

Lemma 3.5. *Let \mathcal{A}, \mathcal{B} be two σ -structures. Then $f^c : \mathcal{B} \rightarrow \mathcal{A}$ is a surjective antihomomorphism if and only if $f : \mathcal{A} \rightarrow \mathcal{B}$ is a surjective homomorphism.*

Proof. Assume that $f : \mathcal{A} \rightarrow \mathcal{B}$ is a surjective homomorphism. As $f : \mathcal{A} \rightarrow \mathcal{B}$ is a surjective function we have that $f^c : \mathcal{B} \rightarrow \mathcal{A}$ is a surjective multifunction. Now suppose that $\neg R_i^{\mathcal{B}}(\underline{b})$. As f must preserve relations, we have $\neg R_i^{\mathcal{A}}(\underline{a})$ whenever $\underline{a}f = \underline{b}$; this is precisely when $\underline{a} \in \underline{b}f^c$. Conversely, suppose that $f^c : \mathcal{B} \rightarrow \mathcal{A}$ is a surjective antihomomorphism; therefore $f : \mathcal{A} \rightarrow \mathcal{B}$ is a surjective function. Suppose also that $R_i^{\mathcal{A}}(\underline{a})$ holds. As f^c is an antihomomorphism, it follows that $\underline{a} \notin \underline{b}f^c$ for every \underline{b} such that $\neg R_i^{\mathcal{B}}(\underline{b})$. Since f is a function, it must be that $\underline{a} \in \underline{b}f^c$ for some \underline{b} such that $R_i^{\mathcal{B}}(\underline{b})$; so f is a homomorphism. \square

Remark. Following this lemma, if $f : \mathcal{A} \rightarrow \mathcal{B}$ is a surjective homomorphism, we write $f^c = \bar{f} : \mathcal{B} \rightarrow \mathcal{A}$ in order to emphasise that the converse of f is both a multifunction and antihomomorphism. Similarly, if $\bar{f} : \mathcal{B} \rightarrow \mathcal{A}$ is a surjective antihomomorphism, we write $\bar{f}^c = f : \mathcal{A} \rightarrow \mathcal{B}$ to emphasise that the converse of \bar{f} is both a function and a homomorphism. The context for when we use this notation should be clear.

If $f : \mathcal{A} \rightarrow \mathcal{B}$ is any homomorphism, we can restrict the codomain to the image to see that $f : \mathcal{A} \rightarrow \mathcal{A}f$ is a surjective homomorphism; and hence $\bar{f} : \mathcal{A}f \rightarrow \mathcal{A}$ is a surjective antihomomorphism by [Lemma 3.5](#). This technique will be used regularly in [Section 4](#).

This result leads to an immediate corollary; an analogue of [Lemma 3.3](#) (2) for σ -structures.

Corollary 3.6. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be σ -structures, and $f : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$ be surjective homomorphisms. Then $\overline{fg} = \overline{g}f$ is a surjective antihomomorphism. \square*

Note that if $f : \mathcal{A} \rightarrow \mathcal{B}$ is a bijective homomorphism, then $\overline{f} : \mathcal{B} \rightarrow \mathcal{A}$ is a bijective function from B to A that preserves non-relations; this is the definition of an *antimonomorphism* (see [\[12\]](#)). Furthermore, if $f : \mathcal{A} \rightarrow \mathcal{B}$ is an isomorphism, then $\overline{f} : \mathcal{B} \rightarrow \mathcal{A}$ is exactly f^{-1} , the inverse isomorphism of f . Having determined that the product of two multifunctions is again a multifunction in [Lemma 3.3](#), an easy composition lemma for antihomomorphisms follows suit.

Lemma 3.7. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be σ -structures. Suppose that $\overline{f} : \mathcal{A} \rightarrow \mathcal{B}$ and $\overline{g} : \mathcal{B} \rightarrow \mathcal{C}$ are antihomomorphisms. Then their composition $\overline{fg} : \mathcal{A} \rightarrow \mathcal{C}$ is an antihomomorphism. \square*

Remark. We note that as every antimonomorphism and isomorphism is also an antihomomorphism, the product $\overline{f}g$ of any antihomomorphism \overline{f} with any antimonomorphism or isomorphism g is again an antihomomorphism. This fact turns out to be crucial in the statement of a suitable amalgamation property for the back part of the back-and-forth argument. Furthermore, the product of two antimonomorphisms (or an antihomomorphism and an isomorphism) is again an antimonomorphism.

4 Proof of [Theorem 1.2](#)

We now move on to discussing extension of finite partial maps of a σ -structure to surjective endomorphisms; this is when $XZ \in \mathfrak{B}$ (when $X \in \{H, M, I\}$, $Z \in \{E, B, A\}$, see [page 6](#), [Figure 2](#)). Due to the lack of symmetry when working with homomorphisms as opposed to isomorphisms, we must provide a backwards condition to achieve the back part of the required back-and-forth argument. Similarly to the more conventional amalgamation properties, this backwards condition is defined on finite structures. This will involve using the concept of antihomomorphisms outlined in [Definition 3.4](#) in three distinct cases; antihomomorphisms (\overline{H}) as converse homomorphisms (H), antimonomorphisms (\overline{M}) as the converse of monomorphisms (M), and inverse isomorphisms (\overline{I}) of isomorphisms (I). Note that the classes I and \overline{I} coincide; we use the barred version when applicable throughout for notational consistency. It can be seen that $\overline{I} \subseteq \overline{M} \subseteq \overline{H}$. We will write $\overline{f} : B \rightarrow A$ to mean some multifunction of type $\overline{X} \in \{\overline{H}, \overline{M}, \overline{I}\}$ from B to A (see [Figure 7](#)).

Type	Map	Converse type	Converse map	\circ	\overline{H}	\overline{M}	\overline{I}
H	homomorphism	\overline{H}	antihomomorphism	\overline{H}	\overline{H}	\overline{H}	\overline{H}
M	monomorphism	\overline{M}	antimonomorphism	\overline{M}	\overline{H}	\overline{M}	\overline{M}
I	isomorphism	\overline{I}	isomorphism	\overline{I}	\overline{H}	\overline{M}	\overline{I}

Figure 7: Types of finite partial maps and multifunctions between structures, and how they are composed.

This notation is used in another manner: if $f : A \rightarrow B$ is a surjective homomorphism of type X , we write $\overline{f} : B \rightarrow A$ to be the corresponding surjective antihomomorphism of type \overline{X} . This is uniquely determined by [Lemma 3.5](#); see [Figure 7](#) for corresponding pairs. The context in which we use this will usually be clear. We also recall [Lemma 3.7](#) and its following remarks;

the composition of two multifunctions of type $\overline{H}, \overline{M}, \overline{I}$ is again a multifunction of type $\overline{H}, \overline{M}, \overline{I}$; a composition table is given in [Figure 7](#).

Let $Y \in \{H, M, I\}$ be as in [Section 2](#). We note that if $Z = E$ then it is a surjective map of type $Y = H$; likewise, when $Z = B$ we have that $Y = M$ and when $Z = A$ we have that $Y = I$. This relation is codified by the following set of pairs:

$$\mathcal{S} = \{(E, H), (B, M), (A, I)\}. \quad (1)$$

It follows that any XZ-homogeneous structure \mathcal{M} is also XY-homogeneous, where the two types of extended map Z and Y are related by the relevant pair $(Z, Y) \in \mathcal{S}$. Therefore, any potential construction for an XZ-homogeneous structure \mathcal{M} must ensure that \mathcal{M} is also XY-homogeneous for the appropriate Y. Following this, the techniques for the forth part of the back-and-forth argument are those used in the proof of [Theorem 1.1](#) (see [Section 2](#)); from these results in [Section 2](#), the use of these properties in the construction will ensure XY-homogeneity of \mathcal{M} .

As mentioned previously, new properties are required to take care of extension and amalgamation in the backwards direction to ensure the map is surjective. This is achieved by writing generalised conditions utilising the concept of antihomomorphisms. Throughout this section, we let $X, Y \in \{H, M, I\}$, $\overline{X}, \overline{Y} \in \{\overline{H}, \overline{M}, \overline{I}\}$, and $Z \in \{E, B, A\}$. To avoid any potential confusion, whenever we refer to a map of type Z being a surjective map of type Y, the symbol Z is always related to Y in the manner illustrated in \mathcal{S} (see [Equation 1](#)).

Motivated by the desire to take care of the ‘back’ part of a back-and-forth argument that would extend a finite partial map of \mathcal{M} to a surjective endomorphism of \mathcal{M} , we can state the *\overline{XY} -extension property* (\overline{XYEP}) along similar lines to the XYEP in [Section 2](#).

(\overline{XYEP}) Suppose that \mathcal{M} is a structure with age \mathcal{C} . For all $A \subseteq B \in \mathcal{C}$ and a multifunction $\bar{f} : A \rightarrow \mathcal{M}$ of type \overline{X} such that $A\bar{f}$ is finite, there exists a multifunction $\bar{g} : B \rightarrow \mathcal{M}$ of type \overline{Y} extending \bar{f} .

Notice that this property differs slightly from previous extension properties as it requires an extra finiteness condition on the image of the multifunction \bar{f} . There could exist multifunctions $\bar{f} : A \rightarrow \mathcal{M}$ of type \overline{H} where the image $A\bar{f}$ is an infinite set (see [Example 3.2](#)); this would cause problems in the proofs of [Proposition 4.1](#) and [Proposition 4.4](#).

We turn our attention to finding necessary and sufficient conditions for XZ-homogeneity, to be used throughout the proof of [Theorem 1.2](#). As stated above, we need to ensure that any XZ-homogeneous structure we construct is also XY-homogeneous for the appropriate Y. It follows that such a structure must satisfy all the conditions outlined in [Proposition 2.2](#); in particular, XY must be in \mathfrak{J} for part (2). With these restrictions in mind, and desiring to obtain the most general result possible, we show that both the XYEP and \overline{XYEP} are necessary conditions for XZ-homogeneity in general, and that it these are also sufficient when the extended map of type Y is also a map of type X. Notice that if a map of type Y is also of type X, then a multifunction of type \overline{Y} is also a multifunction of type \overline{X} .

Proposition 4.1. *Let \mathcal{M} be a σ -structure with age \mathcal{C} and let $XZ \in \mathfrak{B}$.*

- (1) *If \mathcal{M} is XZ-homogeneous, then \mathcal{M} has both the XYEP and the \overline{XYEP} .*
- (2) *Suppose also that $XZ \in \mathfrak{J}$. If \mathcal{M} has the XYEP and the \overline{XYEP} , then \mathcal{M} is XZ-homogeneous.*

Proof. (1) As \mathcal{M} is XZ-homogeneous, it is also XY-homogeneous (where $(Z, Y) \in \mathcal{S}$, as defined in [Equation 1](#)) and so it has the XYEP by [Proposition 2.2](#) (1). Now, suppose that $A, B \in \mathcal{C}$ are such that $A \subseteq B$ and $\bar{f} : A \rightarrow \mathcal{M}$ is a multifunction of type \overline{X} with $A\bar{f}$ finite. As \mathcal{C} is the age of \mathcal{M} , it follows that \mathcal{M} contains copies $A' \subseteq B'$ of A and B and there are isomorphisms $\theta : B \rightarrow B'$ and $\theta^{-1} : B' \rightarrow B$. Restrict the codomain of \bar{f} to its image to find a map $\bar{f}' : A \rightarrow A\bar{f}$; as this is a surjective multifunction of type \overline{X} , we have that $\theta^{-1}\bar{f}' = \bar{h} : A' \rightarrow A\bar{f}$

is also a surjective multifunction of type \bar{X} . By [Lemma 3.5](#), the converse $h : A\bar{f} \rightarrow A'$ of $\theta|_A^{-1}\bar{f}'$ is a surjective map of type X with finite domain $A\bar{f}$; as \mathcal{M} is XZ-homogeneous, extend h to a map $\beta : \mathcal{M} \rightarrow \mathcal{M}$ of type Z. So $\beta\theta^{-1} : \mathcal{M} \rightarrow B$ is a surjective map of type Y; by [Corollary 3.6](#), define $\bar{g} = \theta\beta : B \rightarrow \mathcal{M}$ to be the corresponding surjective multifunction of type \bar{Y} . We need to show it extends \bar{f} . As β extends h , then $\bar{\beta}$ extends \bar{h} . So for all $a \in A$:

$$a\bar{f} = a\theta\theta^{-1}\bar{f} = a\theta\bar{h} = a\theta\bar{\beta}$$

and hence \mathcal{M} has the \overline{XYEP} .

(2) Now suppose that $XZ \in \mathfrak{B} \cap \mathfrak{J}$. Since this happens a map of type Z is also a map of type X; therefore, a multifunction being of type \bar{Y} (where $(Z, Y) \in \mathcal{S}$) implies that it is also a multifunction of type \bar{X} . Suppose also that \mathcal{M} has the XYEP and the \overline{XYEP} , and that $f : A \rightarrow B$ is a map of type X between substructures of \mathcal{M} . We use a back-and-forth argument to show that \mathcal{M} is XZ-homogeneous.

Set $A = A_0$, $B = B_0$ and $f_0 = f$, and assume that we have extended f to a surjective map $f_k : A_k \rightarrow B_k$ of type Y (and hence of type X, by assumption), where each $A_i \subseteq A_{i+1}$ and $B_i \subseteq B_{i+1}$ for all $0 \leq i \leq k-1$. Note also that A_k and B_k are finite for all $k \in \mathbb{N}$. Furthermore, as \mathcal{M} is countable we can enumerate the elements of $M = \{m_0, m_1, \dots\}$.

If k is even, select a point $m_i \in \mathcal{M} \setminus A_k$ where i is the smallest number such that $m_i \notin A_k$, so $A_k \cup \{m_i\} \subseteq \mathcal{M}$. Using the XYEP, extend f_k to a map $f'_{k+1} : A_k \cup \{m_i\} \rightarrow B'_k$ of type Y; by restricting the codomain of f'_{k+1} to its image, it follows that $f_{k+1} : A_k \cup \{m_i\} \rightarrow B_k \cup \{m_i f'_{k+1}\}$ is a surjective map of type Y extending f_k .

If k is odd, choose a point $m_i \in \mathcal{M} \setminus B_k$ where i is the smallest number such that $m_i \notin B_k$; so $B_k \cup \{m_i\} \subseteq \mathcal{M}$. Note that as f_k is a surjective map of type X, we have that $\bar{f}_k : B_k \rightarrow A_k$ is a surjective multifunction of type \bar{X} . As A_k is finite, we can use the \overline{XYEP} to extend \bar{f} to a multifunction $\bar{f}'_{k+1} : B_k \cup \{m_i\} \rightarrow \mathcal{M}$ of type \bar{Y} . Restricting the codomain of \bar{f}'_{k+1} to its image gives a surjective multifunction $\bar{f}_{k+1} : B_k \cup \{m_i\} \rightarrow A_k \cup m_i \bar{f}'_{k+1}$ of type \bar{Y} , where $m_i \bar{f}'_{k+1} = \{y \in \mathcal{M} : (y, m_i) \in \bar{f}'_{k+1}\}$ is a non-empty set. As \bar{f}_{k+1} is a surjective multifunction of type \bar{Y} , we have that $f_{k+1} : A_k \cup m_i \bar{f}_{k+1} \rightarrow B_k \cup \{m_i\}$ is a surjective map of type Y extending f_k .

Since $XZ \in \mathfrak{B} \cap \mathfrak{J}$, a map of type Y is also a map of type X; so we can use the XYEP and \overline{XYEP} to repeat this process infinitely many times. By ensuring that each point of \mathcal{M} appears at both an odd and even step, we extend f to a surjective map β of type Y; which is a map of type Z and so \mathcal{M} is XZ-homogeneous. \square

Remark. Together, [Proposition 2.2](#) and [Proposition 4.1](#) re-prove [[25](#), Lemma 1.1], which states that a countable structure \mathcal{M} is II (MI, HI)-homogeneous if and only if it is IA (MA, HA)-homogeneous. For if a structure \mathcal{M} is HI-homogeneous, then it has the HIEP by [Proposition 2.2](#); this implies that every homomorphism between finite substructures of \mathcal{M} is an isomorphism. Since this happens, it follows that every antihomomorphism between finite substructures of \mathcal{M} is an isomorphism. Finally, as \mathcal{M} has the HIEP it must have the \overline{HIEP} as well and so \mathcal{M} is HA-homogeneous by [Proposition 4.1](#). A similar argument works for the equality concerning MI-homogeneous structures. In the II case, the IIEP is the standard extension property (EP) from Fraïssé's theorem, and so any structure \mathcal{M} with the IIEP is homogeneous by the same result.

We now state our new amalgamation property to accommodate the back portion of a back-and-forth argument; this is the \overline{XY} -amalgamation property (\overline{XYAP}):

(\overline{XYAP}) Let \mathcal{C} be a class of finite σ -structures. We say that \mathcal{C} has the \overline{XYAP} if for all $A, B_1, B_2 \in \mathcal{C}$, multifunction $\bar{f}_1 : A \rightarrow B_1$ of type \bar{X} and embedding $f_2 : A \rightarrow B_2$, there exists a $D \in \mathcal{C}$, embedding $g_1 : B_1 \rightarrow D$ and multifunction $\bar{g}_2 : B_2 \rightarrow D$ of type \bar{Y} such that $\bar{f}_1 g_1 = f_2 \bar{g}_2$ (see [Figure 8](#)).

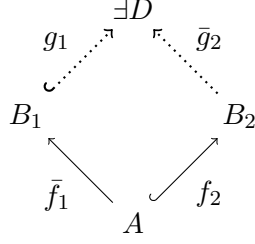


Figure 8: The \overline{XY} -amalgamation property (\overline{XYAP}).

Note that this property represents nine different amalgamation conditions. This corresponds to one for each class $XZ \in \mathfrak{B}$, where $(Z, Y) \in \mathcal{S}$ (see Equation 1 on page 14) and X and \overline{X} are related as in Figure 7. For examples, the \overline{IIAP} is the standard amalgamation property, and the \overline{MMAP} is the BAP of [12].

We can now prove Theorem 1.2 (1). Before we do, we state a straightforward yet important fact about surjective endomorphisms of an infinite first-order structure \mathcal{M} .

Lemma 4.2. *Let \mathcal{M} be a σ -structure, with \mathcal{A} a finite substructure of \mathcal{M} . Then for any $\alpha \in \text{Epi}(\mathcal{M})$, there exists a finite structure $\mathcal{B} \subseteq \mathcal{M}$ such that $\mathcal{B}\alpha = \mathcal{A}$. \square*

Proposition 4.3 (Theorem 1.2 (1)). *Suppose that $XZ \in \mathfrak{B}$. If a structure \mathcal{M} is XZ -homogeneous, then $\text{Age}(\mathcal{M})$ has the $XYAP$ and the \overline{XYAP} .*

Proof. As \mathcal{M} is XZ -homogeneous then it is XY -homogeneous and so has the $XYAP$ by Proposition 2.3. To show that $\text{Age}(\mathcal{M})$ has the \overline{XYAP} , suppose that $A, B_1, B_2 \in \text{Age}(\mathcal{M})$, $\bar{f}_1 : A \rightarrow B_1$ is a multifunction of type \overline{X} and $f_2 : A \rightarrow B_2$ is an embedding. We can assume without loss of generality that A, B_1, B_2 are actually substructures of \mathcal{M} and that f_2 is the inclusion mapping.

By restricting the codomain of \bar{f}_1 to its image, $\bar{f}_1 : A \rightarrow A\bar{f}_1$ is a surjective multifunction of type \overline{X} ; hence the converse $f_1 : A\bar{f}_1 \rightarrow A$ of \bar{f}_1 is a surjective map of type X . Use XZ -homogeneity to extend f_1 to a map $\beta : \mathcal{M} \rightarrow \mathcal{M}$ of type Z ; and so a surjective map of type Y . We see that $B_1\beta$ is a structure containing A , and that $\beta|_{B_1} : B_1 \rightarrow B_1\beta$ extends f_1 . Define $D = B_1\beta \cup B_2$. As β is surjective, there exists a finite substructure C such that $C\beta = D$ by Lemma 4.2. Now, define the map $g_1 : B_1 \rightarrow C$ to be the inclusion map. Since β is a surjective map of type Y , $\bar{\beta} : \mathcal{M} \rightarrow \mathcal{M}$ is a surjective multifunction of type \overline{Y} by Lemma 3.5. Therefore $\bar{\beta}|_{B_2} : B_2 \rightarrow B_2\bar{\beta}$ is a surjective multifunction of type \overline{Y} ; furthermore, $B_2\bar{\beta} \subseteq C$ as $B_2 \subseteq D$. Define $\bar{g}_2 : B_2 \rightarrow C$ to be the multifunction $\bar{\beta}|_{B_2}$ of type \overline{Y} . It is easy to check that $\bar{f}_1 g_1 = f_2 \bar{g}_2$ and so $\text{Age}(\mathcal{M})$ has the \overline{XYAP} . \square

We now show the existence portion of Theorem 1.2. Note that the previously described inductive construction of an infinite structure in Proposition 2.4 used even and odd steps to achieve different stages of the construction at different times. Because we have two amalgamation properties, as well as the JEP to ensure a countable structure exists, we proceed using an inductive argument at steps congruent to 0, 1, 2 mod 3 to accommodate different stages of the construction.

Proposition 4.4 (Theorem 1.2 (2)). *Suppose that $XZ \in \mathfrak{B} \cap \mathfrak{J}$. Let \mathcal{C} be a class of finite σ -structures that is closed under substructures and isomorphism, has countably many isomorphism types and has the JEP, $XYAP$ and the \overline{XYAP} . Then there exists a XZ -homogeneous σ -structure \mathcal{M} with $\text{age } \mathcal{C}$.*

Proof. We build \mathcal{M} over countably many stages, starting with M_0 being some member of \mathcal{C} . As the number of isomorphism types in \mathcal{C} is countable, we can enumerate those types as T_0, T_1, T_2, \dots . Furthermore, we can choose a countable set S of pairs (A, B) with $A \subseteq B \in \mathcal{C}$

so that every such pair of structures in \mathcal{C} is represented by some pair in S . Let $[\mathbf{m}] = \{n \in \mathbb{N} : n \equiv m \pmod{3}\}$ for $m = 1, 2$. Choose two bijections $\beta_m : [\mathbf{m}] \times \mathbb{N} \rightarrow [\mathbf{m}]$ such that $\beta_m(i, j) \geq i$ for all $i, j \in \mathbb{N}$ and where $m = 1, 2$. Assume that M_k has been constructed at some stage $k \in \mathbb{N}$.

Suppose first that $k \equiv 0 \pmod{3}$. Use the JEP to find a structure D that contains both M_k and a copy of some $C \cong T_{k/3} \in \mathcal{C}$; define M_{k+1} to be this structure D . Now suppose that $k \equiv 1 \pmod{3}$. Let $L_k^1 = (A_{kj}, B_{kj}, f_{kj})_{j \in \mathbb{N}}$ be the list of all triples (A, B, f) such that $(A, B) \in S$ and $f : A \rightarrow M_k$ is a map of type X. This list is countable as S is and there are finitely many maps of type X from A into M_k . Let $k = \beta_1(i, j)$. Then as $\beta_1(i, j) \geq i$, the map $f_{ij} : A_{ij} \rightarrow M_i \subseteq M_k$ exists. Therefore, we can use the XYAP to define M_{k+1} such that $M_k \subseteq M_{k+1}$ and the map $f_{ij} : A_{ij} \rightarrow M_k$ of type X extends to some map $g_{ij} : B_{ij} \rightarrow M_{k+1}$ of type Y (see [Figure 9](#) (1)). This ensures that every possible XY-amalgamation occurs. If $k \equiv 2 \pmod{3}$, let $L_k^2 = (P_{kj}, Q_{kj}, \bar{f}_{kj})_{j \in \mathbb{N}}$ be the list of all triples (P, Q, \bar{f}) such that $(P, Q) \in S$ and $\bar{f} : P \rightarrow M_k$ is a multifunction of type \bar{X} with finite image $P\bar{f}$; again, this list is countable. Let $k = \beta_2(i, j)$; as $\beta_2(i, j) \geq i$ it follows that the multifunction $\bar{f}_{ij} : P_{ij} \rightarrow M_i \subseteq M_k$ of type \bar{X} is well-defined. So we can use the $\bar{X}Y$ AP to define M_{k+1} such that $M_k \subseteq M_{k+1}$ and the multifunction $\bar{f}_{ij} : P_{ij} \rightarrow M_k$ extends to some multifunction $\bar{g}_{ij} : Q_{ij} \rightarrow M_{k+1}$ (see [Figure 9](#) (2)). This construction ensures that every possible $\bar{X}Y$ -amalgamation occurs.

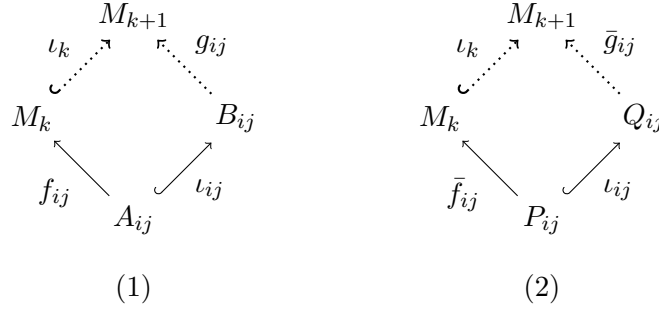


Figure 9: Amalgamations performed in the proof of [Proposition 4.4](#). The ι 's are inclusion mappings.

Define $\mathcal{M} = \bigcup_{k \in \mathbb{N}} M_k$. Our construction ensures that every isomorphism type of \mathcal{C} appears at a $0 \pmod{3}$ stage, so every structure in \mathcal{C} embeds into \mathcal{M} . Conversely, we have that $M_k \in \mathcal{C}$ for all $k \in \mathbb{N}$. As \mathcal{C} is closed under substructures, every structure that embeds into \mathcal{M} is in \mathcal{C} , showing that $\text{Age}(\mathcal{M}) = \mathcal{C}$.

It remains to show that \mathcal{M} is XZ-homogeneous. By [Proposition 4.1](#) and the fact that $XZ \in \mathfrak{B} \cap \mathfrak{J}$, it is enough to show that \mathcal{M} has both the XYEP and the $\bar{X}Y$ EP. Assume that $A \subseteq B \in \mathcal{C}$ and that $f : A \rightarrow \mathcal{M}$ is a map of type X. Using a similar argument to that of [Proposition 2.4](#) (with $j \in [1]$ and $(A_{j\ell}, B_{j\ell}, f_{j\ell}) \in L_j^1$), we can show that as \mathcal{C} has the XYAP, then \mathcal{M} has the XYEP.

Now suppose that $P \subseteq Q \in \mathcal{C}$ and $\bar{f} : P \rightarrow \mathcal{M}$ is a multifunction of type \bar{X} with finite image $P\bar{f}$. As $P\bar{f}$ is finite, it follows that there exists $u \in [2]$ such that $P\bar{f} \subseteq M_u$. Furthermore, there exists a triple $(P_{uv}, Q_{uv}, \bar{f}_{uv}) \in L_u^2$ such that there exists an isomorphism $\eta : Q \rightarrow Q_{uv}$ with $P\eta|_P = P_{uv}$ and $\bar{f} = \eta|_P \bar{f}_{uv}$. Define $w = \beta_2(u, v)$; as $w \geq u$, then $P\bar{f} = P_{uv}\bar{f}_{uv} \subseteq M_u \subseteq M_w$. Here, M_{w+1} is constructed by $\bar{X}Y$ -amalgamating M_w and Q_{uv} over P_{uv} ; this amalgamation extends the multifunction \bar{f}_{uv} of type \bar{X} to the multifunction $\bar{g}_{uv} : Q_{uv} \rightarrow M_{w+1}$ of type \bar{Y} . Consequently, the multifunction $\bar{g} = \eta \bar{g}_{uv} : Q \rightarrow \mathcal{M}$ of type \bar{Y} extends the multifunction \bar{f} of type \bar{X} (see [Figure 10](#) for a diagram). As \mathcal{M} has both the XYEP and the $\bar{X}Y$ EP, it follows that \mathcal{M} is XZ-homogeneous by [Proposition 4.1](#). \square

Finally, we show part (3) of [Theorem 1.2](#). Using the fact that XZ-homogeneous structures

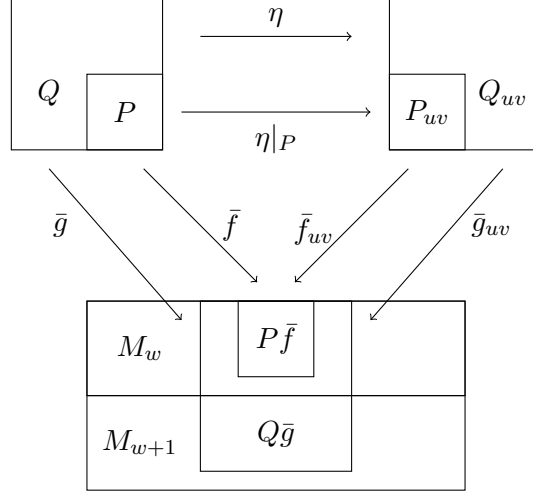


Figure 10: Diagram of maps in the proof of [Proposition 4.4](#). The multifunction $\bar{g} = \eta\bar{g}_{uv} : Q \rightarrow \mathcal{M}$ of type \bar{Y} extends the multifunction \bar{f} of type \bar{X} , proving that \mathcal{M} has the $\bar{X}\bar{Y}$ EP.

have two extension properties, we can ensure that a map between two of them is surjective by using a back-and-forth argument. This motivates a new definition, building on that of Y -equivalence.

Definition 4.5. Let \mathcal{M}, \mathcal{N} be σ -structures, and suppose that $Z \in \{E, B, A\}$ corresponds to the surjective map of type $Y \in \{H, M, I\}$ via the relation \mathcal{S} . Say that \mathcal{M} and \mathcal{N} are Z -equivalent if $\text{Age}(\mathcal{M}) = \text{Age}(\mathcal{N})$ and every embedding $f : A \rightarrow \mathcal{N}$ from a finite substructure A of \mathcal{M} into \mathcal{N} extends to a surjective map $g : \mathcal{M} \rightarrow \mathcal{N}$ of type Y , and vice versa.

For an example, \mathcal{M}, \mathcal{N} are B -equivalent means that they are bi-equivalent in the sense of [\[12\]](#). Note that if two structures \mathcal{M} and \mathcal{N} are Z -equivalent, then they are also Y -equivalent (see [Definition 2.5](#)) where $(Z, Y) \in \mathcal{S}$ given in [Equation 1](#).

Proposition 4.6 ([Theorem 1.2](#) (3)). *Suppose that $XZ \in \mathfrak{B} \cap \mathfrak{I}$.*

- (1) *Assume that \mathcal{M}, \mathcal{N} are Z -equivalent. Then \mathcal{M} is XZ -homogeneous if and only if \mathcal{N} is.*
- (2) *If \mathcal{M}, \mathcal{N} are XZ -homogeneous and $\text{Age}(\mathcal{M}) = \text{Age}(\mathcal{N})$, then \mathcal{M} and \mathcal{N} are Z -equivalent.*

Proof. (1) As \mathcal{M}, \mathcal{N} are Z -equivalent they are also Y -equivalent; as \mathcal{M} is also XY -homogeneous, so is \mathcal{N} by [Proposition 2.6](#). By [Proposition 2.2](#), it follows that \mathcal{N} has the XY EP. We show now that \mathcal{N} has the $\bar{X}\bar{Y}$ EP. Suppose that $A \subseteq B \in \text{Age}(\mathcal{N})$ and there exists a multifunction $\bar{f} : A \rightarrow A' \subseteq \mathcal{N}$ of type \bar{X} with A' finite. Note that A need not be isomorphic to A' . As $\text{Age}(\mathcal{M}) = \text{Age}(\mathcal{N})$ there exists a copy A'' of A' in \mathcal{M} ; fix an isomorphism $e : A' \rightarrow A''$ between the two. Therefore, e is a isomorphism from a finite structure of \mathcal{N} into \mathcal{M} ; as the two are Z -equivalent, we extend this to a surjective map $\alpha : \mathcal{N} \rightarrow \mathcal{M}$ of type Y . This in turn induces a surjective map $\bar{\alpha} : \mathcal{M} \rightarrow \mathcal{N}$ by [Lemma 3.5](#). Note that $\bar{\alpha}$ extends the isomorphism $e^{-1} : A'' \rightarrow A'$. Now, define $\bar{h} = \bar{f}e : A \rightarrow A''$; this is a multifunction of type \bar{X} from A into \mathcal{M} with $A\bar{h} = A''$ finite. Since \mathcal{M} is XZ -homogeneous, it has the $\bar{X}\bar{Y}$ EP by [Proposition 4.1](#) and so we extend \bar{h} to a multifunction $\bar{h}' : B \rightarrow \mathcal{M}$ of type \bar{Y} . Here, the multifunction $\bar{h}'\bar{\alpha} : B \rightarrow \mathcal{N}$ is also of type \bar{Y} ; we need to show it extends \bar{f} . As \bar{h}' extends $\bar{h} = \bar{f}e$, it follows that:

$$a\bar{f} = a\bar{f}ee^{-1} = a\bar{f}e\bar{\alpha} = a\bar{h}'\bar{\alpha}$$

for all $a \in A$. Therefore \mathcal{N} has the XZ EP.

(2) We utilise a back-and-forth argument constructing the surjective map of type Y over infinitely many stages. Let $f : A \rightarrow B$ be a bijective embedding from a finite structure $A \subseteq \mathcal{M}$

to a finite substructure $B \subseteq \mathcal{N}$. Set $A = A_0$, $B = B_0$ and $f = f_0$ and assume that $f_k : A_k \rightarrow B_k$ is a surjective map of type Y (and so of type X by assumption) extending f_k . Note that as both \mathcal{M} and \mathcal{N} are countable, there exist enumerations $\mathcal{M} = \{m_0, m_1, \dots\}$ and $\mathcal{N} = \{n_0, n_1, \dots\}$.

If k is even, select a $m_i \in \mathcal{M} \setminus A_k$, where i is the smallest natural number such that $m_i \notin A_k$. So $A_k \cup \{m_i\} \subseteq \mathcal{M}$, and is also in $\text{Age}(\mathcal{N})$ by assumption. As \mathcal{N} is XZ-homogeneous it has the XYEP by [Proposition 2.2](#) and we use this to extend f_k to a map $f'_{k+1} : A_k \cup \{m_i\} \rightarrow \mathcal{N}$ of type Y. Restricting the codomain of f'_{k+1} to its image yields a surjective map $f_{k+1} : A_k \cup \{m_i\} \rightarrow B_k \cup \{m_i f_{k+1}\}$ of type Y. If k is odd, select a $n_i \in \mathcal{N} \setminus B_k$ such that i is the smallest natural number such that $n_i \notin B_k$. Hence $B_k \cup \{n_i\} \subseteq \mathcal{N}$ and thus it is an element of $\text{Age}(\mathcal{M})$ by assumption. As f_k is a surjective map of type Y, its converse $\bar{f}_k : B_k \rightarrow A_k$ is a surjective multifunction of type \bar{Y} by [Lemma 3.5](#), and of type \bar{X} by assumption. As \mathcal{M} is XZ-homogeneous it has the $\bar{X}\bar{Y}$ EP and so we can extend \bar{f}_k to a multifunction $\bar{f}'_{k+1} : B_k \cup \{n_i\} \rightarrow \mathcal{M}$ of type \bar{Y} . By restricting the codomain of \bar{f}'_{k+1} to its image, we obtain a surjective multifunction $\bar{f}_{k+1} : B_k \cup \{n_i\} \rightarrow A_k \cup n_i \bar{f}'_{k+1}$ of type \bar{Y} , where $n_i \bar{f}'_{k+1} = \{(n_i, y) : y \in \mathcal{M}\}$ is a non-empty set. So by [Lemma 3.5](#), there exists a surjective map $f_{k+1} : A_k \cup n_i \bar{f}'_{k+1} \rightarrow B_k \cup \{n_i\}$ of type Y extending f_k . By our earlier assumption, as a map of type Y is also a map of type X, we can repeat this process infinitely many times. By ensuring all points in \mathcal{M} appear at even stages and all points in \mathcal{N} appear at odd stages, we construct a surjective map $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ of type Y as required. We can use a similar method to show that we can extend any embedding $g : A \rightarrow B$, where $A \in \mathcal{N}$ and $B \in \mathcal{M}$, to a surjective map of type Y; proving that \mathcal{M} and \mathcal{N} are Z-equivalent. \square

Of course, the open problem that arises from [Sections 2 and 4](#) is:

Question 4.7. *With reference to [Figure 2](#), can one prove similar results to [Theorems 1.1 and 1.2](#) for those homomorphism-homogeneity classes in \mathfrak{B} ?*

5 Maximal homomorphism-homogeneity classes

This section is devoted to determining the extent to which well known examples of homogeneous structures are also homomorphism-homogeneous. In some cases, verifying that a structure \mathcal{M} is homogeneous involves using a property of \mathcal{M} to determine that \mathcal{M} has the EP, and so is homogeneous. Good examples of such properties are the density of $(\mathbb{Q}, <)$, and *Alice's restaurant property* characteristic of R (see [Example 5.6](#)). In the homomorphism-homogeneity case, this idea was used by Cameron and Lockett [\[8\]](#) and Lockett and Truss [\[25\]](#) to classify homomorphism-homogeneous posets and determine their position relative to the natural containment order on \mathbb{H} (see [Figure 1](#)). In addition to this, Dolinka [\[14\]](#) used properties of known homogeneous structures to show that they satisfied the *one-point homomorphism extension property* (1PHEP), a necessary and sufficient condition for HH-homogeneity. Our approach in this section is similar to that of [Section 3](#) of [\[14\]](#); by defining necessary and sufficient conditions for XY and XZ-homogeneity and using properties of structures to show that these are satisfied or not satisfied. As in [Section 4](#), we let $X, Y \in \{H, M, I\}$, $\bar{X}, \bar{Y} \in \{\bar{H}, \bar{M}, \bar{I}\}$, and $Z \in \{E, B, A\}$ throughout this section. Furthermore, the pair $(Z, Y) \in \mathcal{S}$ is related as in [Equation 1](#) on page [14](#).

So to begin this section, we define the *one-point XY-extension property*, and the *one-point $\bar{X}\bar{Y}$ -extension property*:

(1PXYEP) We say that a σ -structure \mathcal{M} with age \mathcal{C} has the 1PXYEP if for all $A \subseteq B \in \mathcal{C}$ with $|B \setminus A| = 1$ and maps $f : A \rightarrow \mathcal{M}$ of type X, there exists a map $g : B \rightarrow \mathcal{M}$ of type Y extending f .

(1P $\bar{X}\bar{Y}$ EP) Suppose that \mathcal{M} is a σ -structure with age \mathcal{C} . Say that \mathcal{M} has the 1P $\bar{X}\bar{Y}$ EP if for all $A \subseteq B \in \mathcal{C}$ with $|B \setminus A| = 1$, and a multifunction $\bar{f} : A \rightarrow \mathcal{M}$ of

type \bar{X} with $A\bar{f}$ finite, there exists a multifunction $\bar{g} : B \rightarrow \mathcal{M}$ of type \bar{Y} extending \bar{f} .

For an example, the 1PHHEP is the same thing as the 1PHEP of [14]. These properties, together with the next proposition, provide some of the theoretical basis for the examples that follow.

Proposition 5.1. *Suppose that $XY \in \mathfrak{J}$. A countable σ -structure \mathcal{M} has the XYEP / \overline{XYEP} if and only if it has the 1PXYEP / $1P\overline{XYEP}$.*

Proof. The forward direction for both the XYEP and \overline{XYEP} cases is clear. We now aim to show that if \mathcal{M} has the 1PXYEP then \mathcal{M} has the XYEP. Assume that $A \subseteq B$ with $|B \setminus A| = n$, and $f : A \rightarrow \mathcal{M}$ is a map of type X. We prove the result by induction on the size of this complement; the base case (where $n = 1$) is true by the assumption that \mathcal{M} has the 1PXYEP.

So suppose that for some $k \in \mathbb{N}$, for any $A \subseteq B \in \mathcal{C}$ where $|B \setminus A| = k$ and any map $f : A \rightarrow \mathcal{M}$ of type X can be extended to a map $g : B \rightarrow \mathcal{M}$ of type Y. Take $P \subseteq Q \in \mathcal{C}$ where $|Q \setminus P| = k + 1$ and $f' : P \rightarrow \mathcal{M}$ to be some map of type X. Now, there exists $S \in \mathcal{C}$ containing P such that $|Q \setminus S| = 1$. By the inductive hypothesis, we can extend f' to a map $h : S \rightarrow \mathcal{M}$ of type Y. As $XY \in \mathfrak{J}$, it follows that h is also a map of type X. Now, using the 1PXYEP, extend h to a map $g' : Q \rightarrow \mathcal{M}$ of type Y. Since $P \subseteq S \subseteq Q$ and g' extends h which extends f' , we have that g' extends f' and so we are done. Using a similar argument, we can show that if \mathcal{M} has the $1P\overline{XYEP}$ then it has the \overline{XYEP} . \square

Remark. Let $XY \in \mathfrak{J}$. Together with Proposition 2.2, this result states that a countable structure \mathcal{M} has the 1PXYEP if and only if \mathcal{M} is XY-homogeneous. Similarly, by Proposition 4.1 a countable structure \mathcal{M} is XZ-homogeneous if and only if it has the 1PXYEP and the $1P\overline{XYEP}$, where (Z, Y) are as in \mathcal{S} (Equation 1 on page 14).

By considering properties of partial maps and endomorphisms of structures, our next result places restrictions on certain types of homomorphism-homogeneity. We look at structures known as *cores*; a structure \mathcal{M} is a core if every endomorphism of \mathcal{M} is an embedding [4]. Widely studied examples of cores include the countable dense linear order without endpoints $(\mathbb{Q}, <)$, the complete graph on countably many vertices K^{\aleph_0} , the K^n -free homogeneous graphs for $n \geq 3$ [32] and the Henson digraphs M_T [12]. This straightforward result includes a restatement of Lemma 1.1 of [25].

Lemma 5.2. *Let \mathcal{M} be a countable σ -structure.*

- (1) *\mathcal{M} is MI and MA-homogeneous (HI and HA-homogeneous) if and only if \mathcal{M} is IA-homogeneous and every finite partial monomorphism (homomorphism) of \mathcal{M} is an isomorphism.*
- (2) *If \mathcal{M} is HM or HB-homogeneous, then every finite partial homomorphism of \mathcal{M} is also a monomorphism.*
- (3) *Let \mathcal{M} be a core. If there exists a finite partial monomorphism of \mathcal{M} that is not an isomorphism, then \mathcal{M} is not MH-homogeneous.*

Proof. (1) is contained in Lemma 1.1 of [25]; notice that we cannot extend a map that is not a partial isomorphism of \mathcal{M} to an isomorphism of the entire structure \mathcal{M} . The converse direction is clear. To show (2), note that if h is a finite partial homomorphism of \mathcal{M} that is not injective, then we cannot possibly extend this to an injective map and so \mathcal{M} does not have the HMEP. For (3), let h be a finite partial monomorphism of a core \mathcal{M} that is not an isomorphism. As any endomorphism of \mathcal{M} is an embedding, we cannot extend h . \square

Remark. Note that (1) and (2) also follow from Theorem 1.1 and Theorem 1.2.

Following the approach of [25] in classifying homomorphism-homogeneous posets, the idea of this section is to look at properties of graphs and digraphs to determine “maximal” homomorphism-homogeneity classes with respect to the containment order on \mathfrak{H} . We formally define what we mean by “maximal”.

Definition 5.3. Let \mathcal{M} be a first-order structure. A homomorphism-homogeneity class $PQ \in \mathbb{H}$ is *maximal* for \mathcal{M} if \mathcal{M} is PQ-homogeneous and \mathcal{M} is not FG-homogeneous, where $FG \subset PQ$ in \mathbb{H} . If this happens, we say that PQ is a *maximal homomorphism-homogeneity class* (shortened to *mhh-class*) for \mathcal{M} .

Remark. While this definition describes a *minimal* element in the poset \mathbb{H} , it is so named because of the strengths of different notions of homomorphism-homogeneity. For instance, HA-homogeneity is a stronger condition than IA-homogeneity, but $HA \subseteq IA$ in \mathbb{H} . This reflects the inverse correspondence between the relative strength of notions of homomorphism-homogeneity in \mathfrak{H} and containment of classes in \mathbb{H} (see the discussion on page 5).

For example, if \mathcal{M} is MB-homogeneous but not MA or HB-homogeneous, then MB is a mhh-class for \mathcal{M} . A structure \mathcal{M} may have more than one mhh-class. The set of mhh-classes for \mathcal{M} completely determines the extent of homomorphism-homogeneity satisfied by \mathcal{M} ; we therefore denote this set by $\mathbb{H}(\mathcal{M})$. As an example $\mathbb{H}((\mathbb{Q}, <)) = \{HA\}$; this example arose from the classification of homomorphism-homogeneous posets in [25].

If \mathcal{M} is a countable σ -structure where there exists a finite partial monomorphism of \mathcal{M} that is not an isomorphism, and a finite partial homomorphism of \mathcal{M} that is not a monomorphism, then Lemma 5.2 implies that the “best possible” mhh-classes for \mathcal{M} are IA, MB and HE. As an aside, these classes have important roles to play in the theory of generic endomorphisms [24].

Before we investigate some examples in the context of this article, we note the following direct consequence of Lemma 5.2 (3) with respect to Definition 5.3.

Corollary 5.4. *Let \mathcal{M} be a countable homogeneous core. If there exists a finite partial monomorphism of \mathcal{M} that is not an isomorphism, then $\mathbb{H}(\mathcal{M}) = \{IA\}$.* \square

Remark. It has been shown that every K_n -free graph (for $n \geq 3$) [32], every Henson digraph M_T (for any set of tournaments on more than 3 vertices) and the myopic local order $S(3)$ [12] are cores. By Corollary 5.4, it follows that the mhh-class for each of these structures is IA.

In the rest of this section, we look at a selection of countable homogeneous graphs and digraphs encountered throughout the literature in order to determine sets of mhh-classes for these structures. By restricting ourselves to classes $XY \in \mathfrak{J}$, we can recall Proposition 5.1 and the remark that follows it; to show that \mathcal{M} is XY-homogeneous it suffices to show that \mathcal{M} has the 1PX \overline{Y} EP, and to show that \mathcal{M} is XZ-homogeneous it suffices to show that it has the 1PX \overline{Y} EP and the 1P \overline{X} YEP.

5.1 Graphs

In this article, a *graph* Γ is a set of *vertices* $V\Gamma$ together with a set of *edges* $E\Gamma$, where this edge set interprets an irreflexive and symmetric binary relation E . For $n \in \mathbb{N} \cup \{\aleph_0\}$, recall that a *complete graph* K^n on n vertices is a graph on vertex set VK^n (with $|VK^n| = n$) with edges given by $u \sim v \in K^n$ if and only if $u \neq v \in VK^n$. The complement of the graph K^n is called the *null graph* on n vertices or an *independent set* on n vertices, and is denoted by \overline{K}^n . Recall that the *complement* of a graph $\Gamma = (V, E)$ is the graph $\overline{\Gamma} = (V, [V]^2 \setminus E)$; that is, the graph $\overline{\Gamma}$ on vertex set V where all the edges in $\overline{\Gamma}$ are non-edges of Γ and vice versa. A subset X of V is an *independent set* if $x \not\sim x'$ for all $x, x' \in X$. For more on the basics of graph theory, see [13].

Example 5.5. It is well-known (see [23]) that the complete graph on countably many vertices K^{\aleph_0} is homogeneous. Suppose that $h : A \rightarrow B$ is a homomorphism between two finite substructures of K^{\aleph_0} . Then as h preserves edges, it cannot send two distinct vertices $x_1, x_2 \in VA$ to a single point $v \in VB$; hence h is injective. As there are no non-edges to preserve, it must preserve non-edges and so h is an embedding. It follows from Lemma 5.2 (1) that K^{\aleph_0} is HA-homogeneous and so $\mathbb{H}(K^{\aleph_0}) = \{\text{HA}\}$.

Its complement \bar{K}^{\aleph_0} , the infinite null graph, is also homogeneous and as every finite partial monomorphism of \bar{K}^{\aleph_0} preserves non-edges, it is MA-homogeneous by Lemma 5.2 (1). We note that there exist non-injective finite partial homomorphisms of \bar{K}^{\aleph_0} and hence it is not HM or HB-homogeneous by Lemma 5.2 (3). So if $h : A \rightarrow B$ is any finite partial homomorphism, we can find a bijective map $g : \bar{K}^{\aleph_0} \setminus A \rightarrow \bar{K}^{\aleph_0} \setminus B$ and note that the map $\alpha : \bar{K}^{\aleph_0} \rightarrow \bar{K}^{\aleph_0}$ that acts like h on A and g everywhere else is an epimorphism of \bar{K}^{\aleph_0} ; so \bar{K}^{\aleph_0} is HE. Hence $\mathbb{H}(\bar{K}^{\aleph_0}) = \{\text{MA}, \text{HE}\}$.

Example 5.6. Let R be the random graph (see [7], for instance). Note that there exist finite partial monomorphisms of R that are not isomorphisms and finite partial homomorphisms of R that are not monomorphisms; hence R is not MI or HM-homogeneous by Lemma 5.2. It was shown in [12] that R is MB-homogeneous and in [9] that R is HH-homogeneous; here, we show that R is HE-homogeneous. To do this, we rely on the *Alice's restaurant property* characteristic of R (see [7]), which says:

(ARP) For any finite, disjoint subsets $U, V \subseteq V\Gamma$, there exists $x \in V\Gamma$ such that $x \sim u$ for all $u \in U$ and $x \not\sim v$ for all $v \in V$.

Let $A \subseteq B \in \text{Age}(R)$ with $B \setminus A = \{b\}$ and suppose that $\bar{f} : A \rightarrow R$ is an antihomomorphism such that $A\bar{f}$ is finite. Using ARP, we can find a vertex $v \in VR$ such that v is independent of everything in $A\bar{f}$. Let $\bar{g} : B \rightarrow R$ be the multifunction given by $b\bar{g} = v$ and $\bar{g}|_A = \bar{f}$; this is an antihomomorphism as all non-edges from A to b are preserved. Therefore, R has the 1P $\bar{\text{H}}\text{HEP}$ and so R is HE-homogeneous by Proposition 5.1 and Proposition 4.1. We conclude that $\mathbb{H}(R) = \{\text{IA}, \text{MB}, \text{HE}\}$.

Remark. It was shown in [24, Theorem 5.3] that R has a generic endomorphism. As R is HE-homogeneous, it follows from Theorem 2.1 of the same source that this generic endomorphism must be in $\text{Epi}(R)$.

Example 5.7. Let H be the complement of the homogeneous K_n -free graph for $n \geq 3$. It was previously shown that H is MM but not MB-homogeneous [12]. A result of [14] shows that H has the 1P $\bar{\text{H}}\text{HEP}$, and so is both MM and HH-homogeneous by these two results.

We now show that H does not have the 1P $\bar{\text{H}}\text{HEP}$ and hence cannot be HE-homogeneous. Let $A = \{a\}$ be a single vertex, and let B be the independent pair of vertices $\{a, b\}$. Note that $A \subseteq B \in \text{Age}(H)$. Let $\bar{f} : A \rightarrow H$ be an antihomomorphism sending $A = \{a\}$ to an independent set $A\bar{f}$ of $n - 1$ vertices in H ; such a substructure exists by definition of H . Then as antihomomorphisms preserve non-edges and cannot send two points in a domain to a single point in the codomain, a potential image point for b in H must be a vertex x independent of $A\bar{f}$; this cannot happen as H would then induce an independent n -set. So H does not have the 1P $\bar{\text{H}}\text{HEP}$. Therefore, H is not HE-homogeneous and we see that $\mathbb{H}(H) = \{\text{IA}, \text{MM}, \text{HH}\}$.

The next result, detailing the rest of the disconnected, countably infinite homogeneous graphs, extends results of [9] and [35].

Proposition 5.8. Let $\Gamma = \bigsqcup_{i \in I} K_i^n$ be a disjoint union of $|I|$ many complete graphs, each of which have size $n \in \mathbb{N} \cup \{\aleph_0\}$.

(1) If $|I| = m$ for some $m \in \mathbb{N}$ and $n = \aleph_0$, then $\mathbb{H}(\Gamma) = \{\text{IA}, \text{MM}, \text{HH}\}$.

(2) If $|I| = \aleph_0$ and $n \in \mathbb{N}$, then $\mathbb{H}(\Gamma) = \{IA, HE\}$.

(3) If $|I| = n = \aleph_0$, then $\mathbb{H}(\Gamma) = \{IA, MB, HE\}$.

Proof. (1) It is shown in [12] that the finite disjoint union of infinite complete graphs is MM but not MB-homogeneous. Furthermore, [9, Proposition 1.1] asserts that Γ is HH-homogeneous. It remains to prove that Γ is not HE-homogeneous; here, it is enough to show that Γ does not have the 1P $\overline{\text{HHEP}}$ by Proposition 5.1 and Proposition 4.1. As Γ in this case does not embed an independent $n + 1$ -set, the proof of this is similar to Example 5.7.

(2) In this case, Γ is HH-homogeneous by [9, Proposition 1.1], but not MM-homogeneous by Proposition 2.5 of the same paper. We show that Γ has the 1P $\overline{\text{HHEP}}$ in this case; proving that Γ is HE-homogeneous. Suppose that $A \subseteq B \in \text{Age}(\Gamma)$ with $B \setminus A = \{b\}$ and that $\bar{f} : A \rightarrow \Gamma$ is an antihomomorphism such that $A\bar{f}$ is finite. As $A\bar{f}$ is finite, it follows that $A\bar{f} \subseteq \bigsqcup_{j \in J} K_j^n$ for some finite set $J \subseteq I$. Select a $v \in K_k^n$, where $k \in I \setminus J$; so v is independent of every element of $A\bar{f}$. Define $\bar{g} : B \rightarrow \Gamma$ to act like \bar{f} on A with $b\bar{g} = v$. Then \bar{g} is an antihomomorphism and so Γ has the 1P $\overline{\text{HHEP}}$.

(3) The proof that Γ is MB-homogeneous can be found in [12, Proposition 3.4]; the proof that it is HE-homogeneous is similar to part (2). □

The only countable homogeneous graphs that have not yet been considered are the complements of the disconnected graphs described in Proposition 5.8. The following result deals with these cases. Recall from [35, Theorem 3, Theorem 5] that a countable graph is MH-homogeneous if and only if it is HH-homogeneous; if this graph is also connected, then it is HH-homogeneous if and only if it is MM-homogeneous.

Proposition 5.9. *Let $\bar{\Gamma} = \overline{\bigsqcup_{i \in I} K_i^n}$ be the complement of a disjoint union of $|I|$ many complete graphs, each of which have size $n \in \mathbb{N} \cup \{\aleph_0\}$.*

(1) If $|I| = m$ for some $m \in \mathbb{N}$ and $n = \aleph_0$, then $\mathbb{H}(\bar{\Gamma}) = \{IA\}$.

(2) If $|I| = \aleph_0$ and $n \in \mathbb{N}$, then $\mathbb{H}(\bar{\Gamma}) = \{IA, MM, HH\}$.

(3) If $|I| = n = \aleph_0$, then $\mathbb{H}(\bar{\Gamma}) = \{IA, MB, HE\}$.

Proof. (1) Notice that $\bar{\Gamma}$ does not contain an infinite complete graph; therefore it is not MM-homogeneous by [9, Proposition 2.5]. Since \bar{G} is connected, it is not MH-homogeneous by the results of [35] mentioned above.

(2) In this case, any finite set $U \subseteq \bar{\Gamma}$ of vertices has a vertex x such that x is adjacent to every element of U . (This is property (Δ) of [12].) Therefore, \bar{G} is MM and HH-homogeneous by [9, Proposition 2.1]. If $\bar{\Gamma}$ were MB-homogeneous, then Γ would be MB-homogeneous by [12, Proposition 3.1]; contradicting Proposition 5.8. So Γ is not MB-homogeneous. It remains to show that $\bar{\Gamma}$ is not HE-homogeneous; the proof of this is similar to Proposition 5.8 (1).

(3) This is MB-homogeneous by the remark following [12, Proposition 3.4]. It can be shown that this is HE-homogeneous using a similar argument to part (2) of Proposition 5.8. □

Remark. Notice that part (2) of both Propositions 5.8 and 5.9 show that the complement of a HE-homogeneous graph is not necessarily HE-homogeneous.

These results mean that we have determined the mhh-classes for all countable homogeneous graphs in Lachlan and Woodrow's classification [23]; the results are summarised in Table 2.

mhh-classes	countable IA graph	reference
{HA}	K^{\aleph_0}	(5.5)
{MA, HE}	\bar{K}^{\aleph_0}	(5.5)
{IA, MB, HE}	$R, \bigsqcup_{i \in \mathbb{N}} K_i^{\aleph_0}, \overline{\bigsqcup_{i \in \mathbb{N}} K_i^{\aleph_0}}$	(5.6, 5.8 (3), 5.9 (3))
{IA, HE}	$\bigsqcup_{i \in \mathbb{N}} K_i^n$	(5.8 (2))
{IA, MM, HH}	\bar{K}^n -free ($n \geq 3$), $\bigsqcup_{i=1}^n K_i^{\aleph_0}, \overline{\bigsqcup_{i \in \mathbb{N}} K_i^n}$	(5.7, 5.8 (1), 5.9 (2))
{IA}	K^n -free ($n \geq 3$), $\overline{\bigsqcup_{i=1}^n K_i^{\aleph_0}}$	(5.4, 5.9 (1))

Table 2: Maximal homomorphism-homogeneity classes for countable homogeneous graphs.

5.2 Digraphs

Following the results of [Subsection 5.1](#) and similar work on posets [\[25\]](#), the next natural direction would be to determine mhh-classes for the countable homogeneous digraphs. Unless stated otherwise, in this article a *digraph* $\Delta = (V\Delta, A\Delta)$ is a set of *vertices* $V\Delta$ together with a set $A\Delta \subseteq V\Delta^2$ of ordered pairs, called *arcs* of the digraph. For two vertices x, y of Δ , we write $x \rightarrow y$ if $(x, y) \in A\Delta$, and $x \parallel y$ if neither (x, y) nor (y, x) are in $A\Delta$. All the digraphs in this article are *loopless*; so for all $x \in \Delta$, it follows that $(x, x) \notin A\Delta$. Additionally, we stipulate that they do not contain 2-cycles – there is a *2-cycle* between x and y if and only if $x \rightarrow y$ and $y \rightarrow x$ – with the notable exception of [Example 5.14](#).

We provide a few introductory observations here, and leave the further development of the subject as an open question. Our first example deals with the countable homogeneous tournaments, classified in [\[22\]](#).

Example 5.10. Recall that a *tournament* is defined to be an oriented, loopless complete graph. By a similar argument to that for the complete graph in [Example 5.5](#), every finite partial homomorphism of a tournament is an embedding. It follows from [Lemma 5.2 \(1\)](#) that every countable homogeneous tournament is HA-homogeneous. Therefore, the three countable homogeneous tournaments as classified by Lachlan [\[22\]](#), namely $(\mathbb{Q}, <)$, the random tournament \mathbb{T} , and the local order $S(2)$, are all HA-homogeneous. So HA is the unique mhh-class for these three examples.

Example 5.11. Let D be the *generic digraph without 2-cycles*; for a detailed definition, see [\[1\]](#). This is the unique countable homogeneous structure whose age contains all finite digraphs without 2-cycles. This structure is also MB-homogeneous [\[12, Example 4.4\]](#).

However, D is not HH-homogeneous. First of all, as there exist finite partial homomorphisms of D that are not monomorphisms (such as an independent 2-set being mapped to a single point), D is not HM-homogeneous by [Lemma 5.2 \(2\)](#). To show that D is not HH-homogeneous, it is enough to prove that every endomorphism of D is a monomorphism. Consider $\gamma \in \text{End}(D)$, and suppose there exists $v \parallel w \in VD$ such that $v\gamma = w\gamma$. As D is universal and homogeneous, there exists an oriented graph $A = \{v, w, x\}$ such that $x \rightarrow v$ and $w \rightarrow x$ (see [Figure 11](#)).

The image of A under γ is a 2-cycle and this is a contradiction as D does not embed 2-cycles. It follows that every endomorphism of D is a monomorphism and so D is not HH-homogeneous. We conclude that the mhh-classes of D are IA and MB.

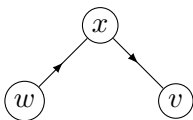


Figure 11: The digraph A described in [Example 5.11](#).

Corollary 5.12. *Let Δ be a countable connected homogeneous digraph that embeds the digraph A as in [Figure 11](#). Then $\text{Mon}(\Delta) = \text{End}(\Delta)$ and Δ is not HH-homogeneous. \square*

Remark. It was shown in [\[12\]](#) that the I_n -free digraph for $n \geq 3$ is MM but not MB-homogeneous. It follows from [Corollary 5.12](#) that the mhh-classes for these digraphs are IA and MM.

In the disconnected case, the situation is slightly different. Here is an example of a HE-homogeneous digraph.

Example 5.13. Let \mathbb{T} be the countable, homogeneous random tournament (see [\[22\]](#)). Using homogeneity of \mathbb{T} , we can show that \mathbb{T} satisfies the following property (\Leftrightarrow):

(Property (\Leftrightarrow)) For all finite, disjoint subsets U, V of $V\mathbb{T}$ there exists $x \in V\mathbb{T}$ such that $x \rightarrow u$ for all u in U and $v \rightarrow x$ for all $v \in V$.

Now, let $\mathcal{T} = \bigsqcup_{i \in \mathbb{N}} \mathbb{T}_i$ be the infinite disjoint union of isomorphic copies \mathbb{T}_i of the countable random tournament. Our aim is to show that \mathcal{T} is HE-homogeneous; so it is enough to show that \mathcal{T} has both the 1PHHEP and 1PHHEP by [Proposition 5.1](#) and [Proposition 4.1](#). Suppose then that $A \subseteq B \in \text{Age}(\mathcal{T})$ is such that $B \setminus A = \{b\}$. As $A \in \text{Age}(\mathcal{T})$, we can write $A = \bigsqcup_{j \in J} T_j$, where each T_j is a finite tournament and J is finite. Let $f : A \rightarrow \mathcal{T}$ be a homomorphism. There are two cases to consider; either b is independent of every tournament in A , or it is not. If b is independent of every tournament in A , then choose any vertex $v \in V\mathcal{T} \setminus Af$; the function $g : B \rightarrow \mathcal{T}$ acting like f on A and sending b to v is a homomorphism. If b is not independent of A , then there is only one tournament $T_j \subseteq A$ that is related to b . Partition T_j into two sets

$$T_j^{\rightarrow}(b) = \{c \in T_j : c \rightarrow b\} \quad \text{and} \quad T_j^{\leftarrow}(b) = \{d \in T_j : b \rightarrow d\}.$$

As T_j is a tournament, it follows that $f|_{T_j} : T_j \rightarrow T_j f$ is bijective and so $T_j^{\rightarrow}(b)f$ and $T_j^{\leftarrow}(b)f$ partition $T_j f$. As $f : A \rightarrow \mathcal{T}$ is a homomorphism, $T_j f$ is a finite subtournament of \mathbb{T}_i for some $i \in \mathbb{N}$; so both $T_j^{\rightarrow}(b)f$ and $T_j^{\leftarrow}(b)f$ are finite. We can use property (\Leftrightarrow) of \mathbb{T}_i to find a vertex w such that $cf \rightarrow w$ for all $cf \in T_j^{\rightarrow}(b)f$ and $w \rightarrow df$ for all $df \in T_j^{\leftarrow}(b)f$. Now, define a function $g : B \rightarrow \mathcal{T}$ that acts like f on A and sends b to w ; here, g preserves all relations and so is a homomorphism. Therefore, \mathcal{T} has the 1PHHEP.

The proof that \mathcal{T} has the 1PHHEP follows from a similar argument to [Proposition 5.8 \(2\)](#). Hence \mathcal{T} is HE-homogeneous by [Proposition 5.1](#) and [Proposition 4.1](#). Adapting this proof for the 1PMMEP and 1PMMEP, we can show that \mathcal{T} is MB-homogeneous; so $\mathbb{H}(\mathcal{T}) = \{\text{IA}, \text{MB}, \text{HE}\}$.

Changing our definition of digraph to include 2-cycles also increases the flexibility of the structure.

Example 5.14. Let D^* be the generic digraph with 2-cycles; similarly to D , it is the unique countable homogeneous digraph whose age contains all finite digraphs with 2-cycles. Recall (from [\[31, ch4\]](#) or [\[11, ch2\]](#)) that D^* has a characteristic extension property known as the *directed Alice's restaurant property (DARP)*, which says:

(DARP) For any finite and pairwise disjoint sets of vertices U, V, W, X of D^* , there exists a vertex z of D^* such that: there is an arc from z to every element of U , an arc to z from every element of V , a 2-cycle between z and every element of W , and z is independent of every vertex in X . (See [Figure 12](#) for a diagram of an example.)

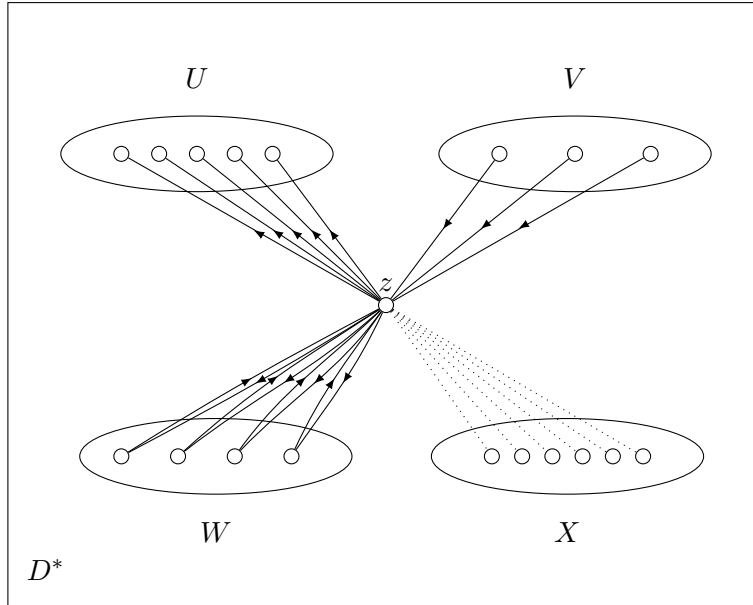


Figure 12: Example of the directed Alice’s restaurant property in the generic digraph with 2-cycles D^* .

It was mentioned in [12] that D^* is MB-homogeneous. Using the DARP, we show that D^* is HE-homogeneous. Let $A \subseteq B \in \text{Age}(D^*)$ with $B \setminus A = \{b\}$ and suppose that $f : A \rightarrow D^*$ is a homomorphism. As Af is finite, we can use DARP to find a vertex $v \in VD^*$ such that there is a 2-cycle between v and every element in Af . Let $g : B \rightarrow D^*$ be the map such that $bg = v$ and $g|_A = f$; this is a homomorphism as all arcs from A to b are preserved. Therefore D^* has the 1PHHEP. The proof to show that D^* has the 1PHHEP is similar; we use DARP to instead find a vertex $w \in VD^*$ that is independent of the finite set Af . The resulting multifunction \bar{g} is an antihomomorphism as it preserves all non-relations. Therefore, D^* is HE-homogeneous by Proposition 5.1 and Proposition 4.1. So $\mathbb{H}(D^*) = \{\text{IA}, \text{MB}, \text{HE}\}$.

Remark. Note the difference between the mhh-classes of D , the generic digraph without 2-cycles, and D^* , the generic digraph with 2-cycles.

The work in this section leads to a natural open question.

Question 5.15. Investigate countable homomorphism-homogeneous digraphs (both with and without 2-cycles) in more detail. In particular, determine the mhh-classes for those countable homogeneous digraphs in Cherlin’s classification [10].

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