# Accepted Manuscript


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PII: S0022-0531(17)30035-2
DOI: http://dx.doi.org/10.1016/j.jet.2017.03.005
Reference: YJETH 4653

To appear in: *Journal of Economic Theory*

Received date: 19 December 2016
Accepted date: 13 March 2017

Please cite this article in press as: Amir, R., et al. On the microeconomic foundations of linear demand for differentiated products. *J. Econ. Theory* (2017), http://dx.doi.org/10.1016/j.jet.2017.03.005

# On the microeconomic foundations of linear demand for differentiated products *

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March 22, 2017

**Abstract**

This paper provides a thorough exploration of the microeconomic foundations for the multivariate linear demand function for differentiated products, which is widely used in industrial organization. The setting is the standard representative consumer with a quasi-linear utility function. A key finding is that strict concavity of the quadratic utility function is critical for the demand system to be well defined. Otherwise, the true demand function may be quite complex: Multi-valued, non-linear and income-dependent. We uncover failures of duality relationships between substitute products and complementary products, as well as the incompatibility between high levels of complementarity and concavity. The two-good case emerges as a special case with strong but non-robust properties. A key implication is that all conclusions derived in applied economic models via the use of linear demand that does not satisfy the Law of Demand ought to be regarded with some suspicion.

JEL codes: D43, L13, C72.

Key words and phrases: linear demand, gross substitutes, gross complements, Edgeworth complements, representative consumer, Law of Demand.

*We gratefully acknowledge helpful feedback frm Wayne Barrett, Robert Becker, Francis Bloch, Sam Burer, Jacques Dreze, Christian Ewerhart, Bertrand Koebel, Laurent Linnemer, Isabelle Maret, Jean-Francois Mertens, Herve Moulin, Heracles Polemarchakis, and Xavier Vives, as well as from two JET referees, and audiences at SAET and PET (Rio de Janeiro, 2016), LAMES-LACEA (Medellin, 2016), and in seminars at the Universities of Glasgow, Indiana, Louvain (CORE), Paris I, Strasbourg, Strathclyde, UNSW-Sydney, UTS-Sydney, and Waseda.

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# 1 Introduction

The emergence of the modern theory of industrial organization owes much to the development of game theory. Due to its priviledged position as the area where novel game theoretic advances found their initial application in an applied setting, industrial organization then served as a further launching ground for these advances to spread to other areas of economics. Yet to explain the success of industrial organization in reaching public policy makers, antitrust practitioners, and undergraduate students, one must mention the role played by the fact that virtually all of the major advances in the theory have relied on an accessible illustration of the underlying analysis using the convenient framework of linear demand.

While this framework goes back all the way to Bowley (1924), it received its first well-known treatment in two visionary books that preceded the revival of modern industrial organization, and yet were quite precocious in predicting the intimate link between modern industrial organization and game theory: Shubik (1959) and Shubik and Levitan (1980). Then early on in the revival period, Dixit (1979), Deneckere (1983) and Singh and Vives (1984) were among the first users of the linear demand setting. Subsequently, this framework has become so widely invoked that virtually no author nowadays cites any of these early works when adopting this convenient setting.[1]

Yet, despite this ubiquitous and long-standing reliance on linear demand, the present paper will argue that some important foundational and robustness aspects of this special demand function remain less than fully understood.[2] Often limiting consideration to the two-good case, the early literature on linear demand offered a number of clear-cut conclusions both on the structure of linear demand systems as well as on its potential to deliver unambiguous conclusions for some fundamental questions in oligopoly theory. Among the former, one can mention the duality features uncovered in the well known paper by Singh and Vives (1984), namely (i) the dual linear structure of inverse and direct demands (along with the use of roman and greek parameters), (ii) the duality between substitute and complementary products and the invariance of the associated cross-slope

[1] Martin (2002) provides an insightful overview of the history of the linear demand system, as well as a comparison between the Bowley and the Shubik specifications.

[2] One is tempted to attribute this oversight to the fact that industrial economists' strong interest in linear demand is not shared by general microeconomists (engaged either in theoretical or in empirical work), as evidenced by the fact that quadratic utility hardly ever shows up in basic consumer theory or in general equilibrium theory.

parameter range of length one for each, and (iii) the resulting dual structure of Cournot and Bertrand competition. In the way of important conclusions, Singh and Vives (1984) showed that, with linear demands, competition is always tougher under Bertrand than under Cournot. In addition, were the mode of competition to be endogenized in a natural way, both firms would always prefer to compete in a Cournot rather than in a Bertrand setting. (Singh and Vives, 1984 inspired a rich literature still active today). Subsequently, Hackner (2000) showed that with three or more firms and unequal demand intercepts, the latter conclusion is not universally valid in that there are parameter ranges for which competition is tougher under a Cournot setting, and that consequently some firms might well prefer a Bertrand world (see also Amir and Jin, 2001, for further qualifications of interest). Hsu and Wang (2005) show that consumer surplus and social welfare are nevertheless higher under Bertrand competition for any number of firms, under Hackner's formulation.

With this as its starting point, the present paper provides a thorough investigation of the micro-economic foundations of linear demand. Following the aforementioned studies, linear demand is derived in the most common manner as the solution to a representative consumer maximizing a utility function that is quadratic in the $n$ consumption goods and quasi-linear in the numeraire. When this utility function is strictly concave in the quantitites consumed, the first order conditions for the consumer problem do give rise to linear demand, as is well known. Our main result is to establish that this is the only way to obtain such a micro-founded linear demand. In other words, we address the novel question of integrability of linear demand, subject to the quasi-linearity restriction on candidate utility functions and find that linear demand can be micro-founded in the sense of a representative consumer if and only if it satisfies the strict Law of Demand in the sense of decreasing operators (see Hildenbrand, 1994), i.e., if and only if the associated substitution/complementarity matrix is positive definite.[3] As a necessary first step, we derive some general conclusions about the consumer problem with quasi-linear preferences that do not necessarily satisfy the convexity axiom. In so doing, we explicitly invoke some powerful results from the theory of monotone operators and

[3]Two studies have addressed related issues under more general conditions. Lafrance (1985) investigates the issue of integrability for an incomplete system of linear demand functions without imposing quasi-linearity of the utility function. He finds that the underlying conditional preferences must be either quadratic or Leontief from a translated origin. Formalizing an idea of Marshall, Vives (1987) provides sufficient conditions for the well known income effect of consumer theory to be negligible as the number of goods increases, so that Marshallian demand can behave like Hicksian demand when the utility function is not necessarily quasi-linear.

convex analysis (see e.g., Vainberg, 1973, Hildenbrand, 1994, and Rockaffelar, 1970), as well as a mix of basic and specialized results from linear algebra.

We also observe that strict concavity of the utility function imposes significant restrictions on the range of complementarity of the $n$ products. For the symmetric substitution matrix of Hackner (2000), we show that the valid parameter range for the complementarity cross-slope is $(-\frac{1}{n-1}, 0)$, which coincides with the commonly reported range of $(-1, 0)$ if and only if there are exactly two goods ($n = 2$). In contrast, the valid range for the cross parameter capturing substitute products is indeed $(0, 1)$, independently of the number of products, which is in line with previous belief.[4]

A closely related point of interest is that, in the case of complements, as one approaches from above the critical value of $-\frac{1}{n-1}$, the usual necessary assumption of enough consumer wealth for an interior solution becomes strained as the amount of wealth actually needed is shown to converge to infinity! This further reinforces, in a sense that is hard to foresee, the finding that linear demand is not robust to the presence of high levels of inter-product complementarity.[5]

In addition, we explore the relationship between the standard notions of gross substitutes and complements and the alternative definitions (due to Edgeworth, 1881) of these relationships as given by the sign of the cross-partial derivative of the utility function. For linear demand for two goods, these two notions are well-known to be pair-wise equivalent (e.g., Singh and Vives, 1984). With three or more goods, the only general fact is that Edgeworth complementarity (or a supermodular utility function) implies that all pairs of goods are gross complements. All the other three possible implications do not generally hold. Here again, these findings bring out another divergence between substitutes and complements as long as one has three or more products.

All together then, the neat duality between substitute and complementary products exhibited by the two-good case breaks down in multiple ways for the case of three or more goods. Nevertheless, we verify that when the demand function is well-founded, the profit functions and the reaction curves in $n$-firm Cournot and Bertrand oligopolies with differentiated products do inherit the familiar

[4]On a related note, recent work by Armstrong and Vickers (2015) shows that a multiproduct demand system can be generated from a discrete choice model with unit demands only if the products are gross substitutes.

[5]For multi-product monopoly pricing under linear demand for differentiated products, Amir, Jin, Troege and Pech (2016) observe that the optimal prices are independent of the inter-product relationships, i.e., all products are priced in the same way irrespective of whether thay are substitutes, complements or independent.

properties from the two-firm (two-good) case.

Since many studies have used linear demand in applied work without sufficient concern for microeconomic foundations, it is natural to explore the nature of linear demand when strict concavity of the utility function does not hold.[6] In other words, we investigate the properties of the solution to the first order conditions of the consumer problem, which is then only a saddle-point with no global optimality properties. (Thus we refer to this solution as a saddle-point demand function.) We find that several rather unexpected exotic phenomena might arise (including negative saddle-point demand and the possibility of Giffen goods). In particular, we explicitly solve for the true (global) solution of the utility maximization problem with a symmetric quadratic utility function that barely fails strict concavity, and show that the resulting demand is multi-valued, highly non-linear and overall quite complex even for the two-good case.

As a final point, we investigate one special case of linear demand with a local interaction structure. This is characterized by two key features: (i) the $n$ products are ranked in terms of some one-dimensional attribute, such as quality, and (ii) the consumer is postulated as viewing the price of any good $i$ as responding to changes in the quantity of every other good $j$ with a magnitude that decreases exponentially with the distance between $i$ and $j$ in characteristic space. The resulting direct demand is then such that two goods are imperfect substitutes if they are direct neighbors in the attribute space and as unrelated products otherwise. We investigate the properties of the resulting linear demands, and show that the model satisfies all the criteria derived in the present paper for a well-founded demand system. Though intended for vertical differentiation, the well-known model of the car industry due to Bresnahan (1987) has the same local interaction structure.

This paper is organized as follows. Section 2 gathers all the microeconomic preliminaries for general quasi-linear preferences. Section 3 specializes to quadratic utility and investigates the integrability properties of linear demand, including in particular the widespread case of symmetric quadratic utility. Section 4 explores the relationship between the notions of gross substitutes/complements and the alternative definition (due to Egdeworth) of these relationships obtained via the utility function. Section 5 considers a special case of a linear demand for vertically differentiated products with a local interaction structure. Finally, Section 6 offers a brief conclusion.

[6] A classical example appears in Okuguchi's (1987) early work on the comparison between Cournot and Bertrand equilibria, which is discussed in some detail in the present paper.

## 2 Some basic microeconomic preliminaries

In this section, we work with the two standard models from the textbook treatment of consumer theory, but allowing for general preferences that are quasi-linear in the numeraire good, but do not necessarily satisfy the convexity axiom. In other words, the utility function is not necessarily strictly quasi-concave here. The main goal is to prove that Marshallian demands are decreasing in the sense of monotone operators (Hildenbrand, 1994), which implies in particular that the demand for each good is also decreasing in own price.

### 2.1 On consumer theory with quasi-linear utility

Let $x \in R^n_+$ denote the consumption levels of the $n$ goods and $y \in R_+$ be the numeraire good. The agent is endowed with a utility function $U : R^n_+ \longrightarrow R$ over the $n$ goods and the numeraire $y$ appears in an additively separable manner in the overall utility. The agent has income $m > 0$ to spend on purchasing the $(n+1)$ goods.

The utility maximization problem is, given a price vector $p \in R^n_+$ and the numeraire price normalized to 1,

$$\max U(x) + y \tag{1}$$

subject to[7]

$$p'x + y \leq m. \tag{2}$$

We shall refer to the solution vector (i.e., the argmax) as the Marshallian demands, denoted $(x^*(p,m), y^*(p,m))$ or simply $(x^*, y^*)$. We shall also use the notation $D(p) = (D_1(p), D_2(p), ..., D_n(p))$ for this direct demand function since the argument $m$ will be immaterial in what follows.

The (dual problem of) expenditure minimization is (with $u$ being a fixed utility level)

$$\min p'x + y \tag{3}$$

subject to

$$U(x) + y \geq u.$$

[7]Throughout the paper, "$'$" will denote the transpose operation, so that $p'x$ denotes the usual dot product between vectors $p$ and $x$. The latter is sometimes also written $p \cdot x$.

We shall refer to the solution vector as the Hicksian demands $(x^h(p,u), y^h(p,u))$ or simply $(x^h, y^h)$. We shall also use the notation $D^h(p)$ for this direct demand function since the argument $u$ will not matter below. Recall that the (minimal) value function for the objective (3) is the so-called expenditure function in standard consumer theory, denoted $e(p,u)$.

The following assumption is maintained throughout the paper.[8]

**(A1)** *The utility function $U$ is twice continuously differentiable and has $U_i \triangleq \frac{\partial U_i}{\partial x_i} > 0$, for all $i$.*

Since $U$ is not necessarily strictly quasi-concave, the solutions to the two problems above, the Marshallian demands $(x^*, y^*)$ and the Hicksian demands $(x^h, y^h)$, may be correspondences in general.[9] By Weirstrass's Theorem, both correspondences are non-empty valued.

### 2.2 On the Law of Demand

In standard microeconomic demand theory, though not always explicitly recognized, the downward monotonicity of multi-variate demand is usually meant in the sense of monotone operators (for a thorough introduction, see Vainberg, 1973). This is a central concept in the theory of demand aggregation in economics (Hildenbrand, 1994) as well as in several different contexts in applied mathematics (Vainberg, 1973).[10] This subsection provides a brief overview of this concept of monotonicity and summary of some of its important, though simple, properties. Further details and proofs may be found e.g., in Vainberg (1973) or [Hildenbrand (1994), Appendix].

We begin with some notation and the definition. Let $S$ be an open convex subset of $R^n$ and $F$ be a function from $S$ into $R^n$. We denote the standard dot product by "$\cdot$".

We shall say that $F$ is (strictly) aggregate-monotonic if

$$[F(s) - F(s')] \cdot (s - s')(<) \leq 0 \text{ for every } s, s' \in S. \tag{4}$$

If $F$ is set-valued (or a correspondence), then (4) is to hold for every selection, i.e., for every $s, s' \in S$,

$$(z - z') \cdot (s - s')(<) \leq 0 \text{ for every } z \in F(s) \text{ and } z' \in F(s').$$

[8] Smoothness is assumed only for convenience here, and is not critical to any of the conclusions of the paper.

[9] It is important to allow for utility functions that do not satisfy the ubiquitous quasi-concavity assumption since we shall be concerned in some parts of this paper with maximizing quadratic, but non-concave, utility functions.

[10] In particular, these notions of monotonicity form the sufficient conditions for the classical local inversion and univalence theorems in multi-variate analysis. They are also relevant for global univalence results such as the Debreu-Gale-Nikaido Theorem (see e.g., Aubin, 2007).

This notion of downward monotonicity is quite distinct from the more prevalent notion of monotonicity in the coordinate-wise (or product) Euclidean order that arises naturally in the theory of supermodular optimization and games (Topkis, 1998, Vives, 1999).[11] Nonetheless, for the special case of a scalar function, both notions boil down to the usual notion of monotonicity, and thus constitute alternative but distinct natural generalizations.

The following characterization of aggregate monotonicity is well known. Let $\partial F(s)$ denote the Jacobian matrix of $F(s)$, i.e., for any $(i, j)$, the $ij$th entry of the matrix $\partial F(s)$ is $\partial_{ij} F(s) = \frac{\partial F_i(s)}{\partial s_j}$, which captures the effect of a change in the price of the $j$th good on the demand for the $i$th good.

**Lemma 1** *Let $S$ be an open convex subset of $R^n$ and $F : S \longrightarrow R^n$ be a continuously differentiable map. Then the following two properties hold.*
*(i) $F$ is aggregate-monotonic if and only if the Jacobian matrix $\partial F(s)$ is negative semi-definite.*
*(ii) If the Jacobian matrix $\partial F(s)$ is negative definite, then $F$ is strictly aggregate-monotonic.*

In Part (ii), the two strict notions are not equivalent. Indeed, there are examples of strictly aggregate-monotonic maps with a Jacobian matrix whose determinant is not everywhere non-zero.

An important direct implication of Lemma 1 is that the diagonal terms of $\partial F(s)$ must be negative. However, this monotonicity concept does not impose restrictions on the signs of the off-diagonal elements of $\partial F(s)$. In contrast, monotonicity in the coordinate-wise order requires that every element of the Jacobian $\partial F(s)$ be (weakly) negative.

**Definition 2** *The Marshallian demand $D(p)$ satisfies the (strict) Law of Demand if $D(p)$ is (strictly) aggregate-monotonic, i.e., for any two price vectors $p$ and $p'$, $D$ satisfies*

$$[D(p) - D(p')] \cdot (p - p')(<) \leq 0 \tag{5}$$

In classical consumer theory, this property of monotonicity of consumer demand is well-known not to hold under very general conditions on the utility function, but sufficient conditions that

[11]In the mathematics literature, functions with this property are simply referred to as monotone functions (or operators). The choice of the terminology "aggregate-monotonic" is ours, and is motivated by two considerations. One is that this is the standard notion of monotonic demand in aggregation theory in economics. The other is a desire to distinguish this monotonicity notion from coordinatewise monotonicity, which is more prevalent in economics.

validate it are available. Unfortunately, these conditions are very restrictive: See Milleron (1974) and Mitjushin and Polterovich (1978), or Hildenbrand (1994) for the associated results and discussion.

Consistent with Lemma 1, a demand function that satisfies the Law of Demand necessarily has the property that each demand component is downward-sloping in own price (in other words, the diagonal elements of the Jacobian matrix are all negative). Put differently, no good can be a Giffen good. In addition, as Lemma 1 makes clear, the Law of Demand entails significantly more multi-variate restrictions on the overall demand function.

### 2.3 A key implication of quasi-linear utility

The following general result reflects a key property of demand, which constitutes the primary motivation for postulating a quasi-linear utility function in industrial organization. This result will prove very useful in our analysis of the foundations of linear demand.

**Proposition 3** *Under Assumption A1, the Marshallian demand $D(p)$ satisfies the Law of Demand.*

**Proof.** We first prove that the Hicksian demand satisfies the Law of Demand. In the expenditure minimization problem, the expenditure function $e(p, u)$, as defined in (3), is defined as the pointwise infimum of a collection of affine functions in $p$. Hence, by a standard result in convex analysis (see e.g., Rockafellar, 1970, Theorem 5.5 p. 35), for an arbitrary such collection, $e(p, u)$ is a concave function of the price vector $p$, for fixed $u$.

The Hicksian demand $D^h(p)$ is the supergradient of $e(p, u)$ with respect to the vector $p$, i.e, $\frac{\partial e(p,u)}{\partial p_i} = D_i^h(p) = x_i^h$ (this is just a version of the standard Shepard's Lemma from basic microeconomic theory). It follows from a well-known result in convex analysis, which characterizes the subgradients of convex functions (Rockafellar, 1966, 1970), that $D^h(p)$ satisfies (5).

Since the overall utility is quasi-linear in the numeraire, it is well known that the Marshallian demand inherits the downward monotonicity of the Hicksian demand (since there are no income effects for Problem (1)-(2)). Hence $D(p)$ too satisfies the Law of Demand (5). ∎

Recall that in the standard textbook treatment of the relationship between Hicksian and Marshallian demands, the utility function is typically assumed to be strictly quasi-concave. The main advantage of using the given general results from convex analysis is to bring to light the fact that quasi-concavity of the utility function is not needed for this basic result. Interestingly, Mc Kenzie

(1957) proved a version of this result with a general class of preference relations, making use of arguments that were not based on the classical results from convex analysis used above (as the latter became available only in the 1960s).[12]

# 3 The case of quadratic utility

In this section, we investigate the implications of the general results from the previous section that hold when the utility function $U$ is a quadratic function in Problem (1)-(2). Along the way, we also review and build on the basic existing results for the case of a concave utility.

Using the same notation as above, the representative consumer's utility function is now given by (here "$'$" denotes the transpose operation)

$$U(x) = a'x - \frac{1}{2}x'Bx, \tag{6}$$

where $a$ is a strictly positive $n$-vector and $B$ is an $n \times n$ matrix. Without loss of generality, we shall keep the following normalization.

**(A2)** *The matrix $B$* is symmetric and has all its diagonal entries $b_{ii}$ equal to 1.

## 3.1 A strictly concave quadratic utility

For this subsection, we shall assume that the matrix $B$ is positive definite, which is equivalent to the property that the utility function is strictly concave. This constitutes the standard case in the broad literature in industrial organization that relies on quadratic utility (see e.g., Amir and Jin, 2001, and Chone and Linnemer, 2008).

It is well known that such a utility function gives rise to a Bowley-type demand function. We allow a priori for the off-diagonal entries of the matrix to have any sign, although different restrictions will be introduced for some more definite results. Thus, this formulation nests different inter-product relationships, including substitute goods, complementary goods, and hybrid cases.

The consumer's problem is to choose $x$ to solve

$$\max\{a'x - \frac{1}{2}x'Bx + y\} \quad \text{subject to} \quad p'x + y = m \tag{7}$$

[12]We are grateful to Bob Becker for pointing out this reference to us.

As a word of caution, we shall follow the standard abuse of terminology in referring to the demand function at hand as linear demand, although a more precise description would clearly refer to it as being an affine function whenever positive and zero otherwise.

**(A3)** *The matrix $B$ and vectors $a$* and *$p$ in (7) satisfy the interiority and feasibility conditions*

$$B^{-1}(a-p) > 0 \quad \textit{and} \quad p'B^{-1}(a-p) \leq m.$$

As will become clear below, this Assumption is needed not only to obtain an interior solution to the consumer problem (in each product), but also to preserve the linear nature of the resulting demand function.

The following result is well known (see e.g, Amir and Jin, 2001), but included for the sake of stressing the need to make explicit the underlying basic assumptions.[13]

**Lemma 4** *Under Assumptions (A2)-(A3), the inverse demand is given by*

$$P(x) = a - Bx \tag{8}$$

*and the direct demand by*

$$D(p) = B^{-1}(a-p) \tag{9}$$

**Proof.** Since the utility function is quasi-linear in $y$, the consumer's problem (1) can be rewritten as $\max\{a'x - \frac{1}{2}x'Bx + m - p \cdot x\}$. Since $B$ is positive definite, this maximand is strictly concave in $x$. Therefore, whenever the solution is interior, the usual first-order condition with respect to $x$, i.e., $a - Bx - p = 0$, is sufficient for global optimality. Solving the latter matrix equation directly yields

[13] It is worth reporting that we are following much of the economics literature on linear demand that considers a restricted linear demand system as being defined over the proper subset of the positive orthant wherein all prices and all quantities are positive. As an interesting exception to this choice, Shubik and Levitan (1980) prove in the Appendix of their book that there is a unique extension of a linear demand system to the positive orthant with the following property: Whenever the demand for any good reaches zero due to its price being high enough (given other prices), any further unilateral price increase for this good leave the entire demand system unaffected. More recently, a shorter and more insightful proof of this important result appeared in Soon, Zhao and Zhang (2009). In some settings, it is important to explicitly specify the entire demand function (defined for all non-negative prices and quantities). One example of such settings is a two-stage game of R&D/product market competition, wherein a firm may be driven out of the market for some subgames (i.e., for some R&D choices): See e.g., Amir et. al. (2011).

the inverse demand function (8). It is easy to check that this solution is interior under Assumption (A3), as the part $B^{-1}(a-p) > 0$ says that each quantity demanded is strictly positive, and the part $p'B^{-1}(a-p) \leq m$ simply says that $p \cdot D(p) \leq m$, i.e., that the optimal expenditure is feasible.

Since $B$ is positive definite, the inverse matrix $B^{-1}$ exists and is also positive definite (see e.g., McKenzie, 1960). Inverting in (8) then yields (9). ■

At this point, it is worthwhile to remind the reader about three hidden points that will play a clarifying role in what follows. The first two points elaborate on the tacit role of Assumption (A3).

**Remark 5** *In the common treatment of the derivation of linear demand in industrial organization, one tacitly assumes that the representative consumer is endowed with a sufficiently high income. The main purpose of Assumption (A3) is simply to provide an explicit lower bound on how much income is needed for an interior solution. We shall see later on that when Assumption (A3) is violated, the resulting demand is not only non-linear, it is also income-dependent. Thus income effects are then necessarily present, a key departure from the canonical case in industrial organization.*

The second point elaborates on the abscence of income effects, and thus captures the essence of the usefulness of a quasi-linear utility for industrial organization.

**Remark 6** *Suppose that we have an interior solution (8) and (9) for some $m$ such that $p'B^{-1}(a-p) \leq m$. Then it can be easily shown that, for any income level $m' > m$, the solution of the consumer problem continues to be (8)-(9). Thus, as long as income $m$ is higher than the threshold level identified in Assumption (A3), the consumer problem reflects no income effect.*

The third point explains the need for the *strict* concavity of $U$.

**Remark 7** *The reason one cannot simply work with a quadratic utility function that is just concave (but not strictly so) is that, then, a matrix $B$ that is just positive semi-definite (and not positive definite) may fail to be invertible. One immediate implication then is that the direct demand need not be well defined (unless one uses some suitable notion of generalized inverse).*

It is well-known that when $B$ is positive definite, direct and inverse demands are both decreasing in own price (see e.g., Amir and Jin, 2001). In fact, we now observe that a stronger property holds.

**Corollary 8** *If the matrix $B$ is positive definite, both the inverse demand and the direct demand satisfy the strict Law of Demand, i.e., (4).*

**Proof.** This follows directly from Lemma 1, since the Jacobian matrices of the inverse demand and the direct demand are clearly $B$ and $B^{-1}$ respectively, both of which are positive definite. ■

The Law of Demand includes joint restrictions on the dependence of one good's price on own quantity as well as on all cross quantities. It captures in particular the well known property that own effect dominates cross effects.

### 3.2 Integrability of linear demand

In this subsection, we consider the reverse question from the one treated in the previous subsection. Namely, suppose one is given a linear inverse demand function of the form $D(p) = d - Mp$, where $d$ is an $n \times 1$ vector and $M$ is an $n \times n$ matrix, along with the corresponding inverse demand. The issue at hand is to identify minimal sufficient conditions on $d$ and $M$ that will guarantee the existence of a utility function of the form (1), a priori satisfying only continuity and quasi-linearity in the numeraire good, such that $D(p)$ can be obtained as a solution of maximizing that utility function subject to the budget constraint (2).

The framing of the issue under consideration here is directly reminiscent of the standard textbook treatment of integrability of demand, but there are two important distinctions. In the present treatment, on the one hand, we limit consideration to quasi-linear utility, but on the other hand, we do not a priori require the underlying utility function to reflect convex preferences.[14] The latter point is quite important in what follows, in view of the fact that one of the purposes of the present paper is to shed light on the role that the concavity of the quadratic utility function (or lack thereof) plays in determining some relevant properties of the resulting linear demand function. The second distinction from the textbook treatment is that the starting primitives here include both the direct and the inverse demand functions. It turns out that this is convenient for a full characterization.

**Proposition 9** *Let there be given a linear demand function $D(p) = d - Mp$ with $d_i \geq 0$ for each $i$, along with the corresponding inverse demand $P(\cdot)$. Then there exists a continuous utility function*

[14] In independent related work, Nocke and Schutz (2017) consider the classic integrability problem for the case where consumers have quasi-linear but otherwise general (non-quadratic) preferences.

*$U : R^n_+ \longrightarrow R$ such that $D(p)$ can be obtained by solving*

$$\max\{U(x) + y\} \quad \textit{subject to} \quad p'x + y \leq m$$

*if and only if $M$ is a symmetric and positive definite matrix and Assumption A3 holds.*

*Then the desired $U$ is given by the strictly concave quadratic function (6) with $B = M^{-1}$, and both the demand and the inverse demand function satisfy the strict Law of Demand.*

**Proof.** The "if" part was already proved in Lemma 4, with $U$ being the quadratic utility in (6).

For the "only if" part, recall that by Proposition 3, every direct demand function that is the solution to the consumer problem when $U$ is continuous and quasi-linear in the numeraire good (but not necessarily quadratic) satisfies the Law of Demand. Therefore, via Lemma 1, the Jacobian of the affine map $(d - Mp)$, which is equal to $M$, must be positive semi-definite.

Now, since both direct and inverse demands are given, the matrix $M$ must be invertible, and hence has no zero eigenvalue. Therefore, since $M$ is Hermitian, $M$ must in fact be positive definite. This implies in turn that the system of linear equations $Ma = d$ possesses a unique solution $a$, such that $a = M^{-1}d$. Finally, identifying $M$ with $B^{-1}$ yields the fact that the demand function can be expressed in the desired form, i.e., $D(p) = d - Mp = M(a - d) = B^{-1}(a - d)$, as given in (9).

Inverting the direct demand $D(p)$ yields the inverse demand (8). Integrating the latter yields the utility function (6), which is then strictly concave since the matrix $B$ is positive definite.

Finally, the fact that both $D(p)$ and $P(x)$ satisfy the strict Law of Demand (i.e., (4) with a "$<$" sign) then follows directly from Corollary 8. ■

The main message of this Proposition is that any linear demand that is micro-founded in the sense of maximizing the utility of a representative consumer necessarily possesses strong regularity properties. Provided the utility function is quasi-linear in the numeraire good (but a priori not even quasi-concave in the other goods), the linear demand must necessarily satisfy the Law of Demand, and originate from a strictly concave quadratic utility function.

This clear-cut conclusion carries some strong implications, some of which are well understood and reflect assumptions that are commonly made in industrial organization. These include in particular that (i) the demand for each product must be downward-sloping in own price (i.e., no Giffen goods are possible), and (ii) demand cross effects must be dominated by own effects.

On the other hand, the following implication is remarkable, and arguably quite surprising.

**Corollary 10** *If a quadratic utility function as given in (6) is not concave, i.e., if the matrix $B$ is not positive semi-definite, then this utility could not possibly give rise to a linear demand function.*

We emphasize that this conclusion holds despite the fact that the utility function is concave in each good separately (indeed, recall that the matrix $B$ is assumed to have all 1's on the diagonal). The key point here is that joint concavity fails. This immediately raises a natural question: What solution is implied by the first order conditions for utility maximization in case the matrix $B$ is not positive semi-definite, and how does this solution fit in with the Corollary? This question is addressed in the next section, in the context of a fully symmetric utility function, postulated as a simplifying assumption, as in Singh and Vives (1984), Hackner (2000) and others.

### 3.3 Symmetric product differentiation: A common special case

A widely used utility specification for a representative consumer foundation is characterized by a fully symmetric substitution/complementarity matrix, i.e. one in which all cross terms are identical for all pairs of goods and represented by a parameter $\gamma \in [-1, 1]$ (e.g., Singh and Vives, 1984 and Hackner, 2000). The substitution matrix is thus

$$B = \begin{bmatrix} 1 & \gamma & \dots & \gamma \\ \gamma & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \gamma \\ \gamma & \dots & \gamma & 1 \end{bmatrix}, \tag{10}$$

which can be reformulated as (here $I_n$ is the $n \times n$ identity matrix and $J_n$ is the $n \times n$ matrix of all 1's)

$$B \equiv (1 - \gamma)I_n + \gamma J_n.$$

It is common in the industrial organization literature to postulate that the meaningful range for the possible values of $\gamma$ is a priori $[-1, 1]$, with $\gamma \in [-1, 0)$ corresponding to (all goods being) complements, $\gamma \in (0, 1)$ to substitutes, and $\gamma = 0$ to independent goods. While we begin with $[-1, 1]$ being the a priori possible range, we shall see below that for the case of complements, further important restrictions will be needed.

As previously stated, strict concavity of $U$ is sufficient for the first-order condition to provide a solution to the consumer's problem. It turns out that for the special substitution matrix at hand, strict concavity of $U$ can easily be fully characterized.

**Lemma 11** *The quadratic utility function in (6) with $B$ as in (10) is strictly concave if only if $\gamma \in (-\frac{1}{n-1}, 1)$.*

**Proof.** For $U$ to be strictly concave, it is necessary and sufficient that $B$ be positive semi-definite. To prove the latter is equivalent to showing that all the eigenvalues of $B$ are strictly positive. To this end, consider

$$\begin{aligned} B - \lambda I_n &= \begin{bmatrix} 1-\lambda & \gamma & \dots & \gamma \\ \gamma & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \gamma \\ \gamma & \dots & \gamma & 1-\lambda \end{bmatrix} \\ &= \gamma J_n + (1-\lambda-\gamma) I_n \end{aligned}$$

By the well known matrix determinant lemma, we have

$$\begin{aligned} \det[B - \lambda I_n] &= \det[\gamma J_n + (1-\lambda-\gamma) I_n] \\ &= (1-\lambda-\gamma)^{n-1}[1-\lambda+(n-1)\gamma] \qquad (11) \end{aligned}$$

The solutions of $\det[B - \lambda I_n] = 0$ are then $\lambda = 1-\gamma$ and $\lambda = 1+(n-1)\gamma$. Since a priori $\gamma \in [-1, 1]$, by simple inspection, these solutions are $> 0$ if and only if $-1/(n-1) < \gamma < 1$. ■

The following observation follows directly from the Proposition and prior results.

**Corollary 12** *Given a linear demand $D(p) = B^{-1}a - B^{-1}p$ with $B$ as given in (10), $D(p)$ can be derived from a quadratic utility function of the form (6) if only if $\gamma \in (-\frac{1}{n-1}, 1)$, in which case both $D(p)$ and the corresponding inverse demand satisfy the Law of Demand.*

It follows that the range of values of the parameter that validate a linear demand function is not $[-1, 1]$, but rather $(-\frac{1}{n-1}, 1)$. One important direct implication is that there is a fundamental asymmetry between the cases of substitutes and complements. For substitutes, the valid range is indeed $(0,1)$, as is widely believed, and this range is independent of the number of goods $n$. However, for complements, the valid range is $(-\frac{1}{n-1}, 0)$.[15] One interesting implication is that this range monotonically shrinks with the number of goods $n$, and converges to the empty set as the number of goods $n \longrightarrow +\infty$.

This Corollary uncovers an exceptional feature of the ubiquitous two-good case.

**Remark 13** *The special case of two goods ($n = 2$) is the only case for which the valid range of the parameter $\gamma$ for a concave utility, and thus for a well-founded demand, i.e. $(-\frac{1}{n-1}, 1)$, is equivalent to the interval $(-1, 1)$, as commonly (and correctly) believed (e.g., Singh and Vives, 1984).*

Before moving on to explore the properties of the solutions of the first order conditions when the latter are not sufficient for global optimality, we report a remarkable result on the level of wealth needed for an interior solution to the consumer problem.

**Proposition 14** *Consider the quadratic utility function in (6) with $B$ as in (10). As $\gamma \downarrow -\frac{1}{n-1}$, the level of income required to obtain an interior linear demand function converges to $\infty$.*

**Proof.** We first derive a simplified version of Assumption (A3) for the case where the matrix $B$ is as in (10). As $a_i = a > p_i = p$, one clearly has $x > 0$.

To check that $p' \cdot x \leq m$, first note that $b_{ii} = 1$ and $b_{ij} = \gamma$ (for $i \neq j$). In addition, for the matrix $B^{-1}$, each diagonal element is equal to $1 + \frac{(n-1)\gamma^2}{(1-\gamma)[1+(n-1)\gamma]}$ , and each off-diagonal term is $\frac{-\gamma}{(1-\gamma)[1+(n-1)\gamma]}$ (for details, see the proof of Lemma 15 below). Therefore, upon a short computation, $p \cdot x = \frac{np(a-p)}{1+(n-1)\gamma}$. The latter fraction converges to $+\infty$ as $\gamma \downarrow -\frac{1}{n-1}$ (since its numerator is $> 0$).

Since Assumption (A3) requires that $p \cdot x = \frac{np(a-p)}{1+(n-1)\gamma} \leq m$, the conclusion follows. ■

This Proposition is a powerful criticism of the assumption that the representative consumer is endowed with a sufficient income level to allow for an interior solution to the utility maximization problem, in cases where the products under consideration are strong complements (i.e., for $\gamma$ close to the maximal allowed value of $-\frac{1}{n-1}$). An interesting implication is that for a given wealth level of the representative consumer, no matter how high, if products are sufficiently complementary *in the permissible range*, i.e., if $\gamma = -\frac{1}{n-1} + \varepsilon$ with $\varepsilon$ small enough, then the consumer's true demand

[15] This point was already noted in some past work, see e.g., Bloch (1995). However, the connections to the well-foundedness of linear demand functions investigated here were not addressed.

will not be the linear demand function. Two examples in Section 4 shed light on what the true demand function might look like in this case.

In conclusion, if one needs to require infinite wealth to rationalize a linear demand system for complements, then perhaps it is time to start questioning the well-foundedness of such demand functions. Put differently, perhaps industrial economists have been overly valuing the analytical tractability of linear demand.

We next move on to illustrate some pathological features that might emerge when one uses linear demand functions in the absence of a strictly concave quadratic utility function.

## 4 Issues with the solution to the first order conditions

Continuing with our investigation of the foundations of the linear demand specification, in this section, we address four key issues that arise when considering linear demand with a matrix $B$ in (10) that is not positive semi-definite.[16] The first critical issue is that this saddle-point demand ends up being negative. The second issue is to clarify that, while the case of perfect substitutes (or $\gamma = +1$) is excluded by the present analysis, it is actually well-defined, but requires a one-good version of our consumer problem as the appropriate foundation, as commonly done for homogeneous goods. The third issue is to consider the boundary case of two perfect complements, i.e., $n = 2$ and $\gamma = -1$, and actually solve for the true demand function, in order to shed light on the nature of saddle-point linear demand functions in the abscence of micro-economic foundations.[17] The fourth issue is an illustration reporting on an older study that relies on an invalid demand function in an attempt to reverse a well-known result in oligopoly theory.

We begin with an intermediate result showing that the direct demand may be derived from the inverse demand, even when $B$ is not positive definite (the proof is in the Appendix). This result is

[16] When $\gamma \in \left(-1, -\frac{1}{n-1}\right)$, we know that the linear demand function that solves the first-order conditions of the consumer problem is not the true demand function. In other words, it is a saddle point of the consumer utility maximization problem, but not the global solution. In particular, as implied by the general results of the previous section, this saddle-point demand function does not satisfy the Law of Demand overall.

[17] In the general theory of quadratic programming, similar features are known to arise when suitable second order conditions do not hold (e.g., Burer and Letchford, 2009). An interesting survey of the literature on quadratic programming without global convexity assumptions (for minimization) is provided by Floudas and Visweswaran (1994).

useful elsewhere in the paper, e.g., in Proposition 14.

**Lemma 15** *If $\gamma \neq 1$ and $\gamma \neq -\frac{1}{n-1}$, the matrix $B = (1-\gamma)I_n + \gamma J_n$ is non-singular and*

$$B^{-1} = \frac{1}{1-\gamma}\left[I_n - \frac{\gamma}{(n-1)\gamma+1}J_n\right]. \tag{12}$$

**Proof.** Let $d$ and $e$ be the diagonal and off-diagonal elements of $B^{-1}$. By definition we have $d + (n-1)\gamma e = 1$ and $\gamma d + b + (n-2)\gamma e = 0$.

Solving these two equations we find:

$$d = \frac{1+(n-2)\gamma}{(1-\gamma)[1+(n-1)\gamma]} \text{ and } e = -\frac{\gamma}{(1-\gamma)[1+(n-1)\gamma]}$$

■

The first issue is the possible emergence of negative demand.

### 4.1 The case of negative demand

The first key issue that is worth pointing out when the matrix $B$ is not positive definite is that the solution to the first-order conditions may lead to negative demand. Indeed, the inverse demand $p_i = a - x_i - \gamma \sum_{j\neq i} x_j$ reduces (when all quantities are equal, $x_i = x_j = x$) to $p = a - [1+\gamma(n-1)]x$. Hence, if $\gamma < -\frac{1}{n-1}$, direct demand is given by $x = \frac{a-p}{1+\gamma(n-1)}$, which is strictly negative for $p < a$.

Thus, the saddle-point demand functions are not even well-defined in a very elementary sense when $\gamma < -\frac{1}{n-1}$ (i.e., goods are strongly complementary).

In what follows, we explore the boundary cases of $\gamma = 1$ (for any $n$), and $\gamma = -1$ and $n = 2$ (for simplicity).

### 4.2 The case of perfect substitutes ($\gamma = 1$)

One surprising outcome of the analysis of Section 3 is that it excludes the case of perfect substitutes or $\gamma = +1$. Here, we shall argue that this is more a matter of representation than of substance, and that perfect substitutes are to be analysed in the setting of a linear demand for homogeneous goods, as is commonly done.

Consider a representative consumer maximizing a utility function for two perfect substitutes (this is only for simplicity, as the case of $n$ perfect substitutes is similarly handled)

$$\max\{x_1 + x_2 - 0.5(x_1 + x_2)^2 + y\} \tag{13}$$

subject to the budget constraint

$$p_1 x_1 + p_2 x_2 + y = m.$$

This utility function $U$ is strictly concave in each of the two goods $x_1$ and $x_2$ separately, as well as jointly concave (but not strictly) in the two goods.

We cannot solve for separate demand functions for $x_1$ and $x_2$, since they are identical goods. Instead, we let $p_1 = p_2 = p$ and $x = x_1 + x_2$, and solve the consumer problem

$$\max\{x - 0.5x^2 + y\} \quad \text{subject to} \quad px + y = m.$$

The direct demand function is then (we include the demand for the numeraire $y$ for completeness)

$$(x^*, y^*) = \begin{cases} (1-p, m-(1-p)p) \text{ if } p < 1 \text{ and } m \geq (1-p)p \\ (m/p, 0) \text{ if } p < 1 \text{ and } m < (1-p)p \\ (0, m) \text{ if } p \geq 1 \end{cases}$$

We thus recover the standard direct/inverse demand for homogeneous goods, namely

$$x^* = D(p) = 1 - p, \text{ or } p = D^{-1}(x) = 1 - x$$

which can be converted to the familiar $p = a - bx$ by suitably inserting constants $a$ and $b$ in (13). Therefore, the fact that the analysis of this paper precludes the case $\gamma = +1$ is simply a reflection of the fact that the utility representation (6) is not suitable for this important special case.

As an ancillary implication of this simple analysis, observe that when the usual ”enough money” condition fails, i.e., when $m < (1-p)p$ in this case, the resulting demand function is not linear, but hyperbolic (with unitary elasticity), i.e., $x^* = m/p$.

### 4.3 The case of two perfect complements ($n = 2$ and $\gamma = -1$)

We have established that, when goods are close to being perfect complements, the ubiquitous ”enough income” condition in the literature is not as innocuous as conventional wisdom suggests.

Indeed the linear demand would require unbounded income when goods are perfect complements. This bring up the natural question, what does the actual demand function look like when goods are perfect complements and the consumer's income is fixed and finite? The answer to this simple question, which seems to have been ignored in the literature so far, will shed light on the crucial role played by the strict concavity of the utility function in linear demand theory.[18]

We restrict attention to the two-good case for simplicity. Nevertheless, it turns out that even with two goods, the demand function is surprisingly complex and violates several of the usual attributes of demand functions in partial equilibrium analysis, starting with linearity itself.

Consider a representative consumer with a utility function for two goods with $\gamma = -1$, i.e.,

$$U = x_1 + x_2 - 0.5x_1^2 - 0.5x_2^2 + x_1 x_2 \tag{14}$$

The consumer problem is then

$$\max\{x_1 + x_2 - 0.5(x_1 - x_2)^2 + y\}$$

subject to the budget constraint

$$p_1 x_1 + p_2 x_2 + y = m.$$

While the utility function $U$ is strictly concave in each of the two goods $x_1$ and $x_2$ separately, it is also jointly concave, but not jointly strictly concave, in the two goods.

Without loss of generality, we assume that $p_1 \leq p_2$. Then the direct demand function is fully described in the next result (a shortened proof is given in the Appendix).

**Proposition 16** *For the case of two goods as perfect complements and the numeraire good $y$ (with*

[18] Actually, in industrial organization, it is not uncommon to find studies that postulate the valid range as being the closed interval $[-1, 1]$, instead of the open $(-1, 1)$.

*utility function (14)), the demand function for the three goods is given by (with $p_1 \leq p_2$) :*

$$(y, x_1, x_2) = \begin{cases} (i)\left(0, \frac{1}{p_1+p_2}\left[m + \frac{(p_1-p_2)p_2}{p_1+p_2}\right], \frac{1}{p_1+p_2}\left[m + \frac{(p_2-p_1)p_1}{p_1+p_2}\right]\right) \\ \quad \text{if } m > \frac{(p_2-p_1)p_1}{p_1+p_2} \text{ and } p_1+p_2 < 2 \\ (ii)\left([0, m-(1-p_1)p_1], \frac{1}{2}\left[m-y+(p_1-1)(2-p_1)\right], \frac{1}{2}\left[m-y+(p_2-1)(2-p_2)\right]\right) \\ \quad \text{if } m > \frac{(p_2-p_1)p_1}{2} \text{ and } p_1+p_2 = 2 \\ (iii)\,(m-(1-p_1)p_1, 1-p_1, 0) \\ \quad \text{if } m \leq \frac{(p_2-p_1)p_1}{p_1+p_2} \text{ or } p_1+p_2 > 2, p_1 < 1 \text{ and } m \geq (1-p_1)p_1 \\ (iv)\,(0, m/p_1, 0) \;\; \text{if } m \leq \frac{(p_2-p_1)p_1}{p_1+p_2} \text{ or } p_1+p_2 > 2, p_1 < 1 \text{ and } m < (1-p_1)p_1 \\ (v)\,(m, 0, 0) \;\; \text{if } m \leq \frac{(p_2-p_1)p_1}{p_1+p_2} \text{ or } p_1+p_2 > 2 \text{ and } p_1 \geq 1 \end{cases}$$

This a priori simple example serves to illustrate quite neatly the fact that even borderline violations of strict concavity of a quadratic utility function lead to drastic departures from the familiar outcome of linear demand. Indeed, the resulting demand is quite complex in many ways; in particular, (i) it is not a single-valued function, but an upper hemi-continuous correspondence (with branch *(ii)* having a continuum of values in the range), (ii) it is highly non-linear in prices (in fact, it reflects a complex dependence on prices), (iii) it is dependent on income in important ways (violating the key feature of abscence of income effects), and (iv) the structure of the demand function changes in significant ways across five different parameter regions.

## 4.4 An example of the use of an unfounded demand function

The next example appears in a classic study on the comparison between Cournot and Bertrand equilibria. This will serve to illustrate that erroneous conclusions may easily be obtained when one uses linear demand functions that are not derived from a strictly concave utility function.

**Example 17** *Okuguchi (1987) uses the following demand specification to show that equilibrium prices may be lower under Cournot than under Bertrand.*

$$p_i = \frac{1}{8}(2 + x_i - 3x_j) \text{ and } x_i = 1 - p_i - 3p_j \text{ (for } i \neq j). \tag{15}$$

*Two violations of standard properties stand out: (i) the inverse demand is upward-sloping, (ii) the two products appear to be complements in the inverse demand function, but substitutes in the direct demand function.*

*The candidate utility function to conjecture as the origin of this demand pair is clearly*

$$U = \frac{1}{8}(2x_1 + 2x_2 - 3x_1x_2 + 0.5x_1^2 + 0.5x_2^2) + y.$$

*In contrast to our treatment so far, this utility function is strictly convex (and not concave) in each good separately, though not jointly strictly convex.*

*It can easily be shown by solving the usual consumer problem with this utility function that the resulting demand solution is not the one given in (15). The true solution includes some of the same complex features encountered in the previous example (the solution is not derived here for brevity). This confirms what the results of the present paper directly imply for this demand pair, namely that it cannot be micro-founded in the sense of maximizing the utility of a representative consumer.*

*Therefore, this demand pair is essentially invalid, and thus the fact that it leads to Bertrand prices that are higher than their Cournot counterparts does not a priori constitute a valid counter-argument to the well known positive result under symmetry (which says that Bertrand prices are lower than Cournot prices; see Vives, 1985; and Amir and Jin, 2001).*

## 5 Cournot and Bertrand Oligopolies

Here we consider the standard models of Cournot and Bertrand oligopolies with linear (fully symmetric) demand, and linear costs normalized to zero (w.l.o.g). We show that under strict concavity of the utility function, the two standard oligopoly models have well-defined profit functions and intuitively well-behaved reaction curves.

The profit functions for firm $i$ under Cournot and Bertrand competition are respectively

$$\Pi_i^C(q) = q_i(a - b_i \cdot q) \tag{16}$$

and

$$\Pi_i^B(p) = p_i b_i^{-1}(a - p) \tag{17}$$

where $b_i$ and $b_i^{-1}$ are the $i$th row of $B$ and $B^{-1}$ respectively.

The first result deals with the strict concavity of the profit functions in the two types of oligopoly.

**Proposition 18** *If $\gamma \in \left(-\frac{1}{n-1}, 1\right)$, the profit functions for Cournot and Bertrand oligopolies, given in (16) and (17), are strictly concave in own action.*

**Proof.** The proofs of concavity come directly from the second-order conditions for maximization with respect to own action in (16) and (17). Indeed, $\frac{\partial^2 \Pi_i^C(q)}{\partial q_i^2} = -2b_{ii} = -2$ and $\frac{\partial^2 \Pi_i^B(p)}{\partial p_i^2} = -2b_{ii}^{-1}$, which from (12) can be seen to be $< 0$ if $\gamma \in \left(-\frac{1}{n-1}, 1\right)$. The details are omitted. ■

It worth observing from the proof that, for Bertrand competition, the profit function is not strictly concave in own action for all values of $\gamma$ in $(-1, 1)$, but essentially requires the same range for $\gamma$ as the utility maximization problem. Hackner (2000) imposes similar restrictions on the range of $\gamma$, as second order conditions for firms' profit maximization.

We now consider a more informal criterion based on the monotonicity of reaction curves for the two oligopolies. A widely held belief about the differences between Bertrand and Cournot oligopolies is that, under broad representative specifications for both models, Bertrand reaction curves should be upward-sloping under substitutes and downward-sloping under complements, while the reverse should hold for Cournot. For the two-good case with linear demands, this is clearly the case, as was elaborated upon by Singh and Vives (1984). We now check whether these intuitive beliefs about the slopes of reaction curves survive in the $n$-good case for the two types of oligopoly. We first see that this is indeed the case for Cournot competition in the $n$-good case.

**Proposition 19** *If $\gamma \in \left(-\frac{1}{n-1}, 1\right)$, under Cournot competition, the reaction curves are*

*(i) always downward-sloping for substitutes ($\gamma > 0$), and*

*(ii) upward-sloping for complements ($\gamma < 0$).*

**Proof.** This is clearly the case since all the off-diagonal elements of $B$ are simply equal to $\gamma$. ■

**Proposition 20** *If $\gamma \in \left(-\frac{1}{n-1}, 1\right)$, under Bertrand competition, the reaction curves are*

*(i) always upward-sloping for substitutes ($\gamma > 0$), and*

*(ii) downward sloping for complements ($\gamma < 0$).*

**Proof.** The reaction function for firm $i$ under Bertrand competition is given by

$$p_i = \frac{b_i^{-1} a}{2b_{ii}^{-1}} - \frac{b_{i,-i}^{-1} p_{-i}}{2b_{ii}^{-1}}$$

with $-i$ indicating that the $i$th element has been removed. We want to show that (see (12))

$$\frac{\partial p_i}{\partial p_j} = -\frac{b_{ij}^{-1}}{2b_{ii}^{-1}} = \frac{\gamma}{2[(n-1)\gamma + 1 - \gamma]} < 0$$

For $\gamma > 0$, the result holds trivially since $\gamma < 1$. For $\gamma < 0$, it is sufficient that $(n-1)\gamma + 1 > 0$ to imply the desired result. ■

Thus, the conventional wisdom about the slopes of reaction curves in Cournot and Bertrand competition is confirmed without any ambiguity.

## 6 Gross substitutes (complements) versus substitutes (complements) in utility

The purpose of this section is to explore the relationship between the standard notions of gross substitutes/complements and the alternative definition of substitute/complement relationships via the utility function. The latter notion of complementarity between goods goes back all the way to Edgeworth (1881), and is quite standard in many different contexts in economics (e.g., Vives, 1999).

The main finding argues that two products may well appear as substitutes in a quadratic utility function, even though they constitute gross complements in demand. On the other hand, we also establish that when all goods are complements in a quadratic utility function, then any two goods necessarily appear as gross complements in demand as well.

We begin with the formal definitions of the underlying notions, along with some general remarks.

**Definition 21** *(a) Two goods $i$ and $j$ are said to be gross substitutes (gross complements) if* $\partial x_i^*/\partial p_j = \partial x_j^*/\partial p_i \geq (\leq) 0$.

*(b) Two goods $i$ and $j$ are said to be substitutes (complements) in utility if the utility function $U$ has increasing differences in $(x_i, x_j)$, or for smooth utility, if $\partial^2 U(x)/\partial x_i \partial x_j \leq (\geq 0)$ for all $x$.*

Part (a) is a standard notion in microeconomics. On the other hand, though a useful and well defined notion, Edgeworth's (1881) definition in part (b) is not as widely used in demand theory.[19]

The following remark will prove useful below.

[19] Note that the condition $\partial^2 U(x)/\partial x_i \partial x_j \leq (\geq 0)$ is simply the smooth equivalent of the property of submodularity (supermodularity) that are increasingly used in various contexts in economics (see e.g., Vives, 1999, for a book treatment, or Amir, 2005, for an elementary survey).

**Remark 22** *One can also define substitutes (complements) with respect to the inverse demand function, in the obvious way: i and j are substitutes (complements) if* $\partial P_i/\partial x_j = \partial P_j/\partial x_i \leq (\geq) 0$*. However, since the inverse demand is simply the gradient of the utility function here, this new definition would simply coincide with part (b).*[20]

It is generally known that for two-good linear demand, the two definitions are equivalent, namely two goods that are gross substitutes (complements) are always substitutes (complements) in utility as well, and vice versa (see e.g., Singh and Vives, 1984). On the other hand, this is not necessarily the case for three or more goods, as we now demonstrate.

**Example 23** *Consider a quadratic utility function* $U(x) = a'x - \frac{1}{2}x'Bx$ *with* $a > 0$ *and*

$$B = \begin{bmatrix} 1 & 3/4 & 0.5 \\ 3/4 & 1 & 3/4 \\ 0.5 & 3/4 & 1 \end{bmatrix}.$$

*It is easy to verify that this matrix is positive definite, so that* $U$ *is strictly concave. Hence the first order conditions define a valid inverse demand function, and we are thus in the standard situation. It is also easy to check that the inverse of* $B$ *is*

$$B^{-1} = \begin{bmatrix} 7/3 & -2 & 1/3 \\ -2 & 4.0 & -2 \\ 1/3 & -2 & 7/3 \end{bmatrix}$$

*Recall that the slopes of both inverse and direct demands do not depend on the vector* $a$ *(though both intercepts do depend on* $a$*). Hence, invoking the above Remark, one sees by inspection that all goods are substitutes in utility (or according to inverse demand), including in particular goods 1 and 3. On the other hand, the latter two goods are clearly gross complements (according to* $B^{-1}$*).*

This possibility is actually quite an intuitive feature, as we shall argue below by providing a basic intuition for it. Nonetheless, this might well appear paradoxical at first sight because we tend

[20] In other words, one always has directly from the first order conditions

$$\frac{\partial^2 U(x)}{\partial x_i \partial x_j} = \frac{\partial P_i}{\partial x_j} = \frac{\partial P_j}{\partial x_i}.$$

to be over-conditioned by observations that hold clearly for the standard two-good case, but are actually not fully robust when moving to a multi-good setting (similar counter-examples are easy to construct whenever $n \geq 3$).

The intuition behind this switch is quite easy to grasp. Assume for concreteness that we consider an exogenous increase in $p_3$. This leads to a lower demand for good 3, but a higher demand for goods 1 and 2 through substitution. The latter effect impacts good 2 relatively more than for good 1 (due to a constant of .75 for 3-2 versus 0.5 for 3-1). A second effect is that the large increase in the consumption of good 2 ends up driving down that of good 1 (as the two are substitutes in utility). The overall effect of the increase in $p_3$ is a decrease in the consumption of both goods 3 and 1, which thus emerge as gross complements.

Consider next a three-good utility function with all goods as complements in utility instead. Adapting the foregoing intuition to this case makes it clear that any two goods will emerge as gross complements. In fact, we now prove that this constitutes a general conclusion for the $n$-good case.

**Proposition 24** *Consider an $n$-good concave quadratic utility $U$ that is supermodular in $x$ (i.e., $\frac{\partial^2 U(x)}{\partial x_i \partial x_j} \geq 0$ for all $i \neq j$). Then all the goods are gross complements.*

**Proof.** Since $\frac{\partial^2 U(x)}{\partial x_i \partial x_j} \geq 0$ for all $i \neq j$, all the off-diagonal elements of $B$ are negative. Since $B$ is positive semi-definite, it follows from a well known result (see e.g., Mc Kenzie, 1960) that all the off-diagonal elements of $B^{-1}$ are positive. This in turn implies directly, via (9), that all goods are gross complements. ■

In this case, all the reactions to a given price change move in the same direction, in a mutually reinforcing manner, so complementarity in utility across all goods always carries over to gross complementarity between every pair of goods.

In conclusion, concerning the relationship between the two different notions of substitutes and complements at hand for the $n$-good case, the only one of the four possible implications that extends from the two-good case is the one given in the preceding Proposition.

## 7 A linear demand with local interaction

In this section, we introduce one more alternative form of the substitution matrix $B$ that may be of interest in particular economic applications. Specifically, we suggest a particular substitution matrix based on product similarities, place it in the context of our broader study of linear demand, and highlight interesting properties of the resulting inverse and direct demands.

Consider a consumer with a preference ordering over goods based on their similarities. The main idea consists in capturing the intuitive notion that the closer products are in their characteristics, the closer substitutes they ought to be. Specifically, the consumer has preferences over $n$ goods horizontally differentiated along one dimension, with the goods uniformly dispersed over a compact interval in that dimension. Without loss of generality, let $i = i, \dots, n$ represent the order of the products over the interval. Consider a quadratic utility function (6) where $B$ is now a Kac-Murdock-Szegö matrix, that is, a symmetric $n$-Toeplitz matrix whose $ij$-th term is

$$b_{ij} = \gamma^{|i-j|}, \quad i, j = 1, \dots, n. \tag{18}$$

As such matrices were first defined in Kac, Murdock, and Szegö (1953), we will refer to this as the KMS model. We focus on the case $\gamma \in (0, 1)$, so that all products are substitutes. In this specification, the price of any good $i$ responds to changes in the quantity of every other good $j$ with a magnitude that decreases with the distance between $i$ and $j$ in characterisitic space.

It is well known[21] that this matrix is positive-definite for $\gamma \in (0, 1)$ and has the inverse[22]

$$B^{-1} = \frac{1}{1-\gamma^2} \begin{bmatrix} 1 & -\gamma & 0 & \dots & 0 \\ -\gamma & 1+\gamma^2 & -\gamma & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & -\gamma & 1+\gamma^2 & -\gamma \\ 0 & \dots & 0 & -\gamma & 1 \end{bmatrix}. \tag{19}$$

While inverse demand facing a given firm is a function of all other goods, direct demand is only a function of the two adjacent substitutes for interior firms, and one adjacent substitute for

[21] See, for example, Horn and Johnson (1985, Section 7.2, Problems 12-13)

[22] Due to the location of two products at the end points of the segment (that can thus have only one neighbor instead of two), direct demand is no longer fully symmetric.

the two firms at the edges. As an example of a demand system with such structure in empirical industrial organization, consider the vertically differentiated model for the automobile industry due to Bresnahan (1987), with equal quality increments. The key idea in this model is to capture the intuitive fact that a given car is in direct competition only with cars of similar qualities.

From the general results of the present paper, we easily deduce that this demand system is well-defined for all $\gamma \in (0, 1)$.

**Corollary 25** *Both inverse and direct demand in the KMS model satisfy the strict Law of Demand.*

**Proof.** The proof follows directly from Corollary 8, since the matrix $B$ is positive definite. ■

This demand formulation has different implications for oligopolistic competition between firms (when each firm sells one of the varieties), depending on the mode of competition. Under Cournot competition, each firm competes with all other firms, but reacts more intensely to those whose products are more similar to its own. In contrast, under Bertrand competition, each firm directly competes only with its one or two adjacent rivals, i.e., those with very similar products (with respect to horizontal differentiation). With respect to those similar firms, previous results still hold. Specifically, as in Singh and Vives (1984), the Bertrand reaction curve for a firm with respect to its direct neighbors is upward sloping.

**Proposition 26** *In the KMS model, each firm $i$ price competes only with its closest substitutes, $i + 1$ and $i - 1$. With respect to these two rivals, firm $i$'s reaction curve is upward sloping.*

**Proof.** Reaction curves can be derived as in Proposition 20, yielding the derivative (here, $b_{ij}^{-1}$ is the $ij$-th term of the matrix $(1 - \gamma^2)B^{-1}$)

$$\frac{\partial p_i}{\partial p_j} = -\frac{b_{ij}^{-1}}{2b_{ii}^{-1}} = \begin{cases} \frac{\gamma}{K} & \text{for } |i - j| = 1 \\ 0 & \text{for } |i - j| > 1 \end{cases}$$

with $K = 2 > 0$ for boundary firms and $K = 2(1+\gamma^2) > 0$ for interior firms. The conclusion follows from the fact that $\frac{\gamma}{K} > 0$. ■

The KMS model thus highlights another interesting lack of duality between oligopolistic price and quantity competition, which is a result of a lack of duality between inverse and direct demands. When firms compete over price, a type of local strategic interaction takes place in that each firm *directly* takes into account the behavior of only their direct neighbors (though in equilibrium, every firm's action will still end up indirectly being a function of all the rivals' actions). However, when firms compete over quantity, they directly take into consideration the behavior of all the other firms in the industry (as they do in the standard cases).

## 8 Conclusion

This paper provides a thorough exploration of the theoretical foundations of the multi-variate linear demand function for differentiated products, which is widely used in industrial organization. For the question of integrability of linear demand, a key finding is that strict concavity of the quadratic utility function of the representative consumer is necessary and sufficient for the resulting demand system to be well defined. Without strict concavity, the true demand function may be quite complex, non-linear and income-dependent, as shown via example. The role of the common, but often tacit, assumption that a representative consumer with quadratic utility must hold sufficient wealth to give rise to a linear demand is explored and clarified in some detail. The relationship between the standard notions of gross substitutes and gross complements on the one hand, and their respective counterparts from the utility function as defined by Edgeworth (1881) are investigated in full detail for any number of goods. The pairwise equivalence between these notions in the two-good case is shown to break down for three or more goods.

The paper uncovers a number of failures of duality relationships between substitute products and complementary products, as well as the incompatibility of high levels of complementarity and the well-foundedness of linear demands. The two-good case often investigated since the pioneering work of Singh and Vives (1984) emerges as a special case with strong but non-robust properties.

A key implication of our results is that all conclusions and policy prescriptions derived via the use of a linear multi-variate demand function that does not satisfy the Law of Demand ought to be regarded a priori with some suspiscion, as such demand functions are saddle-point solutions to the consumer problem. Instead, the latter's global optima would give rise to non-linear, complex and multi-valued demand functions that would be highly intractable for widespread use in economics.

# 9 Appendix: Missing proofs

This Appendix contains the proof of Proposition 16.

**Proof of Proposition 16.**

(i) Assume $(p_2 - p_1)p_1/m < p_1 + p_2 \leq 2$. Given any $y$, if both $x_1$ and $x_2$ are positive, the budget constraint implies $x_1 = (m - y - p_2 x_2)/p_1$. Substituting this into the objective function, we get

$$U + y = y + \frac{m - x_0 - p_2 x_2}{p_1} + x_2 - 0.5[\frac{m - x_0 - p_2 x_2}{p_1} - x_2]^2 \tag{A1}$$

Differentiating (A.1) with respect to $x_2$ yields

$$\frac{\partial U}{\partial x_2} = 1 - \frac{p_2}{p_1} + (\frac{m - y - p_2 x_2}{p_1} - x_2)(1 + \frac{p_2}{p_1})$$

Hence $\frac{\partial U}{\partial x_2} > 0$ if and only if $m - y > \frac{(p_2 - p_1)p_1}{p_1 + p_2} + (p_2 + p_1)x_2$. Since $(p_2 - p_1)p_1/m < p_1 + p_2$, when $y$ is sufficiently small, we always have positive demands for both goods:

$$x_1 = \frac{1}{p_1 + p_2}[m - y + \frac{(p_2 - p_1)p_2}{p_1 + p_2}], \text{ and } x_2 = \frac{1}{p_1 + p_2}[m - y - \frac{(p_2 - p_1)p_1}{p_1 + p_2}].$$

The corresponding value of $U$ is $\widehat{U} = y + \frac{2(m-y)}{p_1+p_2} + 0.5(\frac{p_2-p_1}{p_1+p_2})^2$. Clearly, $\frac{\partial \widehat{U}}{\partial y} = 1 - \frac{2}{p_1+p_2}$.

If $p_1 + p_2 < 2$, we can raise $\widehat{U}$ by lowering $y$. Hence, we get $y = 0$ and

$$x_1 = \frac{1}{p_1 + p_2}[m + \frac{(p_2 - p_1)p_2}{p_1 + p_2}] \text{ and } x_2 = \frac{1}{p_1 + p_2}[m - \frac{(p_2 - p_1)p_1}{p_1 + p_2}].$$

If $p_1 + p_2 = 2$, then $\frac{\partial \widehat{U}}{\partial y} = 0$, so we have $0 \leq y \leq m - (1 - p_1)p_1 = m + (1 - p_2)(2 - p_2)$, and

$$x_1 = 0.5[m - y + (1 - p_1)(2 - p_1)] \text{ and } x_2 = 0.5[m - y + (1 - p_2)(2 - p_2)]$$

If $m \leq \frac{(p_2-p_1)p_1}{p_1+p_2}$, we always have $\frac{\partial \widehat{U}}{\partial y} < 0$. So we should set $x_2 = 0$.

If $m > \frac{(p_2-p_1)p_1}{p_1+p_2}$ but $p_1 + p_2 = 2$, we could a priori have positive $x_1$ and $x_2$, but we also have $\frac{\partial \widehat{U}}{\partial y} > 0$. So we should increase $y$ till $x_2 = 0$. This implies that we cannot have positive $x_1$ and $x_2$ at the same time. It is then straightforward to show that the demands $x_1$ and $x_2$ are as specified in the Proposition (the details are left out).

## References


[1] Amir (2005), Supermodularity and complementarity in economics: An elementary survey, *Southern Economic Journal*, 636-660.

[2] Amir, R., C. Halmenschlager, J. Jin (2011), R&D-induced industry polarization and shake-outs, *International Journal of Industrial Organization* 29, 386-398.

[3] Amir, R., Jin, J. (2001), Cournot and Bertrand equilibria compared: Substitutability, complementarity and concavity, *International Journal of Industrial Organization* 19, 303-317.

[4] Amir, R., J. Jin, M. Troege, G. Pech (2016), Prices, deadweight loss and subsidies in multi-product monopoly, *Journal of Public Economic Theory*, 18, 346–362.

[5] Armstrong, M. and J. Vickers (2015), Which demand systems can be generated by discrete choice?, *Journal of Economic Theory*, 158, 293-307.

[6] Aubin, J.-P.(2007), *Mathematical Methods of Game and Economic Theory*, Dover Publications Inc., Mineola, N.Y.

[7] Bloch, F. (1995), Endogenous structures of association in oligopolies, *The RAND Journal of Economics*, 537-556.

[8] Bowley, A. L. (1924), *The mathematical groundwork of economics: an introductory treatise.* Oxford.

[9] Bresnahan, T. F. (1987), Competition and Collusion in the American Automobile Industry: The 1955 Price War, *The Journal of Industrial Economics*, 35, 457-482.

[10] Burer, S. and Letchford, A. (2009), On non-convex quadratic programming with box constraints, *SIAM Journal on Optimization* 20, 1073-1089.

[11] Choné, P. and L. Linnemer (2008), Assessing horizontal mergers under uncertain efficiency gains, *International Journal of Industrial Organization*, 26, 913–929.

[12] Deneckere, R. (1983), Duopoly supergames with product differentiation, *Economics Letters*, 37-42.

[13] Dixit, A. (1979), A model of duopoly suggesting a theory of entry barriers, *Bell Journal of Economics*, 10, 20-32.

[14] Edgeworth, F. (1881), *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences*, Reprints of Economic Classics, New York: Augustus Kelley Publishers.

[15] Floudas, C. A., Visweswaran, V. (1994), Quadratic Optimization, in *Handbook of Global Optimization*; Horst, R., Pardalos, P. M., Eds.; Kluwer Academic Publishers, 217-270.

[16] Häckner, J. (2000), A note on price and quantity competition in differentiated oligopolies, *Journal of Economic Theory*, 93, 233-239.

[17] Hildenbrand, W. (1994), *Market Demand: Theory and Empirical Evidence*, Princeton University Press, Princeton, N.J.

[18] Horn, R. A., Johnson, C. R. (1985), *Matrix Analysis*, Cambridge University Press.

[19] Hsu, J., X. H. Wang (2005), On welfare under Cournot and Bertrand competition in differentiated oligopolies, *Review of Industrial Organization* 27, 185-191.

[20] Kac, M., Murdock, W. L., Szegö, G. (1953), On the eigenvalues of certain Hermitian forms, *J. Rat. Mech. and Anal.*, 2, 787-800.

[21] LaFrance, J. T. (1985), Linear demand functions in theory and practice, *Journal of Economic Theory*, 37, 147-166.

[22] Martin, S. (2002), *Advanced Industrial Organization*, 2nd Edition, Wiley, Malden, MA.

[23] McKenzie, L. (1957), Demand theory without a utility index, *The Review of Economic Studies*, 24, 185-189.

[24] McKenzie, L. (1960), Matrices with dominant diagonals and economic theory, in Arrow, Karlin and Suppes (eds.) *Mathematical Methods in Social Sciences*, Stanford University Press.

[25] Milleron, J.C. (1974), Unicité et stabilité de l'équilibre en économie de distribution, Séminaire d'Econométric Roy-Malinvaud, Preprint.

[26] Mitjushin, L.G. and V.M. Polterovich (1978), Criteria for monotonicity of demand functions, (in Russian), *Ekonomika i Matematicheskie Metody*, 14, 122–128.

[27] Nocke, V. and N. Schutz (2017), Quasi-linear integrability, *Journal of Economic Theory*, to appear.

[28] Okuguchi, K.(1987), Equilibrium prices in the Bertrand and Cournot oligopolies, *Journal of Economic Theory*, 42, 128-139.

[29] Rockafellar, T. (1966), Characterization of the subdifferentials of convex functions, *Pacific Journal of Mathematics*, 17, 497–510.

[30] Rockafellar, T. (1970), *Convex Analysis*, Princeton University Press, Princeton, N.J.

[31] Shubik, M. (1959), Strategy and Market Structure, Wiley, New York.

[32] Shubik, M. , R. Levitan (1980), Market Structure and Behavior, Harvard University Press, Cambridge, MA.

[33] Singh, N. and X. Vives (1984), Price and quantity competition in a differentiated duopoly, *Rand Journal of Economics* 15, 546-554.

[34] Soon, W., G. Zhao and J. Zhang (2009). Complementarity demand functions and pricing models for multi-product markets, *European Journal of Applied Mathematics,* 20, 399-430.

[35] Topkis, D. (1998), *Supermodularity and Complementarity*, Princeton University Press, Princeton, N.J.

[36] Vainberg, M. M. (1973). *The variational method and the method of monotone operators in the theory of nonlinear equations*, Wiley, New York.

[37] Vives, X. (1987), Small income effects: A Marshallian theory of consumer surplus and downward sloping demand, *Review of Economic Studies*, 40, 87-103.

[38] Vives, X. (1985), On the efficiency of Bertrand and Cournot equilibria with product differentation, *Journal of Economic Theory*, 36, 166-175.

[39] Vives, X. (1999). *Oligopoly Pricing: Old ideas and New Tools*, The MIT Press, Cambridge, MA.