**Entrapping of a vortex pair interacting with a fixed point vortex revisited. Part I:
Point vortices.**

Konstantin V. Koshel,^{1, a)} Jean N. Reinaud,^{2, b)} Giorgio Riccardi,^{3, 4, c)} and Eugene A. Ryzhov^{1, 5, 6, d)}

¹⁾ *V.I.Il'ichev Pacific Oceanological Institute of FEB RAS, 43, Baltiyskaya Street, Vladivostok, 690041, Russia*

²⁾ *Mathematical Institute, University of St Andrews, North Haugh, St Andrews KY169SS, United Kingdom*

³⁾ *Department of Mathematics and Physics, University of Campania "Luigi Vanvitelli", viale A. Lincoln 5, 81100 Caserta, Italy*

⁴⁾ *CNR-INSEAN, National Research Council of Italy Maritime Research Center, via di Vallerano 139, 00128 Rome, Italy*

⁵⁾ *Institute of Applied Mathematics, FEB RAS, 7, Radio Street, Vladivostok, 690022, Russia*

⁶⁾ *Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom*

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The problem of a pair of point vortices impinging on a fixed point vortex of arbitrary strengths [E. Ryzhov and K. Koshel, EPL 102, 44004 (2013)] is revisited and investigated comprehensively. Although the motion of pair of point vortices is established to be regular, the model presents a plethora of possible bounded and unbounded solutions with complicated vortex trajectories. The initial classification [E. Ryzhov and K. Koshel, EPL 102, 44004 (2013)] revealed that pair could be compelled to perform bounded or unbounded motion without giving a full classification of either of those dynamical regimes. The present work capitalizes upon the previous results and introduces a finer classification with a multitude of possible regimes of motion. Regimes of bounded motion for the vortex pair entrapped near the fixed vortex or of unbounded motion, when the vortex pair moves away from the fixed vortex, can be categorized by varying the two governing parameters: (i) the ratio of the distances between pair's vortices and the fixed vortex, and (ii) the ratio of the strengths of the vortices of the pair and the strength of the fixed vortex. In particular, a bounded motion regime where one of pair's vortices does not rotate about the fixed vortex is revealed. In this case, only one of pair's vortices rotates about the fixed vortex, while the other one oscillates at a certain distance. Extending the results obtained with the point-vortex model to an equivalent model of finite size vortices is the focus of a second paper.

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^{a)}Electronic mail: kvkoshel@poi.dvo.ru

^{b)}Electronic mail: jnr1@st-andrews.ac.uk

^{c)}Electronic mail: giorgio.riccardi@unicampania.it

^{d)}Electronic mail: ryzhovea@gmail.com

I. INTRODUCTION

Multipole vortex structures consisting of point or finite size vortices have been extensively studied (e.g.¹⁻⁷). The dynamics of vortex structures may be very involved or even chaotic rendering it often impossible to work out a comprehensive classification of the regimes of vortex motion for a given vortex configuration. Implementing additional constraints into such systems often simplifies the dynamics while the physical interpretation remains possible and reasonable. Establishing steady or periodic states of multipole vortex motion has been the focus of a number of works (e.g.⁸⁻¹¹). One of the most studied multipole vortex configurations consists of three vortices^{6,8,9,12-17}. Studies of vortex motion often resort to the simplest point-vortex model as it is, to some degree, analytically tractable^{6,18-22}. In particular, it is established that the dynamics of three point vortices are completely integrable indicating that all the vortex trajectories are regular¹. However, solutions to the problem featuring only elementary and known special functions can only be obtained for a limited range of special cases¹³. One of these cases is when one of the three vortices is fixed and the other two form a self-propagating pair, i.e. the strengths of the vortices comprising the pair are exactly opposite. In this case²³⁻²⁵, the vortex pair can be trapped near the fixed vortex in a bounded region or otherwise can move off the fixed vortex self-propelling itself to infinity. Fixing one of the vortices may be thought of as a model for topographically trapped eddies systematically observed in the ocean and accordingly modeled²⁶⁻³⁴.

A significant improvement over the simplest point-vortex model towards increasing realism is the model utilizing finite area vortex patches. Since the model allows for vorticity redistribution through the deformation of the vortices and generally more complicated dynamics, many authors look for steady or periodic solutions that ensure that finite area vortices engaged into the interactions remain coherent and retain their initial shape for a reasonably long time. Approaches into finding such solutions include the ones based on contour dynamics^{10,35-39} or analytical ones using complex analysis⁴⁰. In particular, recent studies^{38,41} attest that finite area vortex systems can manifest dynamical regimes very similar to their analogous point-vortex counterparts. Nevertheless, the problem of the stability of the finite area vortex systems needs a separate thorough consideration.

Therefore, the aim of the present work is twofold. The first part focuses on a comprehensive investigation of the possible regimes of motion inferred from the point-vortex model; the

second part will concentrate on finding non-trivial regimes of motion for finite size vortices based on the obtained solutions for the analogous point-vortex model.

The present paper is organized as follows. After the introduction, Section 2 states the problem. Section 3 reiterates and details the results of the symmetric case study²⁴. A full classification of possible regimes of motion and a qualitative analysis of bounded motion regimes in the asymmetric case are given in Section 4. Section 5 concludes the paper with a discussion of the results and poses possible directions for further investigations.

II. PROBLEM FORMULATION

The dynamics of a system of three point vortices in a two dimensional ideal fluid is considered. It is well-established that such a system has a Hamiltonian form^{1,2,6,13,18,24,42,43}. The Hamiltonian of the system is

$$H = -\mu^2 \log r_{12} + \sigma\mu (\log(r_1) - \log(r_2)) \equiv \text{const.} \quad (1)$$

Here, one of the vortices with a circulation $2\pi\sigma$ is fixed at a constant distance r_0 from the origin and at a polar angle θ_0 measured from the x -axis counter-clockwise. The two other vortices form a vortex pair thus ensuring self-propagation. The vortex pair (or vortex dipole) consists of two opposite-signed point vortices with circulations $2\pi\mu$ and $-2\pi\mu$ located at the distances r_1 and r_2 from the fixed vortex and at polar angles $\theta_0 + \theta_1$ and $\theta_0 + \theta_2$, respectively. The distance between the two vortices of the pair is defined by

$$r_{12}^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2). \quad (2)$$

A sketch of the initial vortex configuration is provided in fig. 1.

For convenience, we normalize time by μ , and define the circulation ratio α

$$\tau = \mu t, \quad \frac{\sigma}{\mu} = \alpha. \quad (3)$$

FIG. 1. (Color online) Sketch of the pair moving towards the fixed vortex in the symmetric case ($r_1 = r_2$). The filled circles mark the location of the point vortices or the centers of the finite size vortices. The contours mark the boundary of the finite size vortices. Furthermore, a typical asymmetric configurations ($r_1 \neq r_2$) is shown.

The equations of motion for pair's vortices then become

$$\begin{aligned}
\dot{r}_1 &= -\frac{1}{r_1 \mu^2} \frac{\partial H}{\partial \theta_1} = \frac{r_2 \sin(\theta_1 - \theta_2)}{r_{12}^2}, \\
\dot{r}_2 &= -\frac{1}{r_2 (-\mu^2)} \frac{\partial H}{\partial \theta_2} = \frac{r_1 \sin(\theta_1 - \theta_2)}{r_{12}^2}, \\
\dot{\theta}_1 &= \frac{1}{r_1 \mu^2} \frac{\partial H}{\partial r_1} = -\frac{r_1 - r_2 \cos(\theta_1 - \theta_2)}{r_{12}^2 r_1} + \frac{\alpha}{r_1^2}, \\
\dot{\theta}_2 &= \frac{1}{r_2 (-\mu^2)} \frac{\partial H}{\partial r_2} = \frac{r_2 - r_1 \cos(\theta_1 - \theta_2)}{r_{12}^2 r_2} + \frac{\alpha}{r_2^2},
\end{aligned} \tag{4}$$

where the dot stands for the derivative with respect to the normalized time τ . It is worth noticing that eqs. (4) are invariant under the transformation $t \rightarrow -t$, $r_1 \rightarrow r_2$, $r_2 \rightarrow r_1$, $\theta_1 \rightarrow \theta_2$, $\theta_2 \rightarrow \theta_1$ and $\alpha \rightarrow -\alpha$. It is therefore sufficient to restrict attention to the case $\alpha > 0$. All the cases with $\alpha < 0$ can formally be recovered from the cases with $\alpha > 0$ using the transformation defined above.

Due to symmetry, the angular momentum

$$r_1^2 - r_2^2 = M \equiv \text{const}, \quad (5)$$

is an invariant of motion. Unlike the classical three-vortex problem however, the linear momentum is not conserved in our case. It is convenient to introduce another invariant of motion

$$H_0 = e^{-\frac{H}{\mu^2}} = r_{12} \left(\frac{r_2}{r_1} \right)^\alpha = \text{const} \quad (6)$$

instead of the Hamiltonian. The last equality follows from eqs. (1) and (3).

It is instrumental to introduce new independent variables

$$0 < \rho_1 = \frac{r_2}{r_1} < 1 \quad \text{or} \quad 0 < \rho_2 = \frac{r_1}{r_2} < 1. \quad (7)$$

The system of equations (4) reduces to a system with one degree of freedom,

$$\begin{aligned} \dot{\rho}_1 &= \frac{1}{H_0^2} (\rho_1)^{2\alpha} (1 - \rho_1^2) \sin(\theta_1 - \theta_2), \\ \dot{\theta}_1 - \dot{\theta}_2 &= -\frac{\alpha}{M} \left(\frac{1}{\rho_1} - \rho_1 \right)^2 - \frac{1}{H_0^2} [\rho_1]^{2\alpha} \left(2 - \cos(\theta_1 - \theta_2) \left(\frac{1}{\rho_1} + \rho_1 \right) \right), \quad 0 < r_2 < r_1; \\ \dot{\rho}_2 &= \frac{1}{H_0^2} (\rho_2)^{-2\alpha} (1 - \rho_2^2) \sin(\theta_1 - \theta_2), \\ \dot{\theta}_1 - \dot{\theta}_2 &= -\frac{\alpha}{M} \left(\rho_2 - \frac{1}{\rho_2} \right)^2 - \frac{1}{H_0^2} [\rho_2]^{-2\alpha} \left(2 - \cos(\theta_1 - \theta_2) \left(\rho_2 + \frac{1}{\rho_2} \right) \right), \quad 0 < r_1 < r_2. \end{aligned} \quad (8)$$

From (2),(5) and (6) we obtain the relations

$$\begin{aligned} \cos(\theta_1 - \theta_2) &= \frac{1}{2} \left(\frac{r_1}{r_2} + \frac{r_2}{r_1} - \frac{H_0^2}{M} \left(\frac{r_1}{r_2} - \frac{r_2}{r_1} \right) \left(\frac{r_1}{r_2} \right)^{2\alpha} \right), \\ \sin(\theta_1 - \theta_2) &= \pm \sqrt{1 - \frac{1}{4} \left(\frac{r_1}{r_2} + \frac{r_2}{r_1} - \frac{H_0^2}{M} \left(\frac{r_1}{r_2} \right)^{2\alpha} \left(\frac{r_1}{r_2} - \frac{r_2}{r_1} \right) \right)^2}. \end{aligned} \quad (9)$$

In the case of zero angular momentum $M = 0$, one can obtain an additional invariant. This corresponds to the system having additional symmetry. This symmetric case is addressed in the next section.

III. SYMMETRIC CASE

Here we reconsider certain results from the work²⁴ for the symmetric case $M = 0$ and analyze them in more detail.

The radial symmetry implies that $r_1 = r_2$ for all time. One therefore obtains from eq. (6)

$$r_{12} = r_{120} = \text{const.}$$

We next reformulate the system (8) in the form

$$\begin{aligned} \dot{\rho}_1 = \dot{\rho}_2 &= 0, \\ \dot{\theta}_1 - \dot{\theta}_2 &= -\frac{2}{H_0^2} (1 - \cos(\theta_1 - \theta_2)), \end{aligned} \quad r_2 = r_1. \quad (10)$$

Due to symmetry, one can choose the initial value of $\theta_1 - \theta_2 = \theta_{10} - \theta_{20}$ in the form

$$\theta_{10} = \theta_I - \Delta\theta_0, \quad \theta_{20} = \theta_I + \Delta\theta_0, \quad \theta_{10} - \theta_{20} = -2\Delta\theta_0, \quad \Delta\theta_0 < \frac{\pi}{2}.$$

It is worth mentioning that if $\Delta\theta_0 > 0$, the pair moves towards the fixed vortex, and if $\Delta\theta_0 < 0$, the pair moves away from the fixed vortex. Since the symmetric case allows only for unbounded motion regimes, we only consider the cases with $\Delta\theta_0 > 0$.

The solution to eq. (8) and eq. (4) has a simple form

$$\cot\left(\frac{\theta_1 - \theta_2}{2}\right) = \frac{2}{r_{120}^2} \tau + \cot(-\Delta\theta_0), \quad r_1 = \frac{r_{120}}{2} \sqrt{1 + \left(\frac{2}{r_{120}^2} \tau + \cot(-\Delta\theta_0)\right)^2}. \quad (11)$$

From eq. (4), (10), (11), it follows for the angles θ_1 and θ_2 ,

$$\begin{aligned} \dot{\theta}_1 &= \frac{\alpha - 1/2}{r_1^2} = -(\alpha - 1/2) (\dot{\theta}_1 - \dot{\theta}_2), \quad \dot{\theta}_2 = \frac{\alpha + 1/2}{r_1^2} = -(\alpha + 1/2) (\dot{\theta}_1 - \dot{\theta}_2), \\ \dot{\theta}_1 - \dot{\theta}_2 &= -\frac{1}{r_1^2}. \end{aligned} \quad (12)$$

By using eq. (11), one can show

$$\begin{aligned} \theta_1 &= \theta_I + 2\alpha(-\Delta\theta_0) - (2\alpha - 1) \operatorname{arccot}\left(\frac{2}{r_{120}^2} \tau + \cot(-\Delta\theta_0)\right), \\ \theta_2 &= \theta_I + 2\alpha(-\Delta\theta_0) - (2\alpha + 1) \operatorname{arccot}\left(\frac{2}{r_{120}^2} \tau + \cot(-\Delta\theta_0)\right). \end{aligned} \quad (13)$$

Here care is needed in order to select a correct branch of the inverse trigonometric functions. Another representation of these solutions is given in the Appendix.

Considering the limit $t \rightarrow \infty$, the final result ensues

$$\begin{aligned} \lim_{t \rightarrow \infty} \operatorname{arccot}\left(\frac{2}{r_{120}^2} \tau + \cot(-\Delta\theta_0)\right) &= -\pi, \\ \theta_{1\infty} &= \theta_I - \pi + 2\alpha(\pi - \Delta\theta_0), \quad \theta_{2\infty} = \theta_I + \pi + 2\alpha(\pi - \Delta\theta_0) = \theta_{1\infty} + 2\pi. \end{aligned} \quad (14)$$

FIG. 2. (Color online) Scattering of a point-vortex pair impinging on a fixed point vortex. The ratio of the fixed vortex and pair circulations α is shown in the frames. The fixed vortex is located at $r_0 = 5$, $\theta_0 = -\pi/2$. Pair's vortices are initially located at $x_{10} = -x_{20} = 1.65$, $y_{10} = y_{20} = 0$ or $r_{120} = 3.3$, $\theta_{10} = \frac{\pi}{2} - 0.319$. The analogous finite size vortices have the same circulation ratio, the same initial positions and their areas equal π . $\alpha = 0.3/\pi$ - small deflection as $\Delta = \frac{0.6}{\pi} (\pi - 0.319) - \pi/2$; $\alpha = 1/\pi$ - $\Delta = \frac{2}{\pi} (\pi - 0.319) - \pi/2$; $\alpha = 1.4/\pi$ - $\Delta = \frac{2.8}{\pi} (\pi - 0.319) - \pi/2$; complete reflection as $\alpha = 1/2$ - $\Delta = (\pi - 0.319) - \pi/2 = \theta_{10}$. The black lines indicate scattering angles $\theta_{1\infty} = 2\alpha (\pi - 0.319) - \pi/2$. The black circle with radius 1.65 indicates the minimal permitted distance from the fixed vortex, the filled circles indicate the vortex positions at the given times.

To interpret these results, we write the value of $2\alpha\theta_{10}$ in the form

$$2\alpha (\pi - \Delta\theta_0) = 2\pi n + \Delta. \quad (15)$$

One can infer that one of pair's vortices completes $n - 1/2$ rotations around the fixed vortex, while the other one completes $n + 1/2$ rotations. The scattering angle of the pair equals Δ .

One can also obtain the minimal distance, which can possibly be attained between the moving vortices and the fixed vortex. This extremum of r_1 can be found from eq. (11),

$$\tau_m = \frac{r_{120}^2}{2} \cot(\Delta\theta_0), \quad r_{1m} = \frac{r_{120}}{2}. \quad (16)$$

This means that the minimal distance from the fixed vortex to either of the moving ones is equal to half the initial distance between pair's vortices.

Figures 2–4 illustrate vortex trajectories obtained from eq. (15) and eq. (16). For illustrative purposes, a time evolution of equivalent finite size vortex patches has been superimposed onto the point-vortex trajectories. The parameters of the finite size vortex model are the same as for the point vortices, and area of the vortex patches is set to π . The simulations

FIG. 3. (Color online) Same as in fig. 2 for different values of α . $\alpha = 0.7$ - small deflection after one half of a complete turn as $\Delta = 1.4(\pi - 0.319) - \pi/2$; $\alpha = 1$ - one turn as $\Delta = 2.0(\pi - 0.319) - \pi/2$; $\alpha = 1.3$ - small deflection after a complete turn as $\Delta = 2.6(\pi - 0.319) - \pi/2$; $\alpha = 1.5$ - reflection after a complete turn as $\Delta = 3.0(\pi - 0.319) - \pi/2 = \theta_{10}$.

are performed using the contour dynamic techniques as in the work³⁸. Detailed analysis of the configuration using the finite-size vortex model is reported in the accompanying paper⁴⁴. The parameters used are $r_0 = 5$, $\theta_0 = -\pi/2$, $x_{10} = -x_{20} = 1.65$, $y_{10} = y_{20} = 0$ or $r_{120} = 3.3$, $\theta_{10} = \frac{\pi}{2} - 0.319$, and the values of α are given in fig. 2.

Figure 2 shows a small deflection of the pair as it approaches the fixed vortex, followed by its reflection when $\alpha = 0.5$. Figure 3 illustrates a similar deflection of the pair as it approaches the fixed vortex. In these cases, however, the pair is reflected after one half of a complete turn around the fixed vortex. The last panel of fig. 3 shows reflection after a complete turn. Figure 4 shows the reflection of the pair after two or three complete turns. The minimal distance between pair's vortices and the fixed vortex is also indicated in the figures. It is worth mentioning that the scattering angle of the pair is independent of the initial distance separating its vortices.

The next section addresses the asymmetric case, i.e. when $M \neq 0$. Now, the distance between pair's vortices varies in time. The system therefore possesses no additional invariant.

IV. ASYMMETRIC CASE

When the pair is initially located asymmetrically with respect to the fixed vortex, its dynamics becomes more complicated. The distance between pair's vortices is no longer conserved. This results in the appearance of bounded motion regimes, when the pair oscillates

FIG. 4. (Color online) Same as in fig. 2 for relatively large values of α . $\alpha = 2.53$ near reflection after two complete turns as $\Delta = 5.06(\pi - 0.319) - \pi/2$; $\alpha = 3.59$ near reflection after three complete turns as $\Delta = 7.18(\pi - 0.319) - \pi/2$.

periodically near the fixed vortex for an infinitely long time. This effect has an elegant application in physical oceanography. Indeed, the entrapping of coherent structures by a topographic feature is a common phenomenon in the ocean^{28,31,45–48}. When this happens, bounded motion of vortex structures may lead to effective passive scalar advection^{25,33,49}.

We first analyze a necessary condition for the vortex pair to be entrapped near the fixed vortex. From eq. (4), it follows that pair's vortices reach their extreme distances from the fixed vortex only when $\sin(\theta_1 - \theta_2) = 0$, i.e. when both vortices are aligned along a straight line also passing through the fixed vortex. Without loss of generality, one can initially position pair's vortices at the x -axis (see fig. 1). Furthermore, two configurations are possible: both vortices are initially aligned on one side of the fixed vortex (such that the distance between the vortices $r_{12} < \max(r_1, r_2)$) or the fixed vortex lays between pair's vortices (such that $r_{12} > \max(r_1, r_2)$). Hence,

$$\begin{aligned} r_{120} &= r_{10} + r_{20} = r_1^e + r_2^e, & \theta_{10} - \theta_{20} &= \pi, \\ r_{120} &= |r_{10} - r_{20}| = |r_1^e - r_2^e|, & \theta_{10} - \theta_{20} &= 0. \end{aligned} \quad (17)$$

Here, the superscript 'e' denotes 'extreme value'. We choose the initial angles such that their difference, $\theta_{10} - \theta_{20}$, equals either zero or π and, consequently, there are extreme values r_1^e, r_2^e of the radii for any initial value.

Substituting the extreme points $r_1 = r_1^e, r_2 = r_2^e$ into eq. (6) results in

$$H_2 = \frac{H_0^2}{M} = H_2^\pm = \frac{(r_1^e \pm r_2^e)}{(r_1^e \mp r_2^e)} \left(\frac{r_2^e}{r_1^e} \right)^{2\alpha}, \quad (18)$$

FIG. 5. (Color online) Extreme values of H_2 depending on ρ_1 – left panel; extreme values of $-H_2$ depending on ρ_2 – right panel. The curves corresponding to $r_{12} = r_1 \pm r_2$ are indicated by the associated sign. $\alpha = 0.5$.

the upper (lower) sign corresponds to $\theta_1 - \theta_2 = \pi$ ($\theta_1 - \theta_2 = 0$). This equation can be rewritten as follows

$$\begin{aligned} H_2^\pm &= \frac{(1 \pm \rho_1^e)}{(1 \mp \rho_1^e)} (\rho_1^e)^{2\alpha}, \quad 0 < \rho_1 = \frac{r_2}{r_1} < 1, \\ H_2^\pm &= \frac{(\rho_2^e \pm 1)}{(\rho_2^e \mp 1)} (\rho_2^e)^{-2\alpha}, \quad 0 < \rho_2 = \frac{r_1}{r_2} < 1, \end{aligned} \quad (19)$$

where the superscript 'e' again denotes 'extreme value'. Both cases are further addressed separately: $r_2 < r_1$ and $r_1 < r_2$. In the former case, one obtains

$$\begin{aligned} 0 < H_2^+ &= \frac{(1 + \rho_1^e)}{(1 - \rho_1^e)} (\rho_1^e)^{2\alpha} < \infty, \quad \theta_1 - \theta_2 = \pi, \quad 0 < \rho_1 < 1; \\ 0 < H_2^- &= \frac{(1 - \rho_1^e)}{(1 + \rho_1^e)} (\rho_1^e)^{2\alpha} < \bar{H}_2^-, \quad \theta_1 - \theta_2 = 0, \quad 0 < \rho_1 < 1. \end{aligned} \quad (20)$$

It is worth mentioning that the corresponding extrema of r_1 and r_2 denoted r_1^e and r_2^e respectively, are reached simultaneously at the extrema of the ratios ρ_1 and ρ_2 . It is clearly seen from the general relations

$$\begin{aligned} r_1^2 &= \frac{M}{1 - \rho_1^2}, \quad r_2^2 = \frac{M\rho_1^2}{1 - \rho_1^2}, \quad 0 < r_2 < r_1; \\ r_1^2 &= -\frac{|M|\rho_2^2}{\rho_2^2 - 1}, \quad r_2^2 = -\frac{|M|}{\rho_2^2 - 1}, \quad 0 < r_1 < r_2. \end{aligned}$$

One can easily calculate the second derivatives at the extreme points

$$\ddot{r}_{1,2} = \frac{1}{r_{12}^2 r_{1,2}^e} \left(1 \pm \alpha \left(\frac{1}{r} - r \right) \right), \quad \cos(\theta_1 - \theta_2) = \pm 1, \quad (r = r_2^e \text{ or } r = -r_1^e).$$

Here, in the case $\rho_{1,2} < \rho_1^- = (-1 + \sqrt{1 + 4\alpha^2}) / (2\alpha)$, the second derivatives do not change the sign. The minima of r_1 and r_2 are attained provided $r_{12} = r_1 + r_2$ and exist for any

FIG. 6. Diagram of entrapment of the vortex pair. The grey area corresponds to entrapment, while the white area corresponds to unbounded motion to infinity.

value of $H_2^+ > 0$. The maxima of r_1 and r_2 are then attained when $r_{12} = r_1 - r_2$ and exist only for a finite interval $0 < H_2^- < \bar{H}_2^-$. If either of the ratios ρ_1 or ρ_2 tends to the value ρ_1^- , the maximal point transforms into an inflection point. The maximal value $\bar{H}_2^-$ is

$$\bar{H}_2^- = \frac{(1 - \rho_1^-)}{(1 + \rho_1^-)} (\rho_1^-)^{2\alpha}, \quad \rho_1^- = \frac{-1 + \sqrt{1 + 4\alpha^2}}{2\alpha}. \quad (21)$$

Figure 5 illustrates these dependencies. One can conclude that provided ρ_1 not reaching its maximal value, the pair motion is unbounded. On the contrary, the pair can be entrapped in a bounded region for the parameters chosen under the line

$$H_2(\alpha) = \frac{(2\alpha + 1 - \sqrt{1 + 4\alpha^2})}{(2\alpha - 1 + \sqrt{1 + 4\alpha^2})} \left(\frac{-1 + \sqrt{1 + 4\alpha^2}}{2\alpha} \right)^{2\alpha} \quad (22)$$

in the plane (H_2, α) . The limiting curve of the bounded motion is shown in fig. 6.

For the second case $r_1 < r_2$, we have

$$\begin{aligned} -\infty < H_2^+ &= \frac{(\rho_2^e + 1)}{(\rho_2^e - 1)} (\rho_2^e)^{-2\alpha} < \bar{H}_2^+, \quad \theta_1 - \theta_2 = \pi, \quad 0 < \rho_2 = \frac{r_1}{r_2} < 1; \\ -\infty < H_2^- &= \frac{(\rho_2^e - 1)}{(\rho_2^e + 1)} (\rho_2^e)^{-2\alpha} < 0, \quad \theta_1 - \theta_2 = 0, \quad 0 < \rho_2 = \frac{r_1}{r_2} < 1. \end{aligned} \quad (23)$$

Here, the minima of r_1 and r_2 are attained provided $r_{12} = r_1 - r_2$ and exist when $H_2^- < 0$. The maxima of r_1 and r_2 are then attained when $r_{12} = r_1 + r_2$ and exist only when $-\infty < H_2^+ < \bar{H}_2^+$. The maximal value $\bar{H}_2^+$ is

$$\bar{H}_2^+ = \frac{(\rho_2^+ + 1)}{(\rho_2^+ - 1)} (\rho_2^+)^{-2\alpha}, \quad \rho_2^+ = \rho_1^- = \frac{-1 + \sqrt{1 + 4\alpha^2}}{2\alpha}. \quad (24)$$

FIG. 7. (Color online) The radicand (26) depending on $0 < \frac{r_2}{r_1} < 1$ on the left panel and depending on $0 < \frac{r_1}{r_2} < 1$ on the right one for $\alpha = 0.5$. Regions of bounded motion are filled.

Figure 5 also illustrates these dependencies. Thus, provided ρ_2 not reaching its maximal value, the motion of the pair is unbounded. This happens for the parameters chosen under the line

$$H_2(\alpha) = \frac{(-1 + \sqrt{1 + 4\alpha^2} + 2\alpha)}{(-1 + \sqrt{1 + 4\alpha^2} - 2\alpha)} \left(\frac{2\alpha}{-1 + \sqrt{1 + 4\alpha^2}} \right)^{2\alpha}, \quad (25)$$

in the plane (H_2, α) . The second limiting curve of unbounded motion is also shown in fig. 6.

Figure 7 illustrates the radicand from eq. (9) for values of H_2 indicated in fig. 5 by horizontal lines. The radicand can be written in the form

$$\begin{aligned} R^- &= 1 - \frac{1}{4} \left(\frac{1}{\rho_1} + \rho_1 - \frac{H_0^2}{M} \left(\frac{1}{\rho_1} - \rho_1 \right) (\rho_1)^{-2\alpha} \right)^2, \quad 0 < \rho_1 < 1, \\ R^+ &= 1 - \frac{1}{4} \left(\rho_2 + \frac{1}{\rho_2} + \frac{H_0^2}{M} (\rho_2)^{2\alpha} \left(\frac{1}{\rho_2} - \rho_2 \right) \right)^2, \quad 0 < \rho_2 < 1, \end{aligned} \quad (26)$$

Solutions of system (8) only exist provided ρ_1 and ρ_2 ensuring the positivity of the radicands. It is worth mentioning that if we exclude the symmetric case, the limits $\rho_1 \rightarrow 1$ and $\rho_2 \rightarrow 1$ are equivalent to the limits $r_1, r_2 \rightarrow \infty$. Therefore one may infer from fig. 7 three regimes of motion for the pair. The motion is unbounded only for $H_2 > \bar{H}_2^-$ and $\rho_1 < 1$ or $H_2 < \bar{H}_2^+$ and $\rho_2 < 1$.

Bounded motion of the pair happens given $H_2 < \bar{H}_2^-$ and $\rho_1 < 1$ or $H_2 > \bar{H}_2^+$ and $\rho_2 < 1$. Furthermore, either of ρ_1 or ρ_2 has to be between the first and second zeros of the

radicands (26). The motion is again unbounded if ρ_1 or ρ_2 are between the third zero of the radicands (26) and 1. The cases $H_2 = \bar{H}_2^-$ and $\rho_1 < 1$ or $H_2 = \bar{H}_2^+$ and $\rho_2 < 1$ separate the two types of motion. The critical points ρ_1^- or ρ_2^+ are hyperbolic.

Before considering the trajectories permitted for the pair, we examine the frequencies of the bounded motion. System (4) has two intrinsic periods. The first one is the period of rotation of pair's vortices about the fixed vortex. To estimate this period, we rewrite eq. (4) as

$$\begin{aligned}\dot{\theta}_1 &= -\frac{1}{M} \frac{1 - \rho_1 \cos(\theta_1 - \theta_2)}{r_{12}^2/M} + \frac{\alpha}{M} (1 - \rho_1^2), \quad \dot{\theta}_2 = \frac{1}{M} \frac{\rho_1 - \cos(\theta_1 - \theta_2)}{\rho_1 r_{12}^2/M} + \frac{\alpha}{M} \frac{1 - \rho_1^2}{\rho_1^2}; \\ \dot{\theta}_1 &= -\frac{1}{M} \frac{\rho_2 - \cos(\theta_1 - \theta_2)}{\rho_2 r_{12}^2/M} + \frac{\alpha}{M} \frac{\rho_2^2 - 1}{\rho_2^2}, \quad \dot{\theta}_2 = \frac{1}{M} \frac{1 - \rho_2 \cos(\theta_1 - \theta_2)}{r_{12}^2/M} + \frac{\alpha}{M} (\rho_2^2 - 1).\end{aligned}\tag{27}$$

One can calculate the angular velocities when the vortices attain the extreme points. We start with the case $r_2 < r_1$. Given the initial values of r_1, r_2 such that $\rho_1^+ < \rho_{1\min} < \rho_1 = \rho_{1\max} < \rho_1^-$, the angular velocities of the vortices at the maximal positions are

$$\begin{aligned}\dot{\theta}_1 &= [\alpha(1 - \rho_{1\max}) - 1] \frac{(1 + \rho_{1\max})}{M}, \\ \dot{\theta}_2 &= \left[\alpha \frac{(1 - \rho_{1\max})}{\rho_{1\max}} - 1 \right] \frac{(1 + \rho_{1\max})}{M \rho_{1\max}}.\end{aligned}\tag{28}$$

The angular velocities at the maxima of r_1, r_2 vanish when the parameter α and initial ratio $\rho_{1\max}$ comply with

$$\alpha = \frac{1}{1 - \rho_{1\max}} \quad (\text{or} \quad \rho_{1\max} = \frac{\alpha - 1}{\alpha})\tag{29}$$

for the first vortex, and with

$$\alpha = \frac{\rho_{1\max}}{1 - \rho_{1\max}} \quad (\text{or} \quad \rho_{1\max} = \frac{\alpha}{1 + \alpha})\tag{30}$$

for the second one. It follows that the angular velocities at the maxima of r_1, r_2 change signs when one chooses parameters $\alpha, \rho_{1\max}$ below or above the curves from eqs. (29), (30), shown in fig. 8.

The minimum of ρ_1 lead correspondingly to

$$\begin{aligned}\dot{\theta}_1 &= [\alpha(1 + \rho_{1\min}) - 1] \frac{(1 - \rho_{1\min})}{M}, \\ \dot{\theta}_2 &= \left[\alpha \frac{(1 + \rho_{1\min})}{\rho_{1\min}} + 1 \right] \frac{(1 - \rho_{1\min})}{M \rho_{1\min}}.\end{aligned}\tag{31}$$

FIG. 8. (Color online) Permitted motion regimes of pair's vortices for various values of the parameter α and initial ratio $\rho_{(1,2)\max}$. The pair moves in a bounded area for the parameters taken below the bold black line. The first vortex has a positive angular velocity at the maximal position for the parameters taken below the red (marked as 12) line (29) and, conversely, a negative angular velocity for the parameters taken above the line. At the minimum, this vortex has a positive angular velocity to the right of the magenta (marked as 42) line (32) and, conversely, a negative angular velocity - to the left. The second vortex has a positive angular velocity at the maximum for the parameters taken below the blue (marked as 23) line (30) and, conversely, a negative angular velocity - above it. At the minimum, this vortex has always a positive angular velocity because the demarcating line (33) lays beyond this area. The orange (marked as I) line corresponds to the eq. (35). The dashed line depicts the cases when the first vortex's averaged angular velocity equals zero $\langle \dot{\theta}_1 \rangle_T = 0$ (here T is the period of pair's oscillations). The isolated dots correspond to the vortex trajectories shown further for $r_2 < r_1$ - red dots and for $r_1 < r_2$ - blue dots.

It is worth noticing that $\rho_{1\min}$ is uniquely defined by $\rho_{1\max}$. The same is valid for $\rho_{2\min}$ and $\rho_{2\max}$. This comes from analyzing fig. 5. The angular velocities at the minima of r_1 , r_2 vanish when the parameter α and initial ratio $\rho_{1\max}$ satisfy

$$\alpha = \frac{1}{1 + \rho_{1\min}} \quad (\text{or} \quad \rho_{1\min} = \frac{1 - \alpha}{\alpha}) \quad (32)$$

for the first vortex, and

$$\alpha = -\frac{\rho_{1\min}}{1 + \rho_{1\min}} \quad (\text{or} \quad \rho_{1\min} = -\frac{\alpha}{\alpha + 1}) \quad (33)$$

for the second one. The curve (33) is beyond the physical domain, so the second vortex's angular velocity does not change the sign at the minimum. The angular velocity of the first vortex changes its sign at the minima of r_1 , r_2 when one takes the parameters $\alpha, \rho_{1\max}$ bellow or above the curves from eq. (32), shown in fig. 8.

Furthermore, five possible qualitative types of the pair trajectories ensue depending on the parameter α and the initial value of the ratio $\rho_{1\max} < \rho_1^-$.

1. Both vortices move counter-clockwise (both angular velocities are positive) at the maximal and at the minimal positions (see fig. 9).
2. The angular velocity of the first vortex at the maximal position is negative, while at the minimum position it is positive. The angular velocity of the second vortex is positive at the maximum and minimum positions (see fig. 11).
3. The angular velocity of both vortices are negative at the maximal positions, and, oppositely, positive at the minimal positions (see fig. 12).
4. The angular velocity of the first vortex is negative, while the angular velocity of the second vortex is positive for all time. See fig. 10.
5. The angular velocity of the first vortex is negative for all time, while the angular velocity of the second vortex is negative at the maximal position and positive at the minimal position (see fig. 13).

We next explore the case $r_1 < r_2$. Given the maximal value of $\rho_2^- < \rho_{2\min} < \rho_2 = \rho_{2\max} < \rho_2^+$, one obtains

$$\begin{aligned}\dot{\theta}_1 &= \left(\alpha \frac{\rho_{2\max} + 1}{\rho_{2\max}} - 1 \right) \frac{1 - \rho_{2\max}}{|M| \rho_{2\max}}, \\ \dot{\theta}_2 &= (\alpha (\rho_{2\max} + 1) + 1) \frac{1 - \rho_{2\max}}{|M|}.\end{aligned}\tag{34}$$

These relations suggest that the angular velocity of the first vortex at the maximum changes its sign when one takes the parameters $\alpha, \rho_{2\max}$ above or bellow the line

$$\alpha = \frac{\rho_{2\max}}{\rho_{2\max} + 1}.\tag{35}$$

However, the curve (35) lays above the curve $\rho_2^+(\alpha)$. Thus, the angular velocity of the first vortex never changes its sign at the maximum for in the case of bounded motion. The

angular velocity of the second vortex also never changes its sign at the maximum. The curve (35) is shown in fig. 8.

For the minimal values of ρ_2 , one obtains

$$\begin{aligned}\dot{\theta}_1 &= \frac{1}{|M|} \left(1 + \alpha \frac{1 - \rho_{2\min}}{\rho_{2\min}} \right) \frac{\rho_{2\min} + 1}{\rho_{2\min}}, \\ \dot{\theta}_2 &= \frac{1}{|M|} (1 + \alpha (1 - \rho_{2\min})) (\rho_{2\min} + 1).\end{aligned}\tag{36}$$

The angular velocity of both vortices never changes its sign at the minimal position. Thus, only one possible type of the vortex trajectories occurs. The vortices rotate counter-clockwise for all time, with slightly changing their angular velocities. It is worth noticing that the rotational periods of the vortices, $2\pi / \langle \theta_1 \rangle$, $2\pi / \langle \theta_2 \rangle$, are inherently of the same time-scale. Indeed, one can recast them in the form,

$$\langle \theta_1 \rangle = \langle \theta_2 \rangle + 2\pi n,\tag{37}$$

where n depends on the type of the trajectories shown in fig. 8, and $\langle . \rangle$ stands for the averaging over one period of the first vortex rotation about the fixed vortex.

Furthermore, the system possesses a second time-scale, that is a time period during which r_1 , r_2 pass between their extreme positions. This second time scale can be found by averaging the angular velocity from eq. (8). Provided $\rho_{1\max} \ll 1$ or $\rho_{2\max} \ll 1$, one can estimate these frequencies

$$\begin{aligned}\frac{\dot{\theta}_1 - \dot{\theta}_2}{\dot{\theta}_1} &\sim -\frac{\alpha}{\alpha - 1} \left(\frac{1}{\rho_{1\max}} \right)^2, \quad \frac{\dot{\theta}_1 - \dot{\theta}_2}{\dot{\theta}_2} \sim -1, \quad 0 < r_2 \ll r_1; \\ \frac{\dot{\theta}_1 - \dot{\theta}_2}{\dot{\theta}_1} &\sim 1, \quad \frac{\dot{\theta}_1 - \dot{\theta}_2}{\dot{\theta}_2} \sim \frac{\alpha}{\alpha + 1} \left(\frac{1}{\rho_{2\max}} \right)^2, \quad 0 < r_1 \ll r_2.\end{aligned}\tag{38}$$

The next step is to analyze the angular velocities given the ratio ρ_1 or ρ_2 being close to the separatrix values ρ_1^- or ρ_2^+ . By linearizing eq. (8) near these points and assuming $(\rho_1 = \rho_1^- + \rho, \theta_1 - \theta_2 = \theta)$ or $(\rho_2 = \rho_2^+ + \rho, \theta_1 - \theta_2 = \pi + \theta)$, with ρ and θ being infinitesimal, one obtains the system

$$\begin{aligned}\dot{\rho} &= \frac{1}{M} (1 + \rho_1^-)^2 \theta, \\ \dot{\theta} &= \frac{(1 - \rho_1^{-2})}{M \rho_1^{-2}} (2\alpha \rho_1^- + 1) \rho, \quad 0 < r_2 < r_1; \\ \frac{d}{dt} \rho &= \frac{1}{M} (1 - \rho_2^+)^2 \theta, \\ \dot{\theta} &= \frac{(1 - \rho_2^{+2})}{M \rho_2^{+2}} (2\alpha \rho_2^+ + 1) \rho, \quad 0 < r_1 < r_2.\end{aligned}\tag{39}$$

FIG. 9. (Color online) Vortex trajectories of the first vortex (red) and second vortex (dark blue) for the bounded motion case corresponding to regime 1 in fig. 8; and the first vortex (magenta) and second vortex (blue) for the unbounded motion case associated with the same value of H_2 as is in the bounded case. $\alpha = 3$, $r_{10} = 50.0$, $\theta_{10} = \theta_{20} = 0$. Top row: (left) $-\frac{r_{20}}{r_{10}} = 0.5045$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = 8$); (right) $-\frac{r_{20}}{r_{10}} = 0.52$, ($7 < \frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} < 8$). Bottom row: same as in the top row, except in a coordinate system rotating with the angular velocity $\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle$. The black segments of the trajectories show the first period of r_1 , r_2 .

The relations imply that the points $(\rho_1^-, 0)$ and (ρ_2^+, π) are hyperbolic. Therefore, the frequency $\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle$ is zero at this phase trajectory. Nevertheless, the average rotation frequency of either vortex, $\langle \dot{\theta}_1 \rangle$ and $\langle \dot{\theta}_2 \rangle$, near these hyperbolic points remains finite.

A. Numerical results

The purpose of this section is to illustrate vortex trajectories for different values of the angular velocity. We start with the case $0 < \rho_1 < 1$.

To start, we analyze the simplest types of motion, i.e. types 1 and 4 from the diagram in fig. 8. All these cases correspond to the vortices not altering their direction of motion

(in terms of angular velocity). In case 1, both vortices move counter-clockwise for all time. In case 4, the first vortex moves clockwise, and the second vortex moves counter-clockwise for all time. Probing the ratio of the inherent time-scales, it follows from the estimates eq. (38) that given $\rho_{1\max} \ll 1$, the first vortex moves significantly slower than the second one and even in comparison with the pair rotation period $2\pi / \langle \dot{\theta}_1 - \dot{\theta}_2 \rangle$. One can conclude that there are many maxima and minima of r_1, r_2 while the first vortex completes one turn around the fixed vortex. However, both vortices reach their extreme positions concurrently. The estimates eq. (38) ensure that the second vortex turns with the first one on the same angle and completes another turn around the fixed vortex during one oscillation period of the pair.

This situation is illustrated in fig. 9 and 10. The first period of pair rotation, $2\pi / \langle \dot{\theta}_1 - \dot{\theta}_2 \rangle$, is indicated by a bold black line to show the direction of the vortex rotation. Another feature is that the vortex trajectories are always periodic in the reference frame rotating with the angular velocity $\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle$. This property is also shown in the figures. The next feature worth noticing is the decrease of the angular velocity of the first vortex when $\alpha \sim 1$. This comes from the estimate (38). Although the estimate is not strictly valid for this value of α , it provides a qualitative tendency. The same conclusion can be inferred from eq. (28), (31) or (27). This fact further transpires when considering motion of type 2 shown in fig. 8.

When the parameters correspond to the second and third motion types, the angular velocity of the first vortex changes its sign when the pair passes between its extreme positions. This means that the averaged angular velocity of the first vortex, $\langle \dot{\theta}_1 \rangle$, can be zero. The line in the diagram of fig. 8 corresponding to the $\langle \dot{\theta}_1 \rangle(\alpha, \rho_{1\max}) = 0$ now starts at the point $\alpha = 1, \rho_{1\max} = 0$ and remains in regions 2 and 3. Numerically calculated, this dashed line is plotted in fig. 8. The line has no stagnation points because $\langle \dot{\theta}_2 \rangle(\alpha, \rho_{1\max}) = 2\pi$, provided $\langle \dot{\theta}_1 \rangle(\alpha, \rho_{1\max}) = 0$.

We next consider permitted values of the ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle}$ as the parameters vary. The ratio tends to infinity when $\rho_{1\max} \rightarrow 0$. This case is illustrated in fig. 9 and fig. 10. There is a minimum between the region for small ρ_1 and the line $\langle \dot{\theta}_1 \rangle(\alpha, \rho_1) = 0$. Specifically, when α is sufficiently large, the curve $\langle \dot{\theta}_1 \rangle(\alpha, \rho_1) = 0$ lays in region 3 shown in fig. 8; moreover, the minimum of the ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle}$ coincides with the one attained in region 2. Here only a small range where ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle}$ changes. For example, for $\alpha = 3$ we have $5 < \frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} < 6$ in area 2

FIG. 10. (Color online) Same as in fig. 9 corresponding to region 4 from fig. 8. $\alpha = 0.5$, $r_{10} = 50.0$, $\theta_{10} = \theta_{20} = 0$. Left – $\frac{r_{20}}{r_{10}} = 0.3185$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = 11$); right – same as to the left, but plotted in a rotating reference frame of the angular velocity $\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle$. The black segments of the trajectories indicate the first oscillation period of r_1, r_2 .

shown in fig. 8. Thus, periodic trajectories exist only for specific values of α , provided that it is sufficiently large. Figure 11 depicts this case.

The final feature to notice concerns the ratio of the angular velocities near the separatrix as $\rho_{1\max} \rightarrow \rho_1^-$. The pair rotation period tends to infinity when $\rho_{1\max} \rightarrow \rho_1^-$. Thus, the ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle}$ tends to infinity given $\rho_{1\max} \rightarrow 0$ and near the curve $\langle \dot{\theta}_1 \rangle(\alpha, \rho_{1\max}) = 0$, and, inversely, tends to zero as $\rho_{1\max} \rightarrow \rho_1^-$.

Furthermore, the ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle}$ can in particular be rational or natural. Both cases ensure that the vortex trajectories are periodic in the original non-rotating reference frame. This is also shown in fig. 9 – 13. The case when $\langle \dot{\theta}_1 \rangle(\alpha, \rho_{1\max}) = 0$ belonging to region 2 is also illustrated in fig. 11. The right bottom panel shows the case when the counter-clockwise and clockwise rotation exactly compensate each other. This case is of particular interest because the first vortex never rotates about the fixed vortex and only oscillates in a small bounded region.

We next describe the types of motion close to the separatrix, i.e. types 3 and 5 shown in fig. 8. An example of vortex trajectories characterized by the type 3 motion is illustrated in fig. 12. The top left panel depicts an increase of the ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle}$ compared to the top left panel in fig. 11. The top right panel corresponds to the case $\langle \dot{\theta}_1 \rangle(\alpha, \rho_{1\max}) = 0$, as the

FIG. 11. (Color online) Same as in fig. 9 for region 2 shown in fig. 8. $r_{10} = 50.0$, $\theta_{10} = \theta_{20} = 0$. Top: (left) – $\alpha = 3.1325$, $\frac{r_{20}}{r_{10}} = 0.74$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = 5$), (right) – $\alpha = 0.9$, $\frac{r_{20}}{r_{10}} = 0.4175$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = 21$); bottom: (left) – same as in the top (left) panel but plotted in a rotating reference frame of the angular velocity $\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle$, (right) – $\alpha = 1.2268$, $\frac{r_{20}}{r_{10}} = 0.4$ ($\frac{\langle \dot{\theta}_1 \rangle}{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle} = 0$). The black trajectory segments indicate the first oscillation period of r_1 , r_2 .

first vortex is not rotating about the fixed vortex. The bottom panels show a decrease of the ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle}$ as $\rho_{1\max}$ tends to ρ_1^- .

The motion type corresponding to region 5 differs from the one associated with region 3 as is exemplified by fig. 13. This region corresponds to the first vortex always moving clockwise. As the ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle}$ approaches the separatrix, it decreases leading to the appearance of special trajectories of the second vortex. Indeed, as has been shown above, $\theta_2(T = \frac{2\pi}{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}) = \theta_1(T) + 2\pi$, however, given the parameters $\rho_{1\max} = 0.41421$, $\alpha = 0.5$, one arrives at $\theta_1(T) = -2\pi$ and $\langle \dot{\theta}_2 \rangle = 0$. This implies that the second vortex may not necessarily complete a full rotation about the fixed vortex.

The next case to consider corresponds to $0 < \frac{r_1}{r_2} < 1$. The vortices always rotate counter-clockwise. Therefore, no solutions are permitted given $\langle \dot{\theta}_1 \rangle = 0$ or $\langle \dot{\theta}_2 \rangle = 0$; consequently

FIG. 12. (Color online) Same as in fig. 9 corresponding to region 3 shown in fig. 8. $\alpha = 3.0$, $r_{10} = 50.0$, $\theta_{10} = \theta_{20} = 0$. Top: (left) $-\frac{r_{20}}{r_{10}} = 0.7714$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = 6$), (right) $-\frac{r_{20}}{r_{10}} = 0.83505$ ($\frac{\langle \dot{\theta}_1 \rangle}{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle} = 0$); bottom: (left) $-\frac{r_{20}}{r_{10}} = 0.8465$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = 3$), (right) $-\frac{r_{20}}{r_{10}} = 0.8471$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = 1$). The black trajectory segments mark the first oscillation period of r_1, r_2 in the left top panel and half of a period for the remaining panels.

the ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle}$ decreases at a much faster rate as the parameters approach the separatrix curve $\rho_2^+(\alpha)$. Exemplary trajectories are shown in fig. 14,15. The ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} < 1$ is shown in fig. 15. The right panel of fig. 15 illustrates an example of motion occurring near the separatrix over a long time. When the ratio $\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} < 1$, the vortices move from maximum to minimum in spiralling trajectories. This in turn implies a very slow decrease of the ratio ρ_2 . Analogous dynamics but taking place closer to the separatrix $\rho_1^-(\alpha)$ manifests itself given $0 < r_2 < r_1$.

FIG. 13. (Color online) Same as in fig. 9 corresponding to region 5 shown in fig. 8. $\alpha = 0.5$, $r_{10} = 50.0$, $\theta_{10} = \theta_{20} = 0$. Left $-\frac{r_{20}}{r_{10}} = 0.3508$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = 8$), (right) $-\frac{r_{20}}{r_{10}} = 0.41421$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = 1$). The trajectory black segments mark the first period of r_1, r_2 in the left panel and half a period in the right one.

V. CONCLUSIONS

The dynamics of a vortex pair consisting of two equal counter-rotating point vortices that impinges on a fixed point vortex in a homogeneous barotropic fluid has been analyzed in detail. A paper upon which the present study has been based²⁴ categorizes the possible motion into two types: (i) symmetric case: pair's vortices are located at the same (variable) distance from the fixed vortex, which makes the pair always scatter and move unboundedly when approaching the fixed vortex; (ii) asymmetric case: the distances between pair's vortices and the fixed vortex are not equal, and the motion of pair's vortices can be bounded.

Results for the symmetric case have been refined and preliminary calculations for an analogous finite area vortex model have been illustrated. These calculations have been carried out by the method of contour dynamics³⁸ and agree well with the dynamics of the point-vortex model.

For the asymmetric case, a new detailed classification of the possible bounded motion regimes has been presented. Two parameters are shown to govern the transitions between regimes of the bounded motion: (i) the ratio of the strengths of pair's vortices; (ii) a parameter based on the ratio of the initial positions of the vortices relative to the fixed vortex. An analytic expression comprising these two parameters for the boundary separating the

FIG. 14. (Color online) Trajectories of the first (red) and second (dark blue) vortices for bounded motion, and of the first (magenta) and second (blue) vortices for unbounded motion given the same values of H_2 : $\theta_{10} = \pi, \theta_{20} = 0, r_{20} = 50.0$. Top row: (left) – $\alpha = 0.5, \frac{r_{10}}{r_{20}} = 0.297$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_2 \rangle} = 5, \frac{\langle \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = \frac{1}{5+1}$); (right) – $\alpha = 1.5, \frac{r_{10}}{r_{20}} = 0.296$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_2 \rangle} = 8, \frac{\langle \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = \frac{1}{8+1}$). Bottom row $\alpha = 3.0$: (left) – $\frac{r_{20}}{r_{10}} = 0.482, (\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_2 \rangle} = 3, \frac{\langle \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = \frac{3}{3+1})$, (right) – $\frac{r_{10}}{r_{20}} = 0.561$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_2 \rangle} = 2$). The trajectory black segments denominate the first oscillation period of r_1, r_2 .

bounded and unbounded motion regimes has been derived.

The complete classification of the possible types of the bounded motion is formed on the basis of taking into account the average rotation direction of both vortices around the fixed vortex. When either of the vortices rotates oppositely to the direction induced by the fixed vortex and is located closer to it, the vortices can transiently rotate in either direction. Various combinations of rotation directions lead to the classification of feasible

FIG. 15. (Color online) Same as in fig. 14. $\alpha = 3.0$, $r_{20} = 50.0$, $\theta_{10} = \pi$, $\theta_{20} = 0$. Left - $\frac{r_{10}}{r_{20}} = 0.69$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_2 \rangle} = 1$, $\frac{\langle \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = \frac{1}{1+1}$), right - $\frac{r_{20}}{r_{10}} = 0.82$ ($\frac{\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle}{\langle \dot{\theta}_2 \rangle} = \frac{1}{3}$, $\frac{\langle \dot{\theta}_2 \rangle}{\langle \dot{\theta}_1 \rangle} = \frac{3}{3+1}$). The trajectory black segments mark the first half of the period of r_1 , r_2 .

vortex trajectories. An important feature is that there is a separatrix delineating bounded and unbounded motion regimes. The availability of a hyperbolic point and the separatrix makes it promising to perturb the regular steady-state to look into nascent chaotic dynamics.

Another interesting feature of the pair bounded motion is that the system possesses two proper frequencies. The first one is the pair oscillation frequency and the second one is the frequency of the pair rotation around the fixed vortex. The permitted frequencies range from 0 to infinity, which may be facilitating in order to obtain an abundance of chaotic dynamics regimes^{50–53}.

Making use of the intrinsic frequencies, one can pass over to a rotating reference frame in which one of the vortices will not rotate about the fixed vortex. If the parameters and initial vortex positions result in the fact that the ratio of the frequencies is a rational or natural number, the vortex trajectories are periodic in the original reference frame. A number of configurations that feature one vortex not rotating about the fixed vortex has been found.

We believe that many of the reported features are of great interest to researchers dealing with the real vortex structures, in particular, in the ocean and atmosphere. However, the applicability of point-vortex models are known to be sufficiently restricted, hence our interest in comparing certain regimes of motion reported in the study with a finite area vortex

model that accounts for the fact that the vortices may significantly alter their shape when interacting with each other. Therefore, the second part of the study is intended to demonstrate complicated vortex dynamics found in the finite area vortex model that is similar to the one permitted by the analogous point-vortex model and reported here.

As a final remark, we would like to mention that the model in question might be of use not only for hydrodynamic applications but also in physics of superfluids. For instance, configurations of similar coherent structures have been extensively analyzed in context of Bose-Einstein condensates^{54–56}.

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VI. APPENDIX. ANOTHER REPRESENTATION OF THE SYMMETRIC CASE SOLUTION

Let us derive the equation for r_1 from (4) for the symmetric case, i.e. r_{12} is constant and $r_1 = r_2$. Hence

$$\ddot{r}_1 = \frac{1}{r_{12}^2} \left(\dot{r}_1 \sin(\theta_1 - \theta_2) + r_1 \cos(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) \right).$$

This equation can be rewritten in the form

$$\dot{r}_1 \ddot{r}_1 = \frac{1}{2} \frac{d}{d\tau} \dot{r}_1^2 = \frac{\dot{r}_1}{4r_1^3} = -\frac{d}{d\tau} \frac{1}{8r_1^2}.$$

Integrating this equation results in

$$r_1 = \frac{r_{12}}{2} \left(1 + \frac{4}{r_{12}^4} (\tau - \tau_m)^2 \right)^{1/2} ; \quad \tau_m = -\dot{r}_{10} r_{10} r_{12}^2 = -r_{10}^2 \sin(\theta_{10} - \theta_{20}),$$

where we use the equality $1 + 4\dot{r}_{10}^2 r_{10}^2 = (2r_{10}/r_{12})^2$. From this it follows that r_1 has a minimum at $\frac{r_{12}}{2}$ only if the initial value of $\sin(\theta_{10} - \theta_{20})$ is negative.

By inserting the solution r_1 inside the evolution equation for $(\dot{\theta}_1 - \dot{\theta}_2)$ (12) and integrating, one arrives at

$$\frac{\theta_1(t) - \theta_2(t)}{2} = \frac{\theta_{10} - \theta_{20}}{2} - \arctan\left(\frac{2}{r_{12}^2}(\tau - \tau_m)\right) + (2\alpha - 1) \arctan\left(\frac{2}{r_{12}^2}\tau_m\right),$$

$$\theta_i(t) = \theta_{i0} + (2\alpha \mp 1) \left(\arctan\left(\frac{2}{r_{12}^2}(\tau - \tau_m)\right) + \arctan\left(\frac{2}{r_{12}^2}\tau_m\right) \right), \quad (i = 1, 2).$$

In certain cases, this form of the solutions is more convenient than the one presented in the paper's main body.

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