PLASMA DRIFT WAVES AND INSTABILITIES

William Allan

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

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ABSTRACT

The work of this thesis is concerned with the investigation of the propagation of waves in a magnetized plasma containing various parameter gradients, and with the stability of ion acoustic waves in a weakly collisional plasma with a strong temperature gradient.

The thesis is divided into three sections. In the first section the intention is to derive in a compact and unambiguous tensor form the dispersion relation describing the propagation of waves in a magnetized plasma containing three-dimensional density and temperature gradients, an $\underline{E} \wedge \underline{B}$ drift, and differing temperatures parallel and perpendicular to the magnetic field. This is achieved by introducing and extending the polarized co-ordinate system first proposed by Buneman in 1961, and then carrying through the standard procedure of integration along unperturbed trajectories. The "local" approximation of Krall and Rosenbluth is used in order that an analytic result may be derived. The dispersion relation obtained includes certain moment tensors whose elements may be evaluated independently of the gradients involved in the problem. These elements may then be listed and the list referred to in order to obtain the elements required for a specific problem.

The second section is concerned with the use of the theory and results of J.P. Dougherty to show that in the high-frequency regime the introduction of a small amount of collisions into a plasma is sufficient to disrupt the gyro-resonances which allow the existence of Bernstein waves at multiples of the gyro-frequencies perpendicular and near-perpendicular to the magnetic field. It is shown that a collision frequency ν such that $(k\rho)^{-2} \lesssim \frac{\nu}{\Omega} < (k\rho)^{-1}$ where $k\rho \gg 1$ is sufficient to do this; k is the wave-number, ρ the Larmor radius, and Ω the gyro-frequency. It is also shown that in this case the ion-acoustic dispersion relation is valid even for propagation perpendicular to the magnetic field.

In the final section the result of the second section is used to derive a dispersion relation for high-frequency wave propagation in a weakly-collisional plasma containing an electron temperature gradient. The dispersion relation is solved numerically for various electron-ion temperature ratios and electron temperature gradient drift velocities. Earlier predictions, based on analytic calculations for small temperature ratios and drift velocities, are confirmed and some new results presented. In particular, it is shown that a temperature gradient is a more effective destabilizing agent then a simple drift between ions and electrons. Dispersion plots are given, along with analytic and physical explanations of their form; finally neutral stability curves are presented.

The thesis concludes with a summary of the results obtained.

DECLARATION

I declare that the following thesis is a record of research work carried out by me, that the thesis is my own composition, and that it has not been presented in application for a higher degree previously.

POSTGRADUATE CAREER

I was admitted into the University of St. Andrews as a research student under Ordinance General No. 12 in October 1971, to carry out research work into the theory of waves in magnetized plasmas under the supervision of Dr. J.J. Sanderson. I was admitted under the above resolution as a candidate for the degree of Ph.D. in April 1972.

CERTIFICATE

I certify that William Allan has satisfied the conditions of the Ordinance and Regulations and is thus qualified to submit the accompanying application for the degree of Doctor of Philosophy.

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CONTENTS

INTRODUCTION

Plasma physics is a young science. The use of the term "plasma" to describe a partly or wholly ionized gas is not yet half-a-century old; yet plasma physics is the apex of a pyramid of scientific thought and experiment. Just below the apex lie the ideas of Vlasov and Landau, Alfvén, Tonks and Langmuir, Appleton and Hartree; deepening and broadening the pyramid beneath are the minds of Debye and Larmor, Maxwell and Boltzmann, Faraday, Gauss, Ampère, Volta .... At the base of the pyramid lies the foundation upon which the whole structure is built:- the minds of the Greek philosophers, in which the generic ideas of logical thought and theoretical science were born. Plasma physics may be young, but it has a pedigree that cannot be bettered.

In any fully-ionized plasma there are short-range interactions between a charged test-particle and individual particles close to it; there is also a long-range collective interaction between the test-particle and the averaged electromagnetic field of all the other particles in the plasma (or at least of all the other particles within the Debye sphere of the test-particle, where the radius of the Debye sphere is of order

$\lambda_D = \left(\frac{\kappa T}{4\pi n_0 e^2} \right)^{\frac{1}{2}}$, κ being Boltzmann's constant, T the mean temperature, n_0 the particle number density, and -e the electron charge). Short-range interactions have the effect of changing the trajectory of the test-particle over a relatively short time-scale, while the effect of the averaged Coulomb field is experienced over a much longer time-scale. The collective interaction may be pictured as a smooth, gradual change in the trajectory of the test-particle, with the short-range interactions as a series of small but finite deviations superimposed on the slow collective change.

Depending on the density and temperature of the plasma, one or other of these effects may dominate. If short-range interactions are so important that the collective interaction can be neglected, the plasma is termed a "collision-dominated" plasma; it may be described by a model which considers local values of quantities such as mass density, net flow velocity, mean temperature and so on. This is justified because the test-particle is only affected by particles adjacent to it, and has no significant interaction with particles at a large distance. In essence, this description treats the plasma as a continuous fluid whose properties are averaged over a volume large enough to justify neglect of individual particle motions.

In the limiting case where short-range interactions may be neglected when compared with the collective interaction, the plasma is termed a "collisionless" plasma. Fluid descriptions break down here since they are dependent on collision dominance; however in the so-called "cold-plasma" regime (where the coherent flow velocity of the plasma is much greater than the random thermal velocities of the constituent particles) a quasi-fluid description is possible, although the cold-plasma model is highly idealized.

Models of the plasma state have been developed from the basic idea of an infinite, homogeneous, isotropic, fully-ionized, collisionless plasma with non-zero kinetic temperature. Simple and unrealistic as this may seem, the basic state is still capable of supporting a bewildering variety of waves and disturbances. These may be characterized by deriving a dispersion relation for the plasma. This relation takes the following form:-

$$D(\omega, k, p_1, p_2, p_3 \ldots) = 0$$

where ω is the wave frequency

k is the wave number, that is $(\text{wavelength})^{-1}$

$p_1, p_2, p_3 \ldots$ are parameters such as temperature, density and so forth.

For example, consider the relation

$$\omega^2 = \omega_{pe}^{\ 2} + k^2c^2$$

where $$\omega_{pe}^{\ 2} = \frac{4\pi\ n_0\ e^2}{m}$$

and ω_{pe} is known as the electron plasma frequency.

m is the electron mass.

c is the speed of light in vacuo.

This relation describes the propagation of transverse (or electromagnetic) waves in our basic plasma with zero kinetic temperature.

Situations may arise in a plasma in which the effects of collisions do not dominate the collective interaction mentioned earlier, and yet in which they cannot be neglected entirely. By their random nature, collisions tend to disrupt coherent effects that may take place in a plasma; for instance the dispersion relation for a completely collisionless plasma may contain a resonance occurring at some particular frequency, giving rise to a propagating wave mode. The presence of even a very small amount of collisions may be sufficient to randomize the resonance so that the mode is destroyed; and collisions are always present in any real plasma.

Neither the fluid description nor the cold plasma model is equipped to deal with such a situation. Moreover, both of these descriptions contain averaging processes which lose many properties of the plasma; it is therefore of great interest to investigate a plasma in terms of <u>kinetic theory</u>. This is a more fundamental description than anything discussed so far, in that it tries to deal with the microscopic particle nature of the plasma rather than considering averaged microscopic properties. The use of kinetic theory, along with the introduction of applied electromagnetic fields, spatial gradients in density, temperature, magnetic field and so forth, results in a much more realistic model, but also in a greatly increased complexity of the dispersion relation.

The basic quantity in kinetic theory is the <u>distribution function</u>

of a plasma particle species, denoted by $f(\underline{r}, \underline{v}, t)$. This quantity is such that the product $f \, \underline{dr} \, \underline{dv}$ gives the probable number of particles to be found within an increment $\underline{dr}$ of the point with position vector $\underline{r}$ while travelling with a velocity within an increment $\underline{dv}$ of the velocity $\underline{v}$ at time t. The six-dimensional space including all points with co-ordinates $(\underline{r}, \underline{v})$ is called phase space. The dynamics of a particle species in phase space is normally described by the collisional kinetic equation

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{\underline{F}}{m} \cdot \frac{\partial f}{\partial \underline{v}} = \left(\frac{\partial f}{\partial t} \right)_c$$

where $\underline{F}$ is the macroscopic force on a test-particle due to external fields and to the long-range collective interaction, while $\left(\frac{\partial f}{\partial t} \right)_c$ is the rate of change of f due to microscopic interactions between particles, that is due to collisions. Detailed derivations of this equation are given in many works, for example in reference [1].

A collisionless plasma is described by the Vlasov equation [2], or collisionless kinetic equation, which neglects the term $\left(\frac{\partial f}{\partial t} \right)_c$, giving

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{\underline{F}}{m} \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

This may be linearized by setting $f = f_0 + f_1$ and $\underline{F} = \underline{F}_0 + \underline{F}_1$, where f_0 and $\underline{F}_0$ are equilibrium values, f_1 and $\underline{F}_1$ being small perturbations. Substituting these in the Vlasov equation and neglecting products of small quantities, the linearized Vlasov equation is obtained:-

$$\frac{\partial f_1}{\partial t} + \underline{v} \cdot \frac{\partial f_1}{\partial \underline{r}} + \frac{\underline{F}_0}{m} \cdot \frac{\partial f_1}{\partial \underline{v}} = - \frac{\underline{F}_1}{m} \cdot \frac{\partial f_0}{\partial \underline{v}}$$

Landau [3] solved this using Fourier-Laplace transforms; an expression for the electric field $\underline{E}$ was then obtained from f_1. In principle, this expression may be written in terms of transforms as

$$E(\underline{r},t) = \int_{\underline{k}} \int_{p} E(\underline{k},p)e^{i(\underline{k}\cdot\underline{r} - pt)}\underline{dk}\, dp$$

$$= \int_{\underline{k}} E(\underline{k},t)e^{i\underline{k}\cdot\underline{r}}\, \underline{dk}$$

where $E(\underline{k},t) = \int_{p} E(\underline{k},p)e^{-ipt}\, dp$

Now $E(\underline{k},p)$ has poles $p_j(\underline{k})$ with residues $R_j(\underline{k})$, where p_j is in general complex, say $p_j = \omega_j + i\gamma_j$. Therefore, using Cauchy's Residue Theorem

$$E(\underline{k},t) = \sum_j R_j\, e^{-ip_j t}$$

$$= \sum_j R_j\, e^{-i\omega_j t + \gamma_j t}$$

The Maxwellian distribution function is defined as

$$f_M = n_0 \left(\frac{m}{2\pi\kappa T} \right)^{3/2} \exp\left(- \frac{mv^2}{2\kappa T} \right)$$

This distribution describes a homogeneous, isotropic plasma species in thermal equilibrium. Using the Maxwellian as his f_0 , Landau found that $\gamma_j < 0$, and so $E(\underline{k},t)$ decays as t tends to infinity, since a negative exponential factor is included in it. Any other plasma parameter that can be derived from f_1 decays in a similar manner. Now $\gamma_j > 0$ for any j would imply unlimited wave growth, or instability. Thus waves propagating in a Maxwellian plasma are stable. The decay phenomenon described above is known as Landau damping.

A physical explanation for this effect can be obtained by examining the distribution functions shown in Figure (1a). For a wave with phase velocity v_p, the number of particles of a given species travelling slightly slower than the wave is greater than the number travelling slightly faster. The electric field of the wave tends to accelerate the slower particles and decelerate the faster ones, the net result being that the wave loses energy, and is damped. Thus a negative slope of the distribution function implies damping. In the case of the Maxwellian distributions of Figure (1a) the slope is always negative, and damping occurs for waves of all phase velocities.

If a net drift velocity v_d of the electrons relative to the ions exists, the electron distribution is shifted in the positive v-direction as shown in Figure (1b). The part of the electron distribution from $v = 0$ to $v = v_d$ has positive slope. An isolated positive slope could result in inverse Landau damping, so that waves might extract energy from the electrons, and their amplitude would then grow. A negative ion slope also exists, however; growth occurs where the effect of the positive electron slope is enough to overcome the effect of the negative ion slope. Thus instability occurs if v_d is large enough. Drifts and distortions of the distribution functions occur when the plasma contains applied electric and magnetic fields, and when there are spatial gradients in temperature, density, etcetera. Under these conditions the chance of instability occurring is greatly increased.

Differing ion and electron temperatures have an effect on plasma stability in the following way:- Consider Figure (2a) where $T_i \ll T_e$; the ion gradient is very steep and negative for small positive v, and becomes negligible as v increases. This leaves only the weaker electron damping, so that a small electron drift can cause instability. Figure (2b) shows the distribution functions for $T_i \sim T_e$. The electron gradient is small for phase velocities near zero, but the ion gradient is steep enough to cause significant damping. The ion distribution also has enough spread to cause considerable damping for $v_p \sim v_i$, where the ion thermal velocity v_i is no longer very small. Thus an electron drift of at least $v_d \sim v_i$ would be necessary to cause instability (note that $v_{i,e} = \left(\frac{\kappa T_{i,e}}{m_{i,e}}\right)^{\frac{1}{2}}$),

Thus plasmas that are unstable when $\frac{T_i}{T_e} \ll 1$ tend to stabilize as $\frac{T_i}{T_e}$ increases.

The study of all types of instability has become of the first priority in recent years, since it was realized that most of the troubles of the controlled thermonuclear fusion programme stemmed in one way or another from plasma instabilities. In the first section of this thesis we consider a collisionless plasma with various general particle drifts, and derive a dispersion relation to describe the possible wave modes that may propagate in it. In the second section we investigate one of the cases mentioned earlier, in which the existence of a small collision effect (in magnitude much less than the collective effect) is sufficient to disrupt a resonance leading in the collisionless theory to a set of propagating wave modes, the Bernstein modes. In the third section we investigate in detail the effect of a large temperature gradient drift on the wave-mode known as the ion acoustic wave in a thermal plasma including weak collisions.

We may note here that the method we use to solve the linearized Vlasov equation is equivalent to Landau's Fourier-Laplace transform method, and results in a four-fold integral over three velocity components and time. If the time integral is performed first, followed by the velocity integrals, a solution may be obtained in terms of Bessel functions. If the velocity integrals are performed first, a time integral known as the Gordeyev integral (or some modification of it) is involved in the result. For a general problem the former approach is usually the most profitable. However for a more particular problem, perhaps concerning a simple configuration or a limited parameter range, the Gordeyev integral approach is often to be preferred. For the general drift problem in Section I we use the Bessel function approach; in Sections II and III the Gordeyev integral approach is used in the regime $k\rho >> 1$, where $\rho = \frac{(v_T)_\perp}{\Omega}$ is the Larmor radius, and $(v_T)_\perp$ is the mean thermal velocity perpendicular to the magnetic field $\underline{B}$. The cyclotron frequency Ω is $\frac{qB}{mc}$ where q is the charge of the species considered.

SECTION I :- A dispersion relation for a plasma with various spatial inhomogeneities.

Chapter 1

In this section we derive a dispersion relation for a fully-ionized, collisionless, non-relativistic plasma which includes general gradients in density and temperature, and also differing temperatures parallel and perpendicular to an applied magnetic field.

The kinetic equation describing a particle species in such a plasma is the collisionless kinetic equation, or Vlasov equation:-

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{\underline{F}}{m} \cdot \frac{\partial f}{\partial \underline{v}} = 0 \qquad (I.1)$$

In this equation $\underline{r}$ and $\underline{v}$ denote position and velocity respectively, so that $(\underline{r},\underline{v})$ is a position in phase space. The species distribution function is $f(\underline{r},\underline{v},t)$, while $\underline{F}$ represents the net macroscopic force acting on a particle of the species. This force includes the effect of external forces and of the internal averaged collective force due to particles of all species, but excludes microscopic short-range particle collisions. The particle mass in represented by m.

To solve (I.1) in the linear theory, we must examine the effect of small perturbations on a plasma which is initially in an equilibrium state. We therefore make the following substitution:-

$$f(\underline{r},\underline{v},t) = f_0(\underline{r},\underline{v}) + f_1(\underline{r},\underline{v},t)$$

$$\underline{F}(\underline{r},\underline{v},t) = \underline{F}_0(\underline{r},\underline{v}) + \underline{F}_1(\underline{r},\underline{v},t)$$

where f_0 and $\underline{F}_0$ are equilibrium values, and f_1 and $\underline{F}_1$ are small perturbations such that

$$\left| \frac{f_1}{f_0} \right| << 1 \quad \text{and} \quad \left| \frac{\underline{F}_1}{\underline{F}_0} \right| << 1$$

Using this, equation (I.1) becomes

$$[\frac{\partial f_0}{\partial t} + \underline{v} \cdot \frac{\partial f_0}{\partial \underline{r}} + \frac{\underline{F}_0}{m} \cdot \frac{\partial f_0}{\partial \underline{v}}]$$

$$+ [\frac{\partial f_1}{\partial t} + \underline{v} \cdot \frac{\partial f_1}{\partial \underline{r}} + \frac{\underline{F}_0}{m} \cdot \frac{\partial f_1}{\partial \underline{v}}] = - \frac{\underline{F}_1}{m} \cdot \frac{\partial f_0}{\partial \underline{v}} \qquad \text{(I.1a)}$$

The term $\frac{\underline{F}_1}{m} \cdot \frac{\partial f_1}{\partial \underline{v}}$ is taken to be a product of small quantities, and is therefore neglected compared with the other terms. To find the equation for the equilibrium state, we merely set $f_1 \equiv 0$, giving

$$\frac{\partial f_0}{\partial t} + \underline{v} \cdot \frac{\partial f_0}{\partial \underline{r}} + \frac{\underline{F}_0}{m} \cdot \frac{\partial f_0}{\partial \underline{v}} = 0 \cdot \qquad \text{(I.2)}$$

This is the equation which the equilibrium distribution f_0 must satisfy; the equation giving the perturbation distribution f_1 is therefore, from equation (I.1a)

$$\frac{\partial f_1}{\partial t} + \underline{v} \cdot \frac{\partial f_1}{\partial \underline{r}} + \frac{\underline{F}_0}{m} \cdot \frac{\partial f_1}{\partial \underline{v}} = - \frac{\underline{F}_1}{m} \cdot \frac{\partial f_0}{\partial \underline{v}} \qquad \text{(I.3)}$$

The solution of this equation, $f_1(\underline{r},\underline{v},t)$, contains in principle all the information required to describe the perturbed plasma.

Equation (I.3) may be written in the following form:-

$$[\frac{d}{dt}]_u f_1 = h(\underline{r},\underline{v},t)$$

where $$h(\underline{r},\underline{v},t) = - \frac{\underline{F}_1}{m} \cdot \frac{\partial f_0}{\partial \underline{v}}$$

and $[\frac{d}{dt}]_u$ is a differential operator acting along the characteristic curves of the partial differential equation (I.3). It may be thought of physically as the "convective" time differential operator; that is, the rate of change of a quantity measured along unperturbed particle orbits in phase space. Thus for a given value of $(\underline{r},\underline{v},t)$, equation (I.3) gives the rate of change of f_1 as "seen" by a particle at the phase space point $(\underline{r},\underline{v})$ at time t, but moving along the unperturbed orbit through $(\underline{r},\underline{v})$.

Our solution of (I.3) follows that given by Clemmow and Dougherty [4], except for certain notational changes made for later convenience. If we wish to find the value of f_1 at some point $(\underline{r}',\underline{v}',t')$ in phase-space and time, we must integrate along the unperturbed orbit which passes through $(\underline{r}',\underline{v}')$, arriving at that point at time t'. If $(\bar{\underline{r}}(t),\bar{\underline{v}}(t))$ denotes this orbit, f_1 at $(\underline{r}',\underline{v}',t')$ is given by

$$f_1 \ (\underline{r}',\underline{v}',t') \ = \ \int_{-\infty}^{t'} h \ (\bar{\underline{r}},\bar{\underline{v}},t)dt \qquad (I.4)$$

Obviously $\underline{F}_1$ depends in some way on f_1, and so (I.4) is actually an integral equation with f_1 implicit in h. We therefore require further information in order to specify the function h independently of f_1. This information is contained in Maxwell's equations, which must apply to the plasma as a whole. However, there is no straightforward way of eliminating f_1 from h at this stage, so we adopt the following procedure:-

$\underline{F}_1$ is taken to consist of given perturbing fields, and the response of the plasma to these fields is to be calculated; subsequently the fields are to be made consistent with Maxwell's equations giving a self-consistent overall description. Thus $h(\underline{r},\underline{v},t)$ is a known function at this point.

We now define the Green's function

$$G(\underline{r},\underline{v},t,\underline{r}',\underline{v}',t') \ = \ \delta^3(\underline{r} - \bar{\underline{r}})\delta^3(\underline{v} - \bar{\underline{v}}) \ \varepsilon(t'- t)$$

$$\text{where} \quad \varepsilon(\tau) \ = \ \left.\begin{matrix} 1 \\ 0 \end{matrix}\right\} \begin{matrix} \text{if } \tau > 0 \\ \text{if } \tau < 0 \end{matrix}$$

The equation (I.4) may be written as

$$f_1(\underline{r}',\underline{v}',t') \ = \ \int_{\underline{r}} \int_{\underline{v}} \int_t G(\underline{r},\underline{v},t,\underline{r}',\underline{v}',t')h(\bar{\underline{r}},\bar{\underline{v}},t)\underline{dr} \ \underline{dv} \ \underline{dt}$$

where the integration limits are $(-\infty, \infty)$ in all seven variables.

Let $\underline{X}(t)$ and $\underline{V}(t)$ be the position and velocity functions defining the unperturbed orbit passing through $\underline{r} = 0$ with velocity $\underline{v}$ at $t = 0$. Then the unperturbed orbit passing through $(\underline{r}',\underline{v}')$ at time t' is

$$\bar{\underline{r}} = \underline{r}' - \underline{X}(t' - t)$$

$$\bar{\underline{v}} = \underline{v}' + \underline{v} - \underline{V}(t' - t)$$

Thus

$$G = \delta^3 [\underline{r} - \underline{r}' + \underline{X}(t' - t)] \; \delta^3 [\underline{V}(t' - t) - \underline{v}'] \; \varepsilon \, (t' - t)$$

and therefore

$$f_1(\underline{r}', \underline{v}', t') = \int_{\underline{v}} \int_{t=-\infty}^{t'} h[\underline{r}' - \underline{X}(t'-t), \underline{v}, t] \; \delta^3 [\underline{v}' - \underline{V}(t'-t)] \; \underline{dv} \, dt$$

Setting $t' - t = t''$

$$f_1(\underline{r}', \underline{v}', t') = \int_{\underline{v}} \int_{t''=o}^{\infty} h[\underline{r}' - \underline{X}(t''), \underline{v}, t'-t''] \; \delta^3 [\underline{v}' - \underline{V}(t'')] \; \underline{dv} \, dt''$$

$$= \int_{\underline{v}} \int_{t=o}^{\infty} h[\underline{r}' - \underline{X}(t), \underline{v}, t'-t] \; \delta^3 [\underline{v}' - \underline{V}(t)] \; \underline{dv} \, dt$$

where we have replaced the dummy variable t'' by t.

A function $g(\underline{r}', t')$ may be written in Fourier-Laplace integral form as

$$g(\underline{r}', t') = \int_{\underline{k}} \int_{\omega} g(k, \omega) \exp [i(\underline{k} . \underline{r}' - \omega t')] \; \underline{dk} \, d\omega$$

The equation for $f_1(\underline{r}', \underline{v}', t')$ written in this form is

$$\int_{\underline{k}} \int_{\omega} f_1(\underline{k}, \underline{v}', \omega) \exp [i(\underline{k} . \underline{r}' - \omega t')] \; \underline{dk} \, d\omega$$

$$= \int_{\underline{k}} \int_{\omega} \left\{ \int_{\underline{v}} \int_{t=o}^{\infty} h[\underline{k}, \underline{v}, \omega] \; \delta^3 \; [\underline{v}' - \underline{V}(t)] \; \underline{dv} \, dt \right\}$$

$$. \exp [i(\underline{k} . [\underline{r}' - \underline{X}(t)] - \omega(t' - t))] \; \underline{dk} \, d\omega$$

$$= \int_{\underline{k}} \int_{\omega} \left\{ \int_{\underline{v}}^{\infty} \int_{t=o} h[\underline{k}, \underline{v}, \omega] \; \delta^3 [\underline{v}' - \underline{V}(t)] \exp [i(\omega t - \underline{k} . \underline{X}(t)] \; \underline{dv} \, dt \right\}$$

$$. \exp [\; i(\underline{k} \; . \; \underline{r}' - \omega t')] \; \underline{dk} \, d\omega \quad .$$

Equating Fourier-Laplace components gives

$$f_1(\underline{k},\underline{v}',\omega) = \int_{\underline{v}} \int_{t=o}^{\infty} h\,[\,\underline{k},\underline{v},\omega\,]\;\delta^3\,[\,\underline{v}' - \underline{V}(t)\,]\exp\,[\,i(\omega t-\underline{k}.\underline{X}(t))]\,\underline{dv}\;dt$$

In order to invoke Maxwell's equations, we must work in terms of charge and current densities rather than f_1. They are given respectively in terms of their Fourier-Laplace transforms by

$$\rho_1(\underline{k},\omega) = q\int_{\underline{v}'} f_1(\underline{k},\underline{v}',\omega)\;\underline{dv}'$$

and

$$\underline{j}_1(\underline{k},\omega) = q\int_{\underline{v}'} \underline{v}' f_1(\underline{k},\underline{v}',\omega)\underline{dv}' \qquad (I.5)$$

Considering only the effect of electric and magnetic fields on the plasma, the force $\underline{F}$ in equation (I.1) is given by

$$\underline{F} = q(\underline{E} + \frac{1}{c}\;\underline{v}\wedge\underline{B}) \qquad [\text{ in Gaussian units }]$$

where $\frac{q}{c}\;\underline{v}\wedge\underline{B}$ is the force on a particle due to the interaction between its velocity $\underline{v}$ and the magnetic field $\underline{B}$.

Thus

$$\underline{F}_0 = q(\underline{E}_0 + \frac{1}{c}\;\underline{v}\wedge\underline{B}_0)$$

and

$$\underline{F}_1 = q(\underline{E}_1 + \frac{1}{c}\;\underline{v}\wedge\underline{B}_1)$$

giving $h(\underline{r},\underline{v},t) = -\frac{q}{m}\,(\underline{E}_1(\underline{r},t) + \frac{1}{c}\,\underline{v}\wedge\underline{B}_1(\underline{r},t))\;.\;\frac{\partial}{\partial\underline{v}}\,f_0(\underline{r},\underline{v})$.

The integral for $\underline{j}_1$ is therefore

$$\underline{j}_1(\underline{k},\omega) = -\frac{q^2}{m}\int_{\underline{v}}\int_{t=o}^{\infty}\underline{V}(t)\,[\;\underline{E}_1(\underline{k},\omega) + \frac{1}{c}\,\underline{v}\wedge\underline{B}_1(\underline{k},\omega)\;]\;.\;\frac{\partial}{\partial\underline{v}}\,f_0(\underline{k},\underline{v})$$

$$.\;\exp\,[\,i(\omega t - \underline{k}\,.\,\underline{X}(t))]\;\underline{dv}\;dt \qquad (I.6)$$

where we have substituted for $f_1(\underline{k},\underline{v}',\omega)$ in equation (I.5), using the Fourier-Laplace transformed version of h. Maxwell's equation of electromagnetic induction is

$$\underline{\nabla}\wedge\underline{E} + \frac{1}{c}\,\frac{\partial}{\partial t}\,\underline{B} = 0\;. \qquad (I.7)$$

Suppose we assume fields of the form

$$\underline{E}(\underline{r},t) = \underline{E}_0 + \underline{E}_1(\underline{r},t)$$

$$\underline{B}(\underline{r},t) = \underline{B}_0(\underline{r}) + \underline{B}_1(\underline{r},t)$$

Thus we have a constant equilibrium electric field and a steady equilibrium magnetic field.

Equation (I.7) reduces to

$$\underline{\nabla} \wedge \underline{E}_1 + \frac{1}{c} \frac{\partial}{\partial t} \underline{B}_1 = 0 \qquad \text{(I.8)}$$

The Fourier-Laplace integral forms of $\underline{E}_1$ and $\underline{B}_1$ are

$$\underline{E}_1(\underline{r},t) = \int_{\underline{k}} \int_{\omega} \underline{E}_1(\underline{k},\omega) \exp [\ i(\underline{k} \cdot \underline{r} - \omega t)]\ \underline{dk}\ d\omega$$

and $$\underline{B}_1(\underline{r},t) = \int_{\underline{k}} \int_{\omega} \underline{B}_1(\underline{k},\omega) \exp [\ i(\underline{k} \cdot \underline{r} - \omega t)]\ \underline{dk}\ d\omega$$

Substituting these in (I.8), taking the differential operators inside the integral signs, and then equating Fourier-Laplace components results in the following equation:-

$$B_1(\underline{k},\omega) = \frac{c}{\omega}\ \underline{k} \wedge \underline{E}_1(\underline{k},\omega) \qquad \text{(I.9)}$$

Using this in (I.6), we find that

$$\underline{j}_1(\underline{k},\omega) = -\frac{q^2}{m} \int_{\underline{v}} \int_{t=o}^{\infty} \underline{v}(t) \left[\underline{E}_1(\underline{k},\omega) + \frac{1}{\omega} \left\{ \underline{v} \wedge (\underline{k} \wedge \underline{E}_1(\underline{k},\omega)) \right\} \right] \frac{\partial}{\partial \underline{v}} f_0(\underline{k},\underline{v})$$

$$\cdot \exp [\ i(\omega t - \underline{k} \cdot \underline{X}(t))\ ]\ \underline{dv}\ dt \qquad \text{(I.10)}$$

When the integrations are carried out, the right-hand side of equation (I.10) gives a vector whose components are linear combinations of the components of $\underline{E}_1$. We may therefore write

$$\underline{j}_1(\underline{k},\omega) = \underline{\underline{\sigma}}(\underline{k},\omega) \cdot \underline{E}_1(\underline{k},\omega) \qquad \text{(I.10a)}$$

This is a generalized Ohm's Law, and we define $\underline{\underline{\sigma}}(\underline{k},\omega)$ to be the conductivity tensor for the plasma species concerned.

The total current is given by

$$\underline{j}_1(\underline{k},\omega) = \sum_{\text{species}} \underline{\underline{\sigma}}(\underline{k},\omega) \cdot \underline{E}_1(\underline{k},\omega) \qquad \text{(I.10b)}$$

Chapter 2

In order to derive an algebraic value for $\underline{\underline{\sigma}}$, we must insert a specific expression for the equilibrium distribution $f_0(\underline{k},\underline{v})$ into (I.10). We consider initially the following inhomogeneous equilibrium distribution:-

$$f_0(r,v) = [1 + \{\varepsilon + \delta_\perp [a_\perp v_\perp^2 - 1] + \delta_\parallel [a_\parallel v_\parallel^2 - \tfrac{1}{2}]\}(x + \frac{v_y}{\Omega})] f_M \qquad (I.11)$$

where $\underline{r} = (x,y,z)$ and $\underline{v} = (v_x,v_y,v_z)$ are in orthogonal Cartesian co-ordinates. The quantities $v_\perp$ and $v_\parallel$ are defined by

$$v_\perp^2 = v_x^2 + v_y^2$$

$$v_\parallel^2 = v_z^2$$

We take $\underline{E}_0 = 0$ and $\underline{B}_0 = B_0\,\hat{\underline{z}}$, where B_0 is constant; then $\Omega = \frac{qB_0}{mc}$ is the cyclotron frequency. The quantities $n(x)$, $T^\parallel(x)$ and $T^\perp(x)$ are respectively defined to be the density, and kinetic temperatures parallel and perpendicular to $\underline{B}_0$; n_0, $T_0^\parallel$ and $T_0^\perp$ are their values at $x = 0$, since the origin of our co-ordinate system may be chosen anywhere in space.

We now define f_M to be the Maxwellian distribution for differing parallel and perpendicular temperatures, that is

$$f_M = n_0\left(\frac{a_\perp}{\pi}\right)\left(\frac{a_\parallel}{\pi}\right)^{\frac{1}{2}} \exp\left\{-(a_\perp v_\perp^2 + a_\parallel v_\parallel^2)\right\} \qquad (I.12)$$

where

$$a_\perp = \frac{m}{2\kappa T_0^\perp}$$

$$a_\parallel = \frac{m}{2\kappa T_0^\parallel}$$

f_0 must be chosen in such a way that it satisfies the zero-order Vlasov equation (I.2). We show in Appendix (III a) that the f_0 given by equation (I.11) satisfies (I.2), and also that for small values of ε, $\delta_\perp$ and $\delta_\parallel$ the following relations hold:-

$$n(x) \approx n_0(1 + \varepsilon x)$$

$$T^{\perp}(x) \approx T^{\perp} (1 + \delta_{\perp} x)$$

$$T^{\shortparallel}(x) \approx T^{\shortparallel} (1 + \delta_{\shortparallel} x)$$

Thus we may define ε to be a density gradient

$\delta_{\perp}$ to be a perpendicular temperature gradient

$\delta_{\shortparallel}$ to be a parallel temperature gradient.

[The requirement that the gradients be small is only necessary when a combination of density and temperature gradients is used in f_0; when ε or one of the δ's occurs alone, this requirement is unnecessary (see Appendix (III a))].

The effect of using such a form for f_0 has been treated elsewhere (for example reference [5]). The aim of Section I is to derive a conductivity tensor involving general three-dimensional gradients in density and temperature, with differing temperatures parallel and perpendicular to $\underline{B}_0$, and to derive it in such a way that the result is concise, convenient and easily reducible to a conductivity tensor for a simpler situation. To do this we require a notation which allows the retention of a compact tensor form even when describing an inhomogeneous, anistropic plasma. We use the polarized co-ordinate system originated by Buneman [6] and developed by Dougherty [7]. This system and its associated tensor behaviour is described in detail in Appendix (I); here we merely define the components of a vector in the system. Note that Greek letters are used for indices, and that upper indices denote contravariant indices, while lower indices denote covariant indices.

Suppose that the vector $\underline{b}$ is represented in rectangular Cartesian co-ordinates by the components (b_x, b_y, b_z). Then the contravariant vector b^{μ} in polarized co-ordinates is given by the components

$$b^1 = 2^{-\frac{1}{2}} (b_x + i\, b_y)$$

$$b^0 = b_z$$

$$b^{-1} = 2^{-\frac{1}{2}} (b_x - i\, b_y)$$

The covariant vector b_μ is given by

$$b_1 = 2^{-\frac{1}{2}} (b_x - i\, b_y)$$

$$b_0 = b_z$$

$$b_{-1} = 2^{-\frac{1}{2}} (b_x + i\, b_y)$$

The metric tensor for the system is $\delta_{\lambda,-\mu}$, so that raising and lowering of indices is achieved merely by changing the sign.

The requirement now is to find an f_0, expressed in terms of polarized co-ordinates, which contains general gradients and temperatures, yet which satisfies (I.2). Firstly, suppose we have a steady situation, so that the density and temperatures are given by $n(\underline{r})$, $T^{\perp}(\underline{r})$ and $T''(\underline{r})$. (Note that the position system $\underline{r}$ is not a vector in general, since it does not transform according to tensor laws). If we write $\underline{r}$ as r^ν in polarized co-ordinates, the gradient operator $\frac{\partial}{\partial r^\nu}$ adds a covariant index to any tensor quantity that it operates on (see Appendix I). Define the gradients in n, $T^{\perp}$ and T'' as follows:-

$$(n_0)^{-1} \frac{\partial}{\partial r^\nu} n = \varepsilon_\nu$$

$$(T_0{}^{\perp})^{-1} \frac{\partial}{\partial r^\nu} T^{\perp} = \delta^{\perp}{}_\nu$$

$$(T_0{}'')^{-1} \frac{\partial}{\partial r^\nu} T'' = \delta''{}_\nu$$

where n_0, $T_0{}^{\perp}$ and $T_0{}''$ are n, $T^{\perp}$ and T'' evaluated at the arbitrary origin of the co-ordinate system. Thus ε_ν is the density gradient vector, and $\delta^{\perp}_\nu$, δ''_ν are the temperature gradient vectors.

Considering the simplest non-trivial case, that of constant gradients, we attempt to generalize (I.11) by proposing an f_0 of the following form:-

$$f_0 = \{1 + (\varepsilon_\nu + [a_\perp v_\perp^2 - 1]\ \delta^\perp_\nu + [a_\parallel v_\parallel^2 - \tfrac{1}{2}]\ \delta^\parallel_\nu)(r^\nu + a^\nu_\rho v^\rho)\}\ f_M$$

$$= \{1 + \gamma_\nu(r^\nu + a^\nu_\rho v^\rho)\}\ f_M \qquad \text{(I.13)}$$

where $\gamma_\nu = (\varepsilon_\nu + [a_\perp v_\perp{}^2 - 1]\ \delta^\perp_\nu + [\ a_\parallel v_\parallel{}^2 - \tfrac{1}{2}\ ]\ \delta^\parallel_\nu\ )$

and the Einstein summation convention is used.

As noted before, r^ν is not a vector. However, since ε_ν, $\delta^\perp_\nu$ and $\delta^\parallel_\nu$ are vectors, and f_0 is an invariant, the tensor quotient law implies that the system $(r^\nu + a^\nu_\rho v^\rho)$ must be a vector.

The system a^ν_ρ is chosen in such a way that the f_0 given by (I.13) satisfies (I.2). We now require the quantities $\frac{\partial f_0}{\partial t}$, $\frac{\partial f_0}{\partial \underline{r}}$ and $\frac{\partial f_0}{\partial \underline{v}}$

for substitution in (I.2). Firstly $\frac{\partial f_0}{\partial t}$ is zero. Now consider $\frac{\partial f_0}{\partial r^\mu}$ and $\frac{\partial f_0}{\partial v^\mu}$

$$\frac{\partial f_0}{\partial r^\mu} = \gamma_\nu \frac{\partial r^\nu}{\partial r^\mu} f_M$$

$$= \gamma_\nu\ \delta^\nu_\mu\ f_M$$

$$= \gamma_\mu\ f_M$$

$$\frac{\partial f_0}{\partial v^\mu} = [\ (r^\nu + a^\nu_\rho v^\rho)\ \frac{\partial \gamma_\nu}{\partial v^\mu} + \gamma_\nu\ a^\nu_\mu\ ]\ f_M + [\ 1 + \gamma_\nu(r^\nu + a^\nu_\rho v^\rho)\ ]\ \frac{\partial f_M}{\partial v^\mu}$$

f_M is given by (I.12) so that

$$\frac{\partial f_M}{\partial v^\mu} = -\ f_M\ \frac{\partial}{\partial v^\mu} .\ (a_\perp v_\perp{}^2 + a_\parallel v_\parallel{}^2)$$

Now $\quad v_\perp{}^2 = v_x{}^2 + v_y{}^2 = 2v^1 v^{-1}$

$\quad\quad v_\parallel{}^2 = (v^0)^2$

We define $w_\mu = \tfrac{1}{2}\ \frac{\partial}{\partial v^\mu}\ (a_\perp v_\perp{}^2 + a_\parallel v_\parallel{}^2)$

where $w_1 = a_\perp v^{-1} = a_\perp v_1$

$w_0 = a_\parallel v_0$

$w_{-1} = a_\perp v^1 = a_\perp v_{-1}$

We also define the vector $w^{\perp}_{\mu}$ with components $(a_{\perp}\, v_{1},\ 0,\ a_{\perp}\, v_{-1})$ and the vector w''_{μ} with components $(0,\ a_{\shortparallel}\, v_0,\ 0)$

Thus we have

$$\frac{\partial f_M}{\partial v^{\mu}} = -2\, f_M\, w_{\mu}$$

Now
$$\frac{\partial \gamma_{\nu}}{\partial v^{\mu}} = a_{\perp}\, \delta^{\perp}_{\nu}\, \frac{\partial v_{\perp}{}^2}{\partial v^{\mu}} + a_{\shortparallel}\, \delta''_{\nu}\, \frac{\partial v_{\shortparallel}{}^2}{\partial v^{\mu}}$$

$$= 2(\delta^{\perp}_{\nu}\, w^{\perp}_{\mu} + \delta''_{\nu}\, w''_{\mu})$$

Therefore our final expression for $\frac{\partial f_0}{\partial v^{\mu}}$ is

$$\frac{\partial f_0}{\partial v^{\mu}} = \Big\{ \gamma_{\nu}\, a^{\nu}_{\mu} + 2(r^{\nu} + a^{\nu}_{\rho}\, v^{\rho})(\delta^{\perp}_{\nu}\, w^{\perp}_{\mu} + \delta''_{\nu}\, w''_{\mu}) - 2\,[\,1 + \gamma_{\nu}\,(r^{\nu} + a^{\nu}_{\rho}\, v^{\rho})\,]\, w_{\mu} \Big\} f_M \qquad \text{(I.14)}$$

For the case of a plasma in which the zero-order fields $\underline{E}_0$ and $\underline{B}_0$ are constant, (I.2) takes the form

$$\frac{\partial f_0}{\partial t} + \underline{v} \cdot \frac{\partial f_0}{\partial \underline{r}} + \frac{q}{m}\left(\underline{E}_0 + \frac{1}{c}\,\underline{v} \wedge \underline{B}_0\right) \cdot \frac{\partial f_0}{\partial \underline{v}} = 0 \qquad \text{(I.15)}$$

The existence of a component of $\underline{E}_0$ parallel to $\underline{B}_0$ would result in the acceleration of particles to relativistic velocities in the direction of $\underline{B}_0$, and would also result in arbitrarily large currents and charge separation. The Maxwell equations

$$\underline{\nabla} \wedge \underline{B} = \frac{1}{c}\,\frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c}\,\underline{j} \qquad \text{(I.16)}$$

and
$$\underline{\nabla} \cdot \underline{E} = 4\pi\rho \qquad \text{(I.17)}$$

would then imply large field fluctuations. Thus the assumption of an $\underline{E}_0$ component parallel to $\underline{B}_0$ is inconsistent with our non-relativistic and linear approximations. We therefore take $\underline{E}_0$ perpendicular to $\underline{B}_0$.

The simultaneous existence of $\underline{E}_0$ and $\underline{B}_0$ results in the particle species as a whole drifting with a velocity $\underline{v}_0 = \frac{c(\underline{E}_0 \wedge \underline{B}_0)}{B_0^2}$ (see reference [8]).

The effect on the distribution function is to replace $\underline{v}$ by $\underline{v} - \underline{v}_0$ in the expression for f_0. This replacement and the existence of $\underline{E}_0$ itself in (I.15) greatly complicate the derivation of a conductivity tensor. However, $\underline{v}_0$ may be eliminated from the analysis by transforming to a frame of reference whose origin is moving with velocity $\underline{v}_0$. This transformation eliminates $\underline{v}_0$ from f_0, and also eliminates the zero-order electric field from the convective derivative $[\frac{d}{dt}]_u$. We discuss the transformation in more detail in Appendix (III b).

Our procedure now is to make the above transformation, to derive a conductivity tensor in the transformed frame, and then to carry out the inverse transformation once the final result has been obtained. Details of the inversion will be given at that point.

Under the transformation, equation (I.15) becomes

$$\frac{\partial f_0}{\partial t} + \underline{v} \cdot \frac{\partial f_0}{\partial \underline{r}} + \frac{q}{mc} \underline{v} \wedge \underline{B}_0 \cdot \frac{\partial f_0}{\partial \underline{v}} \equiv 0 \qquad \text{(I.18)}$$

where we have dropped the dashes used in Appendix (III b) to denote transformed variables.

Substitution of the values derived earlier for $\frac{\partial f_0}{\partial t}$, $\frac{\partial f_0}{\partial r^\mu}$ and $\frac{\partial f_0}{\partial v^\mu}$ results in the following equation:-

$$v^\mu \gamma_\mu f_M - \frac{iq}{mc} e^{\mu\beta\gamma} v_\beta (B_0)_\lambda$$
$$\cdot [\gamma_\nu a^\nu_\mu + 2(\dot{r}^\nu + a^\nu_\rho v^\rho)(\delta^\perp_\nu w^\perp_\mu + \delta''_\nu w''_\mu)$$
$$- 2 \{1 + \gamma_\nu (r^\nu + a^\nu_\rho v^\rho)\} w_\mu] f_M \equiv 0 \qquad \text{(I.19)}$$

where $(\underline{v} \wedge \underline{B}_0)^\mu$ has been replaced by its tensor form $- i e^{\mu\beta\lambda} v_\beta (B_0)_\lambda$, $e^{\mu\beta\lambda}$ being the permutation tensor in polarized co-ordinates (see Appendix (I)). We now choose a rectangular Cartesian reference frame such that $\underline{B} = B_0 \hat{\underline{z}}$; that is $\underline{B}_0$ is $(0, B_0, 0)$ in polarized co-ordinates.

Equation (I.19) may now be written as

$$v^\mu \gamma_\mu - \frac{iq}{mc} e^{\mu\beta o} v_\beta B_0 [\gamma_\nu a^\nu_\mu + 2(r^\nu + a^\nu_\rho v^\rho)(\delta^\perp_\nu w^\perp_\mu + \delta''_\nu w''_\mu)$$

$$-2 \{1 + \gamma_\nu (r^\nu + a^\nu_\rho v^\rho)\} w_\mu] \equiv 0 \qquad (I.20)$$

Now $e^{\mu\beta o} v_\beta w^\perp_\mu = e^{1,-1,0} v_1 w^\perp_{-1} + e^{-1,1,0} v_{-1} w^\perp_1$

$$= a_\perp (v_{-1} v_1 - v_1 v_{-1})$$

$$\equiv 0$$

Similarly $e^{\mu\beta o} v_\beta w''_\mu \equiv 0$

and therefore $e^{\mu\beta o} v_\beta w_\mu \equiv 0$

Also $\frac{qB_0}{mc} = \Omega$

Thus (I.20) reduces to

$$v^\mu \gamma_\mu - i\Omega e^{\mu\beta o} v_\beta \gamma_\nu a^\nu_\mu \equiv 0$$

or $$v_\beta (\gamma^\beta - i\Omega e^{\mu\beta o} \gamma_\nu a^\nu_\mu) \equiv 0$$

Since $v_\beta \neq 0$ we have

$$\gamma^\beta - i\Omega e^{\mu\beta o} \gamma_\nu a^\nu_\mu \equiv 0 \qquad (I.21)$$

Now $\gamma_\nu = \varepsilon_\nu + [a_\perp v_\perp^2 - 1] \delta^\perp_\nu + [a_{\shortparallel} v_{\shortparallel}^2 - \frac{1}{2}] \delta''_\nu$

$$= (\varepsilon_\nu - \delta^\perp_\nu - \tfrac{1}{2}\delta''_\nu) + a_\perp \delta^\perp_\nu v_\perp{}^2 + a_{\shortparallel} \delta''_\nu v_{\shortparallel}{}^2$$

or $$\gamma_\nu = g_\nu + h_\nu v_\perp{}^2 + \ell_\nu v_{\shortparallel}{}^2 \qquad (I.22)$$

where

$$g_\nu = \varepsilon_\nu - \delta^\perp_\nu - \tfrac{1}{2}\delta''_\nu$$

$$h_\nu = a_\perp \delta^\perp_\nu$$

$$\ell_\nu = a_{\shortparallel} \delta''_\nu$$

Substitution for γ_ν in (I.21) gives

$$[\, g^\beta - i\Omega e^{\mu\beta o}\, g_\nu\, a^\nu_\mu \,] + [\, h^\beta - i\Omega e^{\mu\beta o}\, h_\nu a^\nu_\mu \,]\, v_0{}^2 + [\, \ell^\beta - i\Omega e^{\mu\beta o}\, \ell_\nu a^\nu_\mu \,]\, v_1\, v_{-1} \equiv 0$$

This equation must be satisfied identically by all possible values of v_0, v_1 and v_{-1}. Such a situation can only occur if the coefficients of 1, $v_0{}^2$ and $v_1\ v_{-1}$ are identically zero. Thus we have three equations of the same form to solve; consider the first equation:-

$$g^\beta - i\Omega e^{\mu\beta o}\, g_\nu\, a^\nu_\mu \equiv 0$$

In component form this is

$$g^1 - i\Omega e^{\mu,1,o}\ g_\nu\, a^\nu_\mu \equiv 0$$

$$g^o - i\Omega e^{\mu,o,o}\ g_\nu\, a^\nu_\mu \equiv 0$$

$$g^{-1} - i\Omega e^{\mu,-1,o}\ g_\nu\, a^\nu_\mu \equiv 0$$

Now $e^{\mu,o,o} \equiv 0$, implying that $g^o \equiv 0$; so we have

$$\left.\begin{array}{l} g^1 - i\Omega\left(g_1 a_{-1}^{\ 1} + g_{-1} a_{-1}^{-1}\right) \equiv 0 \\ g^{-1} + i\Omega\left(g_1 a_1^1 + g_{-1} a^{-1}_{\ 1}\right) \equiv 0 \\ g^o \equiv 0 \end{array}\right\} \qquad (I.23)$$

The equations for h^β and ℓ^β are identical in form, so that

$$h^o \equiv 0 \quad \Rightarrow \quad 2a_\perp\, \delta^\perp_0 \equiv 0$$

$$\Rightarrow \quad \delta^\perp_0 \equiv 0$$

Similarly $\ell^o \equiv 0 \quad \Rightarrow \quad \delta^{\shortparallel}_0 \equiv 0$

Therefore $g^o \equiv 0 \quad \Rightarrow \quad \varepsilon_0 - \delta^\perp_0 - \frac{1}{2}\,\delta^{\shortparallel}_0 \equiv 0$

$$\Rightarrow \quad \varepsilon_0 \equiv 0$$

The elements of a^μ_ν may be chosen in any way that satisfies (I.18). Considering (I.23), the simplest way is to choose $a^1_1 = -\frac{1}{i\,\Omega}$ and $a^{-1}_{-1} = \frac{1}{i\Omega}$, while setting all the other elements equal to zero. This choice also satisfies the equations for h^β and ℓ^β.

The vector $a^{\nu}_{\rho}\, v^{\rho}$ now has the elements

$$a^{1}_{\rho}\, v^{\rho} = a^{1}_{1}\, v^{1} + a^{1}_{0}\, v^{0} + a^{1}_{-1} v^{-1}$$

$$= - \frac{1}{i\Omega}\, v^{1}$$

$$a^{0}_{\rho}\, v^{\rho} = 0$$

$$a^{-1}_{\rho}\, v^{\rho} = \frac{1}{i\Omega}\, v^{-1} .$$

Now consider the elements of the vector $-\frac{i}{\Omega} e^{\nu\rho 0}\, v_{\rho}$:-

$$-\frac{i}{\Omega}\, e^{1,\rho,0}\, v_{\rho} = -\frac{i}{\Omega}\, e^{1,-1,0}\, v_{-1}$$

$$= -\frac{1}{i\Omega} v^{1}$$

$$-\frac{i}{\Omega}\, e^{0,\rho,0}\, v_{\rho} = 0$$

$$-\frac{i}{\Omega}\, e^{-1,\rho,0}\, v_{\rho} = -\frac{i}{\Omega}\, e^{-1,1,0}\, v_{1}$$

$$= \frac{1}{i\Omega} v^{-1}$$

Thus $a^{\nu}_{\rho}\, v^{\rho} \equiv -\frac{i}{\Omega}\, e^{\nu\rho 0}\, v_{\rho}$

Similarly a^{ν}_{μ} can be shown to be identical to $-\frac{i}{\Omega}\, e^{\nu . 0}_{\ \ \mu}$

Substitution for $a^{\nu}_{\rho}\, v^{\rho}$ in (I.13) gives

$$f_{0} = \left\{1 + \gamma_{\nu} \left(r^{\nu} - \frac{i}{\Omega}\, e^{\nu\rho 0}\, v_{\rho}\right)\right\} f_{M} \tag{I.24}$$

where $\gamma_{\nu} = g_{\nu} + h_{\nu}\, v_{1} v_{-1} + \ell_{\nu}\, v_{0}^{2}$

with g_{ν}, h_{ν} and ℓ_{ν} as defined by (I.22).

We note here that the only forms of the gradient vectors ε_{ν}, $\delta^{\perp}_{\nu}$ and $\delta^{\shortparallel}_{\nu}$ which satisfy (I.18) are those with ε_{0}, $\delta^{\perp}_{0}$ and $\delta^{\shortparallel}_{0}$ identically zero. This means that no gradients in density and temperature can exist along the direction of the zero-order magnetic field $\underline{B}_{0}$. The physical explanation for this lies in the fact that $[\frac{d}{dt}]_u f_0$ must be zero. Thus f_0, as "seen" by a particle moving along an unperturbed trajectory, must be independent of time, and therefore must be a function of "constants of the motion." These are quantities which are time-independent when evaluated on an unperturbed trajectory; the quantities $x + \frac{v_y}{\Omega}$ and $y - \frac{v_x}{\Omega}$ arising in f_0 through our choice of a_ρ^ν are constants of the motion. However, the equivalent expression for a z-dependent function would be $z - v_z t$. This is a constant of the motion, but any z-dependence in f_0 would then immediately bring in a time-dependence, so that f_0 would no longer be a steady-state equilibrium distribution. This implies that f_0 has no z-dependence, and therefore that f_0 cannot include a gradient in the z-direction, as we have shown analytically.

By writing f_0 in terms of rectangular Cartesian co-ordinates and using the method outlined in Appendix (III a), it is easily verified that the following expressions hold for small gradients:-

$$n = n_0(1 + \varepsilon_x x + \varepsilon_y y)$$

$$T^\perp = T_0^\perp (1 + \delta_x^\perp x + \delta_y^\perp y)$$

$$T'' = T_0'' (1 + \delta_x'' x + \delta_y'' y)$$

where

$$\varepsilon_x = 2^{-\frac{1}{2}} (\varepsilon^1 + \varepsilon^{-1})$$

$$\varepsilon_y = -i\, 2^{-\frac{1}{2}} (\varepsilon^1 - \varepsilon^{-1})$$

with similar expressions for the other gradients. This shows that the gradients defined in our expression for f_0 can in fact be identified with corresponding gradients in the actual plasma parameters.

Chapter 3

In order to derive an expression for $\underline{j}_1(k,\omega)$ from (I.10), we require a specific value for $\frac{\partial f_0}{\partial \underline{v}}$. On substituting $-\frac{i}{\Omega} e^{\nu.0}_{\mu}$ for a^{ν}_{μ} in (I.14), we have

$$\frac{\partial f_0}{\partial v^{\mu}} = \left\{ -\frac{i}{\Omega} \gamma_{\nu} e^{\nu.0}_{\mu} + 2(r^{\nu} - \frac{i}{\Omega} e^{\nu\rho 0} v_{\rho})(\delta^{\perp}_{\nu} w^{\perp}_{\mu} + \delta''_{\nu} w''_{\mu}) \right.$$
$$\left. -2 \left[1 + \gamma_{\nu}(r^{\nu} - \frac{i}{\Omega} e^{\nu\rho 0} v_{\rho}) \right] w_{\mu} \right\} f_M \qquad (I.25)$$

The $\underline{r}$ dependence in this expression causes great analytic difficulty if it is left in. The result in the electrostatic case is a complicated integral equation; the electromagnetic case is, as usual, much more troublesome. We follow Krall and Rosenbluth [9] in assuming a local approximation in which f_0 is taken as before, but $\frac{\partial f_0}{\partial v^{\mu}}$ (and therefore f_1) is taken as being independent of $\underline{r}$. Krall and Rosenbluth showed that the local approximation is valid if $\frac{\theta}{k_{\perp}} << 1$, where θ is a typical parameter gradient and $k_{\perp}$ is the component of $\underline{k}$ perpendicular to $\underline{B}_0$. This condition is equivalent to saying that the perturbation f_1 goes through many oscillations in the scale length for significant change in $f_0(\underline{r})$, and therefore over a few oscillations there is no $\underline{r}$-dependence of f_1.

We set $r^{\nu} = 0$ in (I.25), so that

$$\frac{\partial f_0}{\partial v^{\mu}} = \left\{ -2w_{\mu} - \frac{i}{\Omega} \gamma_{\nu} e^{\nu.0}_{\mu} - \frac{2i}{\Omega} e^{\nu\rho 0} v_{\rho}(\delta^{\perp}_{\nu} w^{\perp}_{\mu} + \delta''_{\nu} w''_{\mu} - \gamma_{\nu} w_{\mu}) \right\} f_M \qquad (I.26)$$

It is shown in Appendix (IIIc) that the term involving $e^{\nu\rho 0} v_{\rho}$ in (I.26) can be neglected if $k_{\perp}\rho > 1$ where ρ is the Larmor radius. For $k_{\perp}\rho \lesssim 1$, we must consider only small gradients in order that the local approximation holds. These small gradients can be represented approximately by the term $\gamma_{\nu} e^{\nu.0}_{\mu}$ in (I.26).

Our final expression for $\frac{\partial f_0}{\partial v^\mu}$ is therefore

$$\frac{\partial f_0}{\partial v^\mu} = - \left\{ \frac{i}{\Omega} \gamma_\nu e^{\nu.0}_{\mu} + 2w_\mu \right\} f_M \tag{I.27}$$

Substitution of this in (I.10) gives

$$j^\alpha = \frac{q^2}{m} \int_{\underline{v}} \int_{t=o}^{\infty} v^\alpha [\underline{E}_1 + \frac{1}{\omega} \{ \underline{v} \wedge (\underline{k} \wedge \underline{E}_1) \}]^\mu [\frac{i}{\Omega} \gamma_\nu e^{\nu.0}_{\mu} + 2w_\mu] \cdot f_M \exp [i(\omega t - \underline{k} \cdot \underline{X}(t))] \, d\underline{v} \, dt \tag{I.28}$$

We may note here that we have neglected the effects of magnetic field gradients in deriving (I.28). Parameter gradients result in particle drifts, giving equilibrium currents. The Maxwell induction equation (I.16) then implies that a gradient in $\underline{B}$ must exist to balance these currents. We show in Appendix (III d) that it is permissible to neglect magnetic field gradients provided $\beta = \frac{8\pi P}{B_0{}^2} << 1$, where P is the plasma pressure. Other consequences of assuming $\beta << 1$ are also dealt with in Appendix (III d).

In polarized co-ordinates $\underline{k} \wedge \underline{E}_1$ may be written as

$$(\underline{k} \wedge \underline{E}_1)^\lambda = - i \, e^{\lambda\alpha\beta} k_\alpha (E_1)_\beta$$

Therefore

$$[\underline{v} \wedge (\underline{k} \wedge \underline{E}_1)]^\mu = - i \, e^{\mu\rho\lambda} [v_\rho (\underline{k} \wedge \underline{E}_1)_\lambda]$$

$$= - e^{\mu\rho\lambda} e_{\lambda}^{.\tau\beta} v_\rho k_\tau (E_1)_\beta$$

Substitution of this expression in (I.28) gives

$$j^\alpha = \frac{q^2}{m} \int_{\underline{v}} \int_{t=o}^{\infty} v^\alpha \frac{i}{\Omega} [(E_1)^\mu - \frac{1}{\omega} e^{\mu\rho\lambda} e_{\lambda}^{.\tau\beta} v_\rho k_\tau (E_1)_\beta]$$

$$\cdot [\gamma_\nu e^{\nu.0}_{\mu} - 2i\Omega w_\mu] f_M \exp [i(\omega t - \underline{k} \cdot \underline{X}(t))] \, \underline{dv} \, dt$$

$$= \left\{ \frac{iq^2}{m\Omega} \int_{\underline{v}} \int_{t=o}^{\infty} V^{\alpha} [\delta^{\mu}_{\beta} - \frac{1}{\omega} e^{\mu\rho\cdot}{}_{\lambda} e^{\lambda\tau\cdot}{}_{\beta} v_{\rho} k_{\tau}] \right.$$

$$\left. \cdot [\gamma_{\nu} e^{\nu\cdot 0}{}_{\mu} - 2i\Omega w_{\mu}] f_M \exp [i(\omega t - \underline{k} \cdot \underline{X})] \underline{dv}\, dt \right\} (E_1)^{\beta}$$

In polarized co-ordinates (I.10a) takes the form

$$j^{\alpha} = \sigma^{\alpha}_{\beta} (E_1)^{\beta}$$

Comparison of this equation with the preceding one enables us to identify σ^{α}_{β} with the expression in curly brackets. Therefore

$$\sigma^{\alpha}_{\beta} = \frac{iq^2}{m\Omega} \int_{\underline{v}} \int_{t=o}^{\infty} V^{\alpha} [\delta^{\mu}_{\beta} - \frac{1}{\omega} e^{\mu\rho\cdot}{}_{\lambda} e^{\lambda\tau\cdot}{}_{\beta} v_{\rho} k_{\tau}]$$

$$\cdot [\gamma_{\nu} e^{\nu\cdot 0}{}_{\mu} - 2i\Omega w_{\mu}]$$

$$\cdot f_M \exp [i(\omega t - \underline{k} \cdot \underline{X})] \underline{dv}\, dt$$

We now define the vector operator $I^{\alpha} \{ \quad \}$ to be

$$I^{\alpha} \{ \quad \} = \frac{iq^2}{m\Omega} \int_{\underline{v}} \int_{t=o}^{\infty} V^{\alpha} f_M \exp [i(\omega t - \underline{k} \cdot \underline{X})] \{ \quad \} \underline{dv}\, dt$$

and so

$$\sigma^{\alpha}_{\beta} = I^{\alpha} \{ [\delta^{\mu}_{\beta} - \frac{1}{\omega} e^{\mu\rho\cdot}{}_{\lambda} e^{\lambda\tau\cdot}{}_{\beta} v_{\rho} k_{\tau}] [\gamma_{\nu} e^{\nu\cdot 0}{}_{\mu} - 2i\Omega w_{\mu}] \}$$

Now, using equation (I.22)

$$[\delta^{\mu}_{\beta} - \frac{1}{\omega} e^{\mu\rho\cdot}{}_{\lambda} e^{\lambda\tau\cdot}{}_{\beta} v_{\rho} k_{\tau}] [\gamma_{\nu} e^{\nu\cdot 0}{}_{\mu} - 2i\Omega w_{\mu}]$$

$$= [g_{\nu} + h_{\nu} v_{\perp}^2 + \ell_{\nu} v_{\parallel}^2] e^{\nu\cdot 0}{}_{\beta} - 2i\Omega w_{\beta}$$

$$- \frac{1}{\omega} e^{\mu\rho\cdot}{}_{\lambda} e^{\lambda\tau\cdot}{}_{\beta} e^{\nu\cdot 0}{}_{\mu} k_{\tau} [g_{\nu} v_{\rho} + h_{\nu} v_{\perp}^2 v_{\rho} + \ell_{\nu} v_{\parallel}^2 v_{\rho}]$$

$$+ \frac{2i\Omega}{\omega} e^{\mu\rho\cdot}{}_{\lambda} e^{\lambda\tau\cdot}{}_{\beta} k_{\tau} w_{\mu} v_{\rho}$$

Operating on this expression with $I^{\alpha} \{ \quad \}$, we get

$$\sigma^{\alpha}_{\beta} = e^{\nu . 0}_{\ \ \beta} [g_{\nu} A^{\alpha} + h_{\nu} C^{\alpha} + \ell_{\nu} D^{\alpha}] - 2i\Omega F^{\alpha}_{\beta}$$

$$- \frac{1}{\omega} e^{\mu\rho .}_{\ \ \lambda} e^{\lambda\tau .}_{\ \ \beta} e^{\alpha . 0}_{\ \ \mu} k_{\tau} [g_{\nu} G^{\alpha}_{\rho} + h_{\nu} K^{\alpha}_{\rho} + \ell_{\nu} L^{\alpha}_{\rho}]$$

$$+ \frac{2i\Omega}{\omega} e^{\mu\rho .}_{\ \ \lambda} e^{\lambda\tau .}_{\ \ \beta} k_{\tau} M^{\alpha}_{\mu\rho} \qquad (I.29)$$

where we define the following tensor moment integrals

$$\left. \begin{array}{lll} A^{\alpha} = I^{\alpha}\{1\} & C^{\alpha} = I^{\alpha}\{v_{\perp}^{2}\} & D^{\alpha} = I^{\alpha}\{v_{\parallel}^{2}\} \\ F^{\alpha}_{\beta} = I^{\alpha}\{w_{\beta}\} & G^{\alpha}_{\rho} = I^{\alpha}\{v_{\rho}\} & \\ K^{\alpha}_{\rho} = I^{\alpha}\{v_{\perp}^{2} v_{\rho}\} & L^{\alpha}_{\rho} = I^{\alpha}\{v_{\parallel}^{2} v_{\rho}\} & \\ M^{\alpha}_{\mu\rho} = I^{\alpha}\{w_{\mu} v_{\rho}\} & & \end{array} \right\} \qquad (I.30)$$

Thus for any given set of gradients, the conductivity tensor can be expressed in terms of members of the standard set of moment integrals.

These members may be evaluated separately, listed and referred to as and when required for a given problem. In Appendix (II) we evaluate and list the components of some of the simpler tensor moment integrals, and give some idea of how the more complicated integrals are evaluated; considerations of space do not allow us to carry through the evaluations.

The equation (I.29) shows how the use of polarized co-ordinates has enabled us to derive an expression for σ^{α}_{β} with several useful properties. Firstly it is compact, clear and unambiguous. Secondly, the gradients appear as coefficients multiplying moment tensors whose components can be evaluated separately from a given problem, and can be listed for easy reference. Thirdly, by following through the analysis and applying the tensor quotient law, it is easily seen that (I.29) is a tensor equation which holds its form under any tensor transformation. Therefore σ^{α}_{β} may be evaluated in another co-ordinate system merely by transforming the necessary tensors according to the appropriate transformation law, and then substituting them into (I.29).

The total conductivity tensor for a multi-species plasma is given by S^{α}_{β} where

$$S^{\alpha}_{\beta} = \sum_{\text{species}} \sigma^{\alpha}_{\beta}$$

A dispersion relation describing possible waves in such a plasma is obtained as follows:-

We have the Maxwell equations

$$\underline{\nabla} \wedge \underline{E}_1 = -\frac{1}{c}\frac{\partial \underline{B}_1}{\partial t}$$

$$\text{and} \quad \underline{\nabla} \wedge \underline{B}_1 = \frac{1}{c}\frac{\partial \underline{E}_1}{\partial t} + \frac{4\pi}{c}\underline{j}_1$$

The Fourier-Laplace transformed version of the second equation is obtained in the same way as we obtained equation (I.9) from the first equation. The transformed versions are

$$\underline{B}_1(\underline{k},\omega) = \frac{c}{\omega}\underline{k} \wedge \underline{E}_1(\underline{k},\omega)$$

$$\text{and} \quad \underline{k} \wedge \underline{B}_1(\underline{k},\omega) = -\frac{\omega}{c}\left[\underline{E}_1(\underline{k},\omega) + \frac{4\pi i}{\omega}\underline{j}_1(\underline{k},\omega)\right] \qquad \text{(I.31)}$$

Substituting for $\underline{B}_1$ and $\underline{j}_1$ in (I.31) and writing the result in polarized co-ordinates gives

$$\left[\underline{k} \wedge (\underline{k} \wedge \underline{E}_1)\right]^{\alpha} = -\frac{\omega^2}{c^2}\left[(E_1)^{\alpha} + \frac{4\pi i}{\omega} S^{\alpha}_{\beta}(E_1)^{\beta}\right]$$

$$= -\frac{\omega^2}{c^2}\left[\delta^{\alpha}_{\beta} + \frac{4\pi i}{\omega} S^{\alpha}_{\beta}\right](E_1)^{\beta}$$

$$= -\frac{\omega^2}{c^2}\mathcal{E}^{\alpha}_{\beta}(E_1)^{\beta} \qquad \text{(I.32)}$$

where we have defined the <u>dielectric tensor</u> $\mathcal{E}^{\alpha}_{\beta}$ by

$$\mathcal{E}^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} + \frac{4\pi i}{\omega} S^{\alpha}_{\beta}$$

We show as an example in Appendix (I) that

$$[\underline{k} \wedge (\underline{k} \wedge \underline{E}_1)]^\alpha = [k^\alpha k_\beta - k^2 \delta^\alpha_\beta] (E_1)^\beta$$

Substitution in (I.32) gives

$$[\mathcal{E}^\alpha_\beta + \frac{c^2}{\omega^2} \{ k^\alpha k_\beta - k^2 \delta^\alpha_\beta \}] (E_1)^\beta = 0$$

This is a system of linear equations in the components of $(E_1)^\beta$; the condition for a non-trivial solution for $(E_1)^\beta$ is the following:-

$$\det [\mathcal{E}^\alpha_\beta + \frac{c^2}{\omega^2} \{ k^\alpha k_\beta - k^2 \delta^\alpha_\beta \}] = 0$$

or, in terms of S^α_β

$$\det [\frac{4\pi i}{\omega} S^\alpha_\beta + (1 - \frac{c^2 k^2}{\omega^2}) \delta^\alpha_\beta + \frac{c^2}{\omega^2} k^\alpha k_\beta] = 0 \quad (I.33)$$

We include the effects of $\underline{E} \wedge \underline{B}$ drift velocities by making appropriate Lorentz transformations of the individual species conductivity tensors σ^α_β making up S^α_β. Suppose for a given species the $\underline{E} \wedge \underline{B}$ drift velocity is $\underline{v}_0$. We define an orthogonal Cartesian frame of reference moving with velocity $\underline{v}_0$, and carry out our derivation of σ^α_β as previously. We use $\underline{k}'$ and ω' to represent the wave-vector and frequency in this frame, so that we have a four-vector $(\underline{k}', \frac{i\omega'}{c})$. We now transform to a frame in which the species considered is moving with velocity $\underline{v}_0$. In this frame we take the four-vector to be $(\underline{k}, \frac{i\omega}{c})$. The transformation of the four-vector is given by the following equations:-

$$\underline{k}' = \underline{k} + (\gamma - 1) \frac{\underline{k} \cdot \underline{v}_0}{v_0^2} \underline{v}_0 - \gamma \frac{\underline{v}_0}{c^2} \omega$$

$$\omega' = \gamma (\omega - \underline{k} \cdot \underline{v}_0)$$

where $\gamma = (1 - \frac{v_0^2}{c^2})^{-\frac{1}{2}}$ [see reference (10)]

In our non-relativistic case $\gamma \approx 1$ so that

$$\underline{k}' \simeq \underline{k} - \frac{\underline{v}_0}{c^2}\,\omega$$

$$\omega' \simeq \omega - \underline{k}\cdot\underline{v}_0$$

The first equation may be written as

$$\underline{k}' \simeq \underline{k} - k\left(\frac{\underline{v}_0}{c}\right)\cdot\left(\frac{\omega}{ck}\right)$$

Now $\frac{v_0}{c} << 1$, and we normally consider the regime $\frac{\omega}{ck} \lesssim 1$. Thus we are justified in using the approximation $\underline{k}' \simeq \underline{k}$, so that the transformation becomes the simple Doppler shift

$$\left.\begin{array}{l} \underline{k}' \simeq \underline{k} \\ \omega' \simeq \omega - \underline{k}\cdot\underline{v}_0 \end{array}\right\} \qquad \text{(I.34)}$$

So, to modify equation (I.33) to include $\underline{E} \wedge \underline{B}$ drift velocities, we merely Doppler shift the expression for σ^{α}_{β} according to (I.34), for each species separately. We then denote the resulting total conductivity tensor by $\{S^{\alpha}_{\beta}\}_L$. Our final expression for the dispersion relation is

$$\det\left[\frac{4\pi i}{\omega}\{S^{\alpha}_{\beta}\}_L + \left(1 - \frac{c^2k^2}{\omega^2}\right)\delta^{\alpha}_{\beta} + \frac{c^2}{\omega^2}k^{\alpha}k_{\beta}\right] = 0 \qquad \text{(I.35)}$$

SECTION II :- the effect of weak collisions on the Bernstein modes.

Chapter 1

In Section I we derived a general dispersion relation in the context of a completely collisionless plasma involving various spatial parameter gradients. As observed in the Introduction, such a collisionless dispersion relation may contain descriptions of wave modes which depend on resonance effects, and which may be destroyed by the presence of even a very small amount of collisions. In this Section we introduce such collisions, and in the high-frequency regime we investigate their effect on particular resonance modes which are present in the final dispersion relation of Section I, namely the Bernstein modes. These occur at multiples of the ion and electron gyro-frequencies, propagating perpendicular and near-perpendicular to the magnetic field $\underline{B}_0$ in the plasma.

In this context the general dispersion relation of Section I is far too complicated to be dealt with as it stands; we therefore introduce a small collision frequency and investigate the effect of this on the Bernstein modes that exist in an otherwise collisionless, homogeneous, magnetized plasma.

The intention is then to apply the results of this investigation to a particular case of inhomogeneity, namely that of a temperature gradient in a magnetized plasma. If the cyclotron resonances which generate the Bernstein modes are destroyed in this particular case, it is reasonable to assume that they will not be significant in the general dispersion relation derived in Section I; therefore in using any reduced form of equation (I.35) in the high-frequency regime, we need not concern ourselves with possible effects due to instabilities in the Bernstein modes. In the following work, we make use of techniques and results published in 1964 by J.P. Dougherty [7].

The Bernstein (or electron-cyclotron) modes were first described in 1958 by I.B. Bernstein [11], who solved the linearized Vlasov equation by the Fourier-Laplace transform method. The method of integration along unperturbed trajectories as used in Section I is equivalent to this, and will be used in the subsequent analysis.

Following Section I, we note that the perturbation charge density is given by

$$\rho_1 \ (\underline{r}', t') = q \int_{\underline{v}} \int_{t=-\infty}^{t'} h \ (\bar{\underline{r}}, \bar{\underline{v}}, t) \ \underline{dv} \ dt$$

where $(\bar{\underline{r}}, \bar{\underline{v}})$ is the unperturbed trajectory passing through $(\underline{r}', \underline{v}')$ when $t = t'$. In general

$$h \ (\underline{r}, \underline{v}, t) = - \frac{q}{m} \ (\underline{E}_1 + \frac{\underline{v} \wedge \underline{B}_1}{c}) . \ \frac{\partial f_0}{\partial \underline{v}}$$

To derive the Bernstein modes, we follow Clemmow and Dougherty [4], using the electrostatic approximation (in effect letting c tend to infinity) and replacing $\underline{E}$ by $-\underline{\nabla}\phi$, where ϕ is a scalar potential. This gives

$$\rho_1 \ (\underline{r}', t') = \frac{q^2}{m} \int_{\underline{v}} \int_{t=-\infty}^{t'} \underline{\nabla}\phi \ . \ \frac{\partial f_0}{\partial \underline{v}} \ \underline{dv} \ dt$$

By a similar procedure to that used in Section I it is possible to take Fourier-Laplace components of this (equivalent to assuming that the variables are harmonic functions, that is they are proportional to the function exp [$i(\underline{k} \ . \ \underline{r} - \omega t)$].) The resulting equation is

$$\rho_1 \ (\underline{k}, \omega) = \frac{i q^2 \phi}{m} \int_{\underline{v}} \int_{t=o}^{\infty} \exp [\ i \ (\omega t - \underline{k} . \ \underline{X}(t)) \ ] \ \underline{k} \ . \ \frac{\partial f_0}{\partial \underline{v}} \ \underline{dv} \ dt$$

where $\underline{X}(t)$ is the unperturbed trajectory passing through $\underline{r} = 0$ with velocity $\underline{v}$ when $t = 0$.

The components of $\underline{X}(t)$ are linear in those of $\underline{v}$ so that

$$\underline{k} \cdot \underline{X}(t) \equiv \underline{v} \cdot \underline{p}(t)$$

where $\underline{p}(t)$ is easily obtained, and is the same for all particles since $\underline{v}$ has been extracted. Integrating by parts with respect to $\underline{v}$ we find that

$$\rho_1(k,\omega) = -\frac{q^2\phi}{m} \int_{\underline{v}} \int_{t=o}^{\infty} \underline{k} \cdot \underline{p}(t) \exp[\ i(\omega t - \underline{p} \cdot \underline{v})\ ] f_0 \, \underline{dv} \, dt$$

Taking f_0 to be the Maxwellian distribution

$$f_0 = n_0 \left(\frac{m}{2\pi\kappa T} \right)^{\frac{3}{2}} \exp\left[-\frac{mv^2}{2\kappa T} \right]$$

the $\underline{v}$ integration is the Fourier transform of a Gaussian distribution, which gives when carried out

$$\rho_1 = -\frac{n_0 \, q^2\phi}{m} \int_{t=o}^{\infty} \underline{k} \cdot \underline{p}(t) \exp\left\{ i\omega t - \frac{\kappa T}{2m} p^2 \right\} dt$$

No generality is lost if we choose our axes such that $\underline{k} = (k_\perp, 0, k_\parallel)$; particle orbit theory gives for $\underline{X}(t)$

$$\underline{X}(t) = \left(\frac{v_x}{\Omega} \sin \Omega t + \frac{v_y}{\Omega}(1 - \cos \Omega t), \frac{-v_x}{\Omega}(1 - \cos \Omega t) + \frac{v_y}{\Omega} \sin \Omega t, v_z t \right)$$

Thus $\underline{p}(t) = \left(\frac{k_\perp}{\Omega} \sin\Omega t, \frac{k_\perp}{\Omega}(1 - \cos \Omega t), k_\parallel t \right)$

and

$$\rho_1 = -\frac{n_0 q^2 \phi}{m} \int_{t=o}^{\infty} \left[\frac{k_\perp \sin \Omega t}{\Omega} + k_\parallel^2 t^2 \right] \exp\left\{ i\omega t - g(t) \right\} dt$$

where $g(t) \equiv \frac{\kappa T}{2m} \left[\frac{2k_\perp^2}{\Omega^2} (1 - \cos\Omega t) + k_\parallel^2 t^2 \right]$

$$= k_\perp^2 \rho^2 (1 - \cos\Omega t) + \tfrac{1}{2} k_\parallel^2 \rho^2 t^2$$

(ρ is the Larmor radius as defined previously)

Thus

$$\rho_1 = -\frac{n_0 q^2 \phi}{\kappa T} \int_{t=o}^{\infty} \frac{dg}{dt} \exp\left\{i\omega t - g(t)\right\} dt$$

$$= -\frac{n_0 q^2 \phi}{\kappa T} \left[1 + i\omega \int_{t=o}^{\infty} \exp\left\{i\omega t - g(t)\right\} dt \right]$$

Poisson's equation is $\nabla^2\phi = -4\pi \sum_{\text{species}} \rho_1 (\underline{r},t)$

or, in terms of Fourier-Laplace components

$$k^2\phi = 4\pi \sum_{\text{species}} \rho_1(\underline{k},\omega)$$

where $k^2 = k_{\parallel}^2 + k_{\perp}^2$

Suppose we consider a plasma with thermal electrons and a cold, stationary background of ions. The ion distribution function is $f_0 = n_0\ \delta(\underline{v})$, and the electron distribution is Maxwellian. Using Poisson's equation and our final expression for ρ_1, the dispersion relation for this plasma is

$$1 + \frac{k_i^2}{k^2} + \frac{k_e^2}{k^2} \left[1 + i\omega \int_{t=o}^{\infty} \exp\left\{ i\omega t - g(t)\right\} dt\right] = 0 \qquad \text{(II.1)}$$

where $k_{i,e}^2 = \dfrac{4\pi n_0 e^2}{T_{i,e}}$

The integral $\int_{t=o}^{\infty} \exp\{i\omega t - g(t)\}\,dt$ is the <u>Gordeyev integral</u> [12].

We may define a dimensionless form of this integral by setting $\tau = \Omega t$. Then the dimensionless Gordeyev integral is

$$G = \int_{\tau=o}^{\infty} \exp\{i\omega'\tau - g(\tau)\}\, d\tau \qquad \text{(II.2)}$$

where $\omega' = \dfrac{\omega}{\Omega}$.

Unfortunately, for general parameter values, the integral has no concise analytic result. In certain limited parameter ranges, however, analytic expressions can be derived. The results obtained in Appendix (IIIc) suggest that gradient effects are most significant within the local approximation in the regime $(k_\perp \rho)^2 >> 1$; it would therefore seem to be of interest to examine the Gordeyev integral in this regime.

To derive the Bernstein modes, we must look at wave propagation perpendicular to the magnetic field; that is we must set $k_\parallel = 0$. The dimensionless Gordeyev integral in this case is

$$G = \int_{\tau=0}^{\infty} \exp \left\{ i\omega\tau - k^2\rho^2 (1 - \cos \tau \right\} d\tau \quad . \qquad (II.3)$$

where $k^2 = k_\perp{}^2$

The usual way of deriving the Bernstein modes from this integral is to use the identity

$$\exp (\lambda \cos \Omega t) = \sum_{n=-\infty}^{\infty} I_n(\lambda) \exp (in\Omega t)$$

where I_n is the Bessel function of imaginary argument and λ in this case is $k^2\rho^2$. The integral may then be easily carried out, and the asymptotic forms of I_n used. However, we intend to investigate the effect on the Bernstein modes of introducing a small collision frequency; the results involve modified Gordeyev integrals of greater complexity, for which the Bessel function approach is much more difficult. We shall therefore apply an approximation technique to equation (II.1) to indicate the origin of the Bernstein modes, and then use the same technique to examine the effect of a small collision frequency on these modes.

Chapter 2

The dimensionless Gordeyev integral for $k_{\parallel} = 0$ is

$$G_1 = \int_{\tau=0}^{\infty} \exp\{i\omega'\tau - k^2 \rho^2(1 - \cos\tau)\}\, d\tau$$

and we intend to examine the regime $k^2\rho^2 >> 1$. The function $1 - \cos\tau$ has the form shown in Figure (3a). Our basic assumption in the approximation technique we now use is that the integrand in G_1 contributes significantly to the integral only in regions near $1 - \cos\tau = 0$, since the integrand contains the factor $\exp\{- k^2 \rho^2 (1 - \cos\tau)\}$ and $k^2 \rho^2 >> 1$. This assumption is supported by later computational results for the case with a collision frequency included. Thus we need only examine regions where $\cos\tau \simeq 1$; that is where $\tau = 2n\pi + \phi$ with $|\phi| << 1$ for $n = 0, 1, 2, \ldots\ldots$

Define the number δ_n $(n = 0, 1, 2, \ldots.)$ to be the size of a <u>domain of significance</u> around the point $\tau = 2n\pi$. By this we mean that δ_n is large enough for the following inequalities to hold:-

$$\int_0^{\delta_0} \exp[\, s(\tau)\,]\, d\tau \;>>\; \int_{\delta_0}^{2\pi-\delta_1} \exp[\, s(\tau)\,]\, d\tau$$

and

$$\int_{2\pi n-\delta_n}^{2\pi n+\delta_n} \exp[\, s(\tau)\,]\, d\tau \;>>\; \int_{2\pi n+\delta_n}^{2\pi(n+1)-\delta_{n+1}} \exp[\, s(\tau)\,]\, d\tau$$

for $n \geq 1$, where $s(\tau) = i\omega'\tau - k^2 \rho^2 (1 - \cos\tau)$.

G_1 may now be written as

$$G_1 \simeq \int_0^{\delta_0} \exp[\, s(\tau)\,]\, d\tau + \sum_{n=1}^{\infty} \int_{2\pi n-\delta_n}^{2\pi n+\delta_n} \exp[\, s(\tau)\,]\, d\tau$$

or, replacing τ by $2\pi n + \phi$ so that

$$1 - \cos\tau \simeq \frac{\phi^2}{2}$$

the following approximation holds:-

$$G_1 \simeq \int_0^{\delta_0} \exp\left\{i\omega'\phi - k^2\rho^2\frac{\phi^2}{2}\right\} d\phi$$

$$+ \sum_{n=1}^{\infty} \exp\left\{2n\pi i\omega'\right\} \int_0^{\delta_n} \exp\left\{i\omega'\phi - k^2\rho^2\frac{\phi^2}{2}\right\} d\phi$$

$$+ \sum_{n=1}^{\infty} \exp\left\{2n\pi i\omega'\right\} \int_0^{\delta_n} \exp\left\{-i\omega'\phi - k^2\rho^2\frac{\phi^2}{2}\right\} d\phi \qquad \text{(II.4)}$$

Now $1 - \cos\tau = \frac{\phi^2}{2} - \frac{\phi^4}{4!}$ ....., so that the integral involving $\exp\left\{-k^2\rho^2\frac{\phi^2}{2}\right\}$ is more convergent than the one involving $\exp\left\{-k^2\rho^2(1-\cos\tau)\right\}$; the δ_n's must also be the sizes of domains of convergence for the integrals in equation (II.4).

Therefore

$$\int_0^{\delta_n} \exp\left\{i\omega'\phi - k^2\rho^2\frac{\phi^2}{2}\right\} d\phi \simeq \int_0^{\infty} \exp\left\{i\omega'\phi - k^2\rho^2\frac{\phi^2}{2}\right\} d\phi$$

for $n = 0, 1, 2, \ldots$

Consider the integral

$$I = \int_0^{\infty} \exp\left\{\pm i\omega'\phi - k^2\rho^2\frac{\phi^2}{2}\right\} d\phi.$$

Change the variables as follows:-

$$\zeta = \frac{\pm\omega'}{\sqrt{2}k\rho} \qquad \phi = \frac{\sqrt{2}}{k\rho}(i\zeta - p).$$

$$\text{Then } I = \frac{\sqrt{2}}{k\rho} e^{-\zeta^2} \int_{i\zeta-\infty}^{i\zeta} e^{-p^2} dp$$

$$= -\frac{i}{\sqrt{2}k\rho} Z(\zeta)$$

where $Z(\zeta)$ is the plasma dispersion function of Fried and Conte [13a]

Thus from (II.4)

$$G_1 \approx -\frac{i}{\sqrt{2}k\rho} Z\left(\frac{\omega'}{\sqrt{2}k\rho}\right) \sum_{n=o}^{\infty} \exp\{2n\pi i\omega'\}$$

$$-\frac{i}{\sqrt{2}k\rho} Z\left(\frac{-\omega'}{\sqrt{2}k\rho}\right) \sum_{n=1}^{\infty} \exp\{2n\pi i\omega'\}$$

$$= -\frac{i}{\sqrt{2}}\left[Z\left(\frac{\omega'}{\sqrt{2}k\rho}\right) \frac{(k\rho)^{-1}}{(1-\exp[2\pi i\omega'])} + Z\left(\frac{-\omega'}{\sqrt{2}k\rho}\right) \left\{\frac{(k\rho)^{-1}}{(1-\exp[2\pi i\omega'])} - (k\rho)^{-1}\right\}\right]$$

If $(1 - \exp[2\pi i\omega'])$ is of order unity, then the contribution that G_1 makes to a dispersion relation such as (II.1) is quite small because of the factor $(k\rho)^{-1}$. However, if $(1 - \exp[2\pi i\omega'])$ is of order $(k\rho)^{-1}$, then the contribution is much more significant. This condition results in the following

$$\cos 2\pi i\omega' \simeq 1$$

$$\Rightarrow \quad \omega \simeq n\Omega \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

Bernstein showed that dispersion relations involving G_1 have solutions with real ω and k for $\omega \simeq n\Omega$ $(n \neq 0)$. These are known as the Bernstein modes, and they are undamped for propagation perpendicular to $\underline{B}_0$.

Let us examine the regime $k_\perp^2\rho^2 \gg 1$ and $k_\parallel^2\rho^2 \gtrsim 1$. We have

$$G = \int_o^{\infty} \exp\{i\omega'\tau - k_\perp^2\rho^2 (1 - \cos\tau) - \tfrac{1}{2}k_\parallel^2\rho^2\tau^2\}\, d\tau$$

Making the change of variable $\tau = 2n\pi + \phi$ as before and using the same approximations gives

$$G \approx \int_o^{\infty} \exp\{i\omega'\phi - \tfrac{1}{2}k^2\rho^2\phi^2\}\, d\phi \quad + \sum_{n=1}^{\infty} \exp\{2\pi n i\omega' - \tfrac{1}{2}k_\parallel^2\rho^2(2\pi n)^2\}$$

$$\cdot \left[\int_o^{\infty} \exp\{\phi(i\omega' - k_\parallel^2\rho^2 .\, 2n\pi) - \tfrac{1}{2}k^2\rho^2\phi^2\}\, d\phi\right.$$

$$\left. + \int_o^{\infty} \exp\{-\phi(i\omega' - k_\parallel^2\rho^2 .\, 2n\pi) - \tfrac{1}{2}k^2\rho^2\phi^2\}\, d\phi\right]$$

For $n = 1$, $(2\pi n)^2 \approx 40$; we also have $k_{\shortparallel}^2\rho^2 \gtrsim 1$. Thus $\exp\ [-\frac{1}{2}\ .\ (2\pi n)^2\ k_{\shortparallel}^2\rho^2\ ] \leq \exp\ [-20\ ]$ for $n \geq 1$, and so the only significant term in G is the $n = 0$ term. Therefore

$$G \simeq \int_0^{\infty} \exp\left\{i\omega'\phi - \tfrac{1}{2}k^2\rho^2\phi^2\right\} d\phi$$

$$= -\frac{i}{\sqrt{2}k\rho}\ Z\left(\frac{\omega}{\sqrt{2}kv_T}\right)$$

as before, where v_T is the mean thermal velocity.

For cold, stationary ions the following dispersion relation results:-

$$1 + \frac{k_i^2}{k^2} - \frac{k_e^2}{2k^2}\ Z'\left(\frac{\omega}{\sqrt{2}kv_T}\right) = 0.$$

This is the dispersion relation for ion acoustic waves in an unmagnetized plasma with cold stationary ions. Figure (4) shows the angular regions relative to the magnetic field in which the different types of waves are important. For $k_{\shortparallel} = 0$, there are undamped Bernstein waves at $\omega \simeq n\Omega$ for $n \neq 0$, and the damped $n = 0$ wave is in fact the ion acoustic wave. For $k_{\shortparallel}^2\rho^2 << 1$, the Bernstein waves are damped, but still of the same order as the ion-acoustic wave. For $k_{\shortparallel}^2\rho^2 \gtrsim 1$, the ion acoustic wave dominates; the Bernstein waves are damped so quickly that they can be ignored.

Chapter 3

To investigate the effect on the Bernstein modes of introducing a small amount of collisions into the plasma, we make use of theory developed by Dougherty [7]. We provide here an outline of his procedure.

Dougherty begins with a Boltzmann equation as follows:-

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial r_j} + a_j \frac{\partial f}{\partial v_j} = \left(\frac{\partial f}{\partial t}\right)_c \qquad \text{(II.5)}$$

where Einstein sums of Cartesian tensors are used, and a_j is the macroscopic acceleration vector. $\left(\frac{\partial f}{\partial t}\right)_c$ is a Fokker-Planck collision term given by

$$\left(\frac{\partial f}{\partial t}\right)_c = \frac{\partial}{\partial v_i} \left\{ - A_i f + \tfrac{1}{2} \frac{\partial}{\partial v_j} (B_{ij}\, f) \right\}$$

where $$A_i = -\nu(v_i - u_i)$$

and $$B_{ij} = \left(\frac{2\nu\kappa T}{m}\right) \delta_{ij}$$

ν is an inverse time, independent of velocity

u_i and T are the local drift velocity

and temperature respectively, given by

$$nu_i = \int v_i\, f\, \underline{dv}$$

$$3\, n\kappa T = \int m(\underline{v} - \underline{u})^2\, f\, \underline{dv}$$

where n is the local number density of particles, defined by $n = \int f\, \underline{dv}$.

Equation (II.5) is linearized, and written in the form

$$Df_1 = h$$

where D is a linear differential operator, and f_1 is given by $f = f_0 + f_1$, f_0 being an equilibrium distribution function. Thus

$$f_1 = D^{-1} h$$

where D^{-1} is an inverse differential operator. Dougherty defines a set of quantities

$$ {}_{i\ldots\ldots}H_{j\ldots\ldots} = \int v_{i\ldots\ldots} D^{-1}(v_{j\ldots\ldots}f_0)\, \underline{dv} $$

where at most two suffixes are needed before or after H. Each suffix (if any) labels the component of $\underline{v}$ to be inserted in the appropriate place in the integral.

The theory gives for f_1

$$ f_1 = \frac{m}{\kappa T_0}(a_j + \nu u_j)D^{-1}(v_j f_0) + \frac{\nu T_1}{T_0}\left[\frac{m}{\kappa T_0} D^{-1}(v^2 f_0) - 3D^{-1} f_0\right] \qquad \text{(II.6)} $$

and the following expressions for the perturbation quantities n_1, $\underline{u}$, and T_1 (n_0 and T_0 are equilibrium values of n and T and a_j is now the perturbed macroscopic acceleration vector)

$$ \left.\begin{aligned} n_1 &= \frac{m}{\kappa T_0}(a_j + \nu u_j)H_j + \frac{\nu T_1}{T_0}\left(\frac{m}{\kappa T_0} H_{jj} - 3H\right) \\ n_0 u_i &= \frac{m}{\kappa T_0}(a_j + \nu u_j)\,{}_iH_j + \frac{\nu T_1}{T_0}\left(\frac{m}{\kappa T_0}\,{}_iH_{jj} - 3\,{}_iH\right) \\ n_1 + n_0\frac{T_1}{T_0} &= \frac{m}{3\kappa T_0}\left\{\frac{m}{\kappa T_0}(a_j + \nu u_j)\,{}_{ii}H_j + \frac{\nu T_1}{T_0}\left(\frac{m}{\kappa T_0}\,{}_{ii}H_{jj} - 3\,{}_{ii}H\right)\right\} \end{aligned}\right\} \text{(II.7)} $$

We write $\frac{\kappa T_0}{m}$ as $v_T{}^2$, where v_T is the mean thermal velocity of the particle species considered. Our procedure now differs from that of Dougherty in that we derive an expression for the charge density $\rho(\underline{r},t)$, while Dougherty solves for the perturbation velocity $\underline{u}$.

Solving equations (II.7) for u_i gives

$$ u_i = \frac{M_{ij}}{[\, n_0 - \nu\, M_{kk}]}\, a_j $$

where

$$ M_{ij} = \frac{1}{v_T^2}\left\{ {}_iH_j + \frac{3\nu\left[\frac{1}{3v_T{}^2}\,{}_{ii}H_j - H_j\right]\left[\frac{1}{v_T{}^2}\,{}_iH_{jj} - 3\,{}_iH\right]}{n_0 + \nu\left(\frac{1}{v_T^2}\left[{}_{ii}H + H_{jj}\right] - 3H - \frac{1}{3v_T{}^4}\,{}_{ii}H_{jj}\right)} \right\} $$

Substitution of u_i back in (II.7) gives

$$\frac{T_1}{T_0} = \frac{\frac{1}{v_T^2}\left[1 + \frac{\nu M_{jj}}{n_0 - \nu M_{jj}}\right]\left[\frac{1}{3v_T^2}\, {}_{ii}H_j - H_j\right] a_j}{n_0 + \nu\left(\frac{1}{v_T^2}\left[{}_{ii}H + H_{jj}\right] - 3H - \frac{1}{3v_T^2}\, {}_{ii}H_{jj}\right)}$$

Note that n_1 is implicit in u_i and $\frac{T_1}{T_0}$, even though it is not explicitly involved in the expression for f_1.

To derive a dispersion relation, f_1 itself is not required; the perturbed charge density $\rho(\underline{r},t)$ is sufficient, and this is given as follows, using equation (II.6):-

$$\rho(\underline{r},t) = q \int f_1(\underline{r},\underline{v},t)\, \underline{dv}$$

$$= q\left[\frac{1}{v_T^2}\left(1 + \frac{\nu M_{jj}}{n_0 - \nu M_{jj}}\right) \int a_i\, D^{-1}(v_i f_0)\, \underline{dv}\right.$$

$$\left. + \nu\, \frac{T_1}{T_0}\left(\frac{1}{v_T^2} \int D^{-1}(v^2 f_0)\, \underline{dv} - 3\int D^{-1} f_0\, \underline{dv}\right)\right] \qquad \text{(II.8)}$$

In general $a_i = \frac{q}{m}\left[\underline{E} + \frac{1}{c}\underline{v} \wedge \underline{B}\right]_i$

where $\underline{E}$ and $\underline{B}$ are the perturbed electric and magnetic fields respectively. To investigate Bernstein modes, we follow Chapter 1 of this Section by making the electrostatic approximation and taking Fourier-Laplace components, which is again equivalent to assuming a harmonic dependence of the form $\exp[\,i(\underline{k}\cdot\underline{r} - \omega t)]$.

The expression for a_i becomes

$$a_i = \frac{q}{m} E_i$$

$$= -\frac{q}{m}\frac{\partial}{\partial r_i}\phi$$

where ϕ is an electrostatic potential independent of $\underline{v}$. Thus

$$a_i = -\frac{iq}{m} k_i\, \phi \qquad \text{(II.9)}$$

taking a harmonic dependence for ϕ as in Chapter 1. Therefore

$$\int a_i D^{-1} (v_i f_0) \underline{dv} = \frac{-iq}{m} k_i \phi \int D^{-1} (v_i f_0) \underline{dv}$$

$$= \frac{-iq}{m} k_i \phi H_i$$

Also $\int D^{-1} (v^2 f_0) \underline{dv} = H_{jj}$

and $\int D^{-1} (f_0) \underline{dv} = H$

Substitution of these values and the expression for $\frac{T_1}{T_0}$ into (II.8) gives in terms of $\underline{k}$ and ω

$$\rho(\underline{k},\omega) = \frac{-iq^2 k_j \phi}{m v_T^2} \left\{ (1 + \frac{\nu M_{jj}}{n_0 - \nu M_{jj}}) H_j + \frac{\nu [1 + \frac{\nu M_{jj}}{n_0 - \nu M_{jj}}] [\frac{1}{3v_T^2}\, {}_{ii}H_j - H_j] [\frac{1}{v_T^2} H_{jj} - 3H]}{n_0 + \nu(\frac{1}{v_T^2} [{}_{ii}H + H_{jj}] - 3H - \frac{1}{3v_T^4}\, {}_{ii}H_{jj})} \right\}$$

$$= \frac{-iq^2 \phi K_j}{m v_T^2} \left[1 + \frac{\nu M_{jj}}{n_0 - \nu M_{jj}} \right] \left[H_j + \frac{\nu(\frac{1}{3v_T^2}\, {}_{ii}H_j - H_j)(\frac{1}{v_T^2} H_{jj} - 3H)}{n_0 - \nu(3H + \frac{1}{3v_T^4}\, {}_{ii}H_{jj} - \frac{1}{v_T^2} [{}_{ii}H + H_{jj}])} \right]$$

We now convert all Cartesian tensors to the polarized co-ordinate tensor form developed in Section I. In terms of these tensors, M_{ij} becomes

$$M^\lambda_\mu = \frac{1}{v_T^2} \left\{ {}^\lambda H_\mu + \frac{3\nu [{}^\lambda H - \frac{1}{3v_T^2}\, {}^\lambda H^\delta_\delta] [H_\mu - \frac{1}{3v_T^2}\, {}^\varepsilon_\varepsilon H_\mu]}{n_0 - 3\nu [H - \frac{1}{3v_T^2} ({}^\varepsilon_\varepsilon H + H^\delta_\delta) + \frac{1}{9v_T^4}\, {}^\varepsilon_\varepsilon H^\delta_\delta} \right\}$$

where upper indices are contravariant and lower indices are covariant; repeated indices (one contravariant and the other covariant) are again Einstein sums.

The charge density is

$$\rho(\underline{k},\omega) = \frac{-iq^2\phi}{m{v_T}^2}\left[1 + \frac{\nu M^\delta_\delta}{n_0 - \nu M^\delta_\delta}\right]\left[H_\lambda + \frac{\nu(\frac{1}{3{v_T}^2}\,{}^\varepsilon_\varepsilon H_\lambda - H_\lambda)(\frac{1}{{v_T}^2}\,H^\delta_\delta - 3H)}{n_0 - \nu\,(3H + \frac{1}{3v_T^4}\,{}^\varepsilon_\varepsilon H^\delta_\delta - \frac{1}{v_T^2}\,[\,{}^\varepsilon_\varepsilon H + H^\delta_\delta])}\right]k^\lambda$$

$$\text{Define } A_\lambda = \frac{\nu\,(\frac{1}{3v_T^2}\,{}^\varepsilon_\varepsilon H_\lambda - H_\lambda)}{n_0 - \nu(3H + \frac{1}{3v_T^4}\,{}^\varepsilon_\varepsilon H^\delta_\delta - \frac{1}{v_T^2}\,[\,{}^\varepsilon_\varepsilon H + H^\delta_\delta\,])}$$

$$\text{Then } M^\lambda_\mu = \frac{1}{v_T^2}\,\{\,{}^\lambda H_\mu - 3\,[\,{}^\lambda H - \frac{1}{3v_T^2}\,{}^\lambda H^\delta_\delta\,]\,A_\mu\,\} \qquad \text{(II.10)}$$

and

$$\rho(\underline{k},\omega) = \frac{-iq^2\phi}{mv_T^2}\left[1 + \frac{\nu M^\delta_\delta}{n_0 - \nu M^\delta_\delta}\right]\left[H_\lambda + (\,\frac{1}{v_T^2}\,H^\delta_\delta - 3H)A_\lambda\right]k^\lambda \qquad \text{(II.11)}$$

These expressions involve a $\underline{k}$ with general direction. We would like to investigate the effect of collisions on Bernstein modes in the region where they are most important, namely with $k_\parallel = 0$ so that they are undamped in the collisionless case. Thus in order to find $\rho(\underline{k},\omega)$, we must evaluate the following set of H-functions with $k_\parallel = 0$:-

$$H,\ H_\lambda,\ {}^\lambda H,\ {}^\varepsilon_\varepsilon H,\ H^\delta_\delta,\ {}^\delta H_\delta,\ {}^\varepsilon_\varepsilon H_\lambda,\ {}^\lambda H^\delta_\delta,\ {}^\varepsilon_\varepsilon H^\delta_\delta$$

Dougherty shows that the general H-function ${}^{\lambda\ldots}H_{\mu\ldots}$ can be evaluated as follows:-

$$^{\lambda\ldots}H_{\mu\ldots} = \left[\,\frac{1}{i}\,\frac{\partial}{\partial\sigma_\lambda}\,\ldots\,\frac{1}{i}\,\frac{\partial}{\partial\rho^\mu}\,\ldots\,I\,\right]_{\underline{\sigma} = \underline{\rho} = 0} \qquad \text{(II.12)}$$

where

$$I = n_0 \int_0^\infty \exp\,[\,-\Phi(t) - \Psi(t) + i\omega t\,]\,dt \qquad \text{(II.13)}$$

$$\Phi(t) = \frac{1}{v_T{}^2} \left\{ \frac{k_\parallel^2}{\nu^2} \ (\nu t - 1 + e^{-\nu t}) + \frac{k_\perp{}^2}{\nu^2+\Omega^2} [\ \cos\chi + \nu t - e^{-\nu t}\cos(\Omega t - \chi)] \right\}$$

$$\Psi(t) = \frac{1}{v_T{}^2} \left\{ \tfrac{1}{2}\,\sigma_\lambda\sigma^\lambda + \tfrac{1}{2}\,\rho_\lambda\rho^\lambda + k_\lambda\sigma^\lambda \quad \left(\frac{1 - e^{-[\nu - i\Omega\lambda]t}}{\nu - i\Omega\lambda}\right) \right.$$

$$\left. + \ k_\lambda\rho^\lambda \ \left(\frac{1 - e^{-[\nu + i\Omega\lambda]t}}{\nu + i\Omega\lambda}\right) + \rho^\lambda\sigma_\lambda \ e^{-(\nu + i\Omega\lambda)t} \right\}$$

$\chi = 2\tan^{-1}\left(\frac{\nu}{\Omega}\right)$ and $0 \leq \chi \leq \pi$

Note that Greek indices are used as algebraic quantities at some points in the expression for $\Psi(t)$; see Appendix (Ib).

For $k_\parallel = 0$

$$\Phi(t) = \frac{1}{v_T^2} \cdot \frac{k^2}{\nu^2 + \Omega^2} \ [\ \cos\chi + \nu t - e^{-\nu t}\cos(\Omega t - \chi)\ ] \qquad \text{(II.14)}$$

Since σ and ρ are eventually set equal to zero, $\Psi(t)$ disappears from the final result of any integration.

The simplest H-function is H itself, given by

$$H = n_0 \int_0^\infty \exp\{-\Phi(t) + i\omega t\}\,dt$$

and in general

$${}^{\lambda\ldots}H_{\mu\ldots} = n_0 \int_0^\infty {}^{\lambda\ldots}F_{\mu\ldots}(t)\exp\{-\Phi(t) + i\omega t\}\,dt$$

where ${}^{\lambda\ldots}F_{\mu\ldots}(t) = \left[\frac{1}{i}\frac{\partial}{\partial\sigma_\lambda} \cdots \frac{1}{i}\frac{\partial}{\partial\rho^\mu} \cdots e^{-\Psi(t)}\right]_{\underline{\sigma} = \underline{\rho} = 0}$

We now proceed to derive one of the required F's and to list the others, in order to prevent tedious repetition. The necessary F's are the following:-

$$F_\lambda,\ {}^\lambda F,\ {}^\varepsilon_\varepsilon F,\ F^\delta_\delta,\ {}^\delta F_\delta,\ {}^\varepsilon_\varepsilon F_\lambda,\ {}^\lambda F^\delta_\delta,\ {}^\varepsilon_\varepsilon F^\delta_\delta$$

where all the functions are to be evaluated with $k_\parallel = 0$, that is with $k^0 = k_0 = 0$.

Consider the case of ${}^{\lambda}F_{\mu}$:-

$$ {}^{\lambda}F_{\mu} = - \left[\frac{\partial}{\partial\sigma_{\lambda}} \frac{\partial}{\partial\rho^{\mu}} e^{-\Psi} \right]_{\underline{\sigma} = \underline{\rho} = 0} $$

$$ = \left[\frac{\partial^2\Psi}{\partial\sigma_{\lambda}\partial\rho^{\mu}} - \frac{\partial\Psi}{\partial\sigma_{\lambda}} \frac{\partial\Psi}{\partial\rho^{\mu}} \right]_{\underline{\sigma} = \underline{\rho} = 0} $$

Now

$$ \frac{\partial\Psi}{\partial\rho^{\mu}} = v_T^2 \left\{ \rho_{\mu} + k_{\mu} \frac{(1 - e^{-[\nu+i\Omega\mu]t})}{\nu + i\Omega\mu} + \sigma_{\mu} e^{-[\nu+i\Omega\mu]t} \right\} $$

Therefore $$ \left[\frac{\partial\Psi}{\partial\rho^{\mu}} \right]_{\underline{\sigma} = \underline{\rho} = 0} = v_T^2 k_{\mu} \left\{ \frac{1 - e^{-[\nu+i\Omega\mu]t}}{\nu + i\Omega\mu} \right\} $$

Similarly

$$ \left[\frac{\partial\Psi}{\partial\sigma_{\lambda}} \right]_{\underline{\sigma} = \underline{\rho} = 0} = v_T^2 k^{\lambda} \left\{ \frac{1 - e^{-[\nu+i\Omega\lambda]t}}{\nu + i\Omega\lambda} \right\} $$

$$ \left[\frac{\partial^2\Psi}{\partial\sigma_{\lambda}\partial\rho^{\mu}} \right]_{\underline{\sigma} = \underline{\rho} = 0} = v_T^2 \delta^{\lambda}_{\mu} e^{-[\nu+i\Omega\mu]t} $$

Therefore

$$ {}^{\lambda}F_{\mu} = v_T^2 \delta^{\lambda}_{\mu} e^{-(\nu+i\Omega\mu)t} - v_T^4 k^{\lambda}k_{\mu} \left\{ \frac{1 - e^{-[\nu+i\Omega\lambda]t}}{\nu + i\Omega\lambda} \right\} \left\{ \frac{1 - e^{-[\nu+i\Omega\mu]t}}{\nu + i\Omega\mu} \right\} $$

and

$$ {}^{\delta}F_{\delta} = v_T^2 \left[e^{-(\nu+i\Omega)t} + e^{-\nu t} + e^{-(\nu - i\Omega)t} \right] $$

$$ - v_T^4 \left[k^1k_1 \frac{(1 - e^{-(\nu+i\Omega)t})^2}{(\nu+i\Omega)^2} + k^{-1}k_{-1} \frac{(1 - e^{-(\nu-i\Omega)t})^2}{(\nu-i\Omega)^2} \right] $$

Now $k^1k_1 = k^{-1} k_{-1} = \frac{1}{2}k^2$ where $k^2 = k_{\perp}^2$

Therefore

$$^{\delta}F_{\delta} = v_T^2 \,[\, e^{-\nu t}(1 + 2\cos\Omega t)$$

$$- \frac{\frac{1}{2}k^2 v_T^2}{(\nu^2+\Omega^2)^2} \{ (\nu - i\Omega)^2 \,[1-e^{-(\nu+i\Omega)t}]^2 + (\nu+i\Omega)^2 \,[1-e^{-(\nu-i\Omega)t}]^2 \}]$$

Finally

$$^{\delta}F_{\delta} = v_T^2 \,[\, e^{-\nu t}(1 + 2\cos\Omega t)$$

$$- \frac{k^2 v_T^2}{\nu^2+\Omega^2} \{ \frac{\nu^2-\Omega^2}{\nu^2+\Omega^2} (1 - 2e^{-\nu t}\cos\Omega t + e^{-2\nu t}\cos 2\Omega t)$$

$$+ \frac{2\nu\Omega}{\nu^2+\Omega^2} e^{-\nu t} (2\sin\Omega t - e^{-\nu t}\sin 2\Omega t)\}]$$

Similarly, the following expressions may be derived:-

$$F_{\lambda} = i\, v_T^2\, k_{\lambda} \,[\, \frac{1 - e^{-(\nu+i\Omega\lambda)t}}{\nu + i\Omega\lambda} \,]$$

$$^{\lambda}F = i\, v_T^2\, k^{\lambda} \,[\, \frac{1 - e^{-(\nu+i\Omega\lambda)t}}{\nu + i\Omega\lambda} \,]$$

$$^{\varepsilon}F_{\varepsilon} = v_T^2 \,(3 - \frac{k^2 v_T^2}{\nu^2+\Omega^2} [\, 1 - 2e^{-\nu t}\cos\Omega t + e^{-2\nu t}\,])$$

$$= F^{\delta}_{\delta}$$

$$^{\varepsilon}_{\varepsilon}F_{\lambda} = i\, v_T^4 \{ 2e^{-(\nu+i\Omega\lambda)t} \frac{[1 - e^{-(\nu-i\Omega\lambda)t}]}{\nu - i\Omega\lambda}$$

$$- \frac{[1 - e^{-(\nu+i\Omega\lambda)t}]}{\nu + i\Omega\lambda} \,[\, \frac{k^2 v_T^2}{\nu^2+\Omega^2} (1 - 2e^{-\nu t}\cos\Omega t + e^{-2\nu t})$$

$$- 3] \} k_{\lambda}$$

$^{\lambda}F^{\varepsilon}_{\varepsilon}$ is the same as $^{\varepsilon}_{\varepsilon}F_{\lambda}$ except that k^{λ} replaces k_{λ}.

$$^{\varepsilon}_{\varepsilon}F^{\delta}_{\delta} = v_T^4 \,[\, v_T^4 U^2 - 2v_T^2(3U + 2V) + (9 + 2e^{-2\nu t} \,[2 + \cos 2\Omega t]\,)]$$

where $U = \frac{k^2}{\nu^2+\Omega^2} (1 - 2e^{-\nu t}\cos\Omega t + e^{-2\nu t})$

and $$V = \frac{k^2}{\nu^2+\Omega^2} \; e^{-\nu t} \; \left\{ \frac{\nu^2-\Omega^2}{\nu^2+\Omega^2} \; [\cos \Omega t(1 + e^{-2\nu t}) - 2e^{-\nu t} \;] \right.$$

$$\left. + \frac{2\nu\Omega}{\nu^2+\Omega^2} \; [\; \sin \Omega t(1 - e^{-2\nu t}) \;] \right\}$$

If we define

$$^{\lambda}C = {}^{\lambda}H - \frac{1}{3v_T^{\,2}} \; {}^{\lambda}H^{\delta}_{\delta}$$

$$C_{\lambda} = H_{\lambda} - \frac{1}{3v_T^{\,2}} \; {}^{\varepsilon}_{\varepsilon}H_{\lambda}$$

and $$D = H - \frac{1}{3v_T^{\,2}} \; ({}^{\varepsilon}_{\varepsilon}H + H^{\delta}_{\delta}) + \frac{1}{9v_T^{\,4}} \; {}^{\varepsilon}_{\varepsilon}H^{\delta}_{\delta}$$

then $$A_{\lambda} = \frac{\nu C_{\lambda}}{3\nu D - n_0}$$

and $$M^{\lambda}_{\mu} = \frac{1}{v_T^{\,2}} \; \{ \, {}^{\lambda}H_{\mu} - 3 \, {}^{\lambda}C \, A_{\mu} \, \}$$

implying $$M^{\delta}_{\delta} = \frac{1}{v_T^{\,2}} \; \{ \, {}^{\delta}H_{\delta} - 3 \, {}^{\delta}C \, A_{\delta} \, \}$$

Examining the expression for $\rho(\underline{r},t)$, we see that the following functions are required:-

$$k^{\lambda}C_{\lambda}, \; k^{\lambda}H_{\lambda}, \; {}^{\lambda}H_{\lambda}, \; {}^{\lambda}CC_{\lambda}, \; D$$

F-functions for all of these except ${}^{\lambda}CC_{\lambda}$ are easily derived by combining the F-functions for the component parts of each. ${}^{\lambda}CC_{\lambda}$ is a product of two integrals, and therefore must be treated separately.

We list the appropriate F-functions below:-

$$k^{\lambda}C_{\lambda} \rightarrow \frac{ik^2v_T^{\,2}}{3(\nu^2+\Omega^2)} \; \left\{ \frac{k^2v_T^{\,2}}{(\nu^2+\Omega^2)} \, (1-2e^{-\nu t} \cos \Omega t+e^{-2\nu t})(\nu \, [1-e^{-\nu t} \cos\Omega t] +\Omega e^{-\nu t}\sin \Omega t) \right.$$

$$\left. - 2e^{-\nu t} \; (\nu \, [\, e^{-\nu t} \cos \Omega t - 1 \,] + \Omega \sin \Omega t) \right\}$$

$$k^{\lambda}H_{\lambda} \rightarrow \frac{ik^2v_T^{\,2}}{\nu^2+\Omega^2} \; \{ \; \nu \; (1 - e^{-\nu t} \cos \Omega t) + \Omega e^{-\nu t} \sin \Omega t \}$$

$$^{\lambda}H_{\lambda} \to v_T^{\,2}\ (e^{-\nu t}\ [\,1+2\cos\Omega t\,] - \frac{k^2 v_T^{\,2}}{\nu^2+\Omega^2}\left\{ \frac{\nu^2-\Omega^2}{\nu^2+\Omega^2}\ (1-2e^{-\nu t}\cos\Omega t + e^{-2\nu t}\cos 2\Omega t)\right.$$

$$\left. + \frac{2\nu\Omega}{\nu^2+\Omega^2}\ e^{-\nu t}\ (2\sin\Omega t - e^{-\nu t}\sin 2\Omega t)\right\}$$

$$D \to \frac{1}{9}\left\{2e^{-2\nu t}\ [\,2+\cos 2\Omega t\,] - 4\,\frac{k^2 v_T^{\,2}}{\nu^2+\Omega^2}\,e^{-\nu t}\ \left[\ \frac{\nu^2+\Omega^2}{\nu^2-\Omega^2}\ (\cos\Omega t\ [\,1+e^{-2\nu t}\,] - 2e^{-\nu t})\right.\right.$$

$$\left. + \frac{2\nu\Omega}{\nu^2+\Omega^2}\sin\Omega t(1 - e^{-2\nu t})\ \right]$$

$$\left. + \frac{k^4 v_T^{\,4}}{(\nu^2+\Omega^2)^2}(1 - 2e^{-\nu t}\cos\Omega t + e^{-2\nu t})^2\right\}$$

Using the definitions of $^{\lambda}C$ and C_{λ}, the function $^{\lambda}CC_{\lambda}$ may be written as follows:-

$$^{\lambda}CC_{\lambda} = \frac{n_0^{\,2}k^2}{2}\ \left[\ \left(\int_0^{\infty}\mathcal{F}_1\ \exp\{i\omega t - \Phi(t)\}\ dt\right)^2 + \left(\int_0^{\infty}\mathcal{F}_{-1}\ \exp\{i\omega t - \Phi(t)\}dt\right)^2\ \right] \qquad \text{(II.15)}$$

where $\mathcal{F}_1$ and $\mathcal{F}_{-1}$ are given by

$$\mathcal{F}_1 = \frac{iv_T^{\,2}}{3(\nu^2+\Omega^2)}\ \left\{\ \frac{k^2 v_T^{\,2}(\nu-i\Omega)}{\nu^2+\Omega^2}\ (1 - 2e^{-\nu t}\cos\Omega t + e^{-2\nu t})(1 - e^{-(\nu+i\Omega)t}) - 2(\nu + i\Omega)(e^{-(\nu+i\Omega)t} - e^{-2\nu t})\right\}$$

$$\mathcal{F}_{-1} = \frac{iv_T^{\,2}}{3(\nu^2+\Omega^2)}\ \left\{\frac{k^2 v_T^{\,2}(\nu+i\Omega)}{\nu^2+\Omega^2}\ (1 - 2e^{-\nu t}\cos\Omega t + e^{-2\nu t})(1 - e^{-(\nu-i\Omega)t}) - 2(\nu - i\Omega)(e^{-(\nu-i\Omega)t} - e^{-2\nu t})\right\}$$

It is easily (if laboriously) verified from the expressions listed above that all the integrals involved in $\rho(\underline{k},\omega)$ have F- or $\mathcal{F}$- functions which are linear sums of the following exponentials:-

(1) $e^{-\nu t}$, $e^{-(\nu \pm i\Omega)t}$

(2) $e^{-2\nu t}$, $e^{-(2\nu \pm i\Omega)t}$, $e^{-(2\nu \pm 2i\Omega)t}$

(3) $e^{-(3\nu \pm i\Omega)t}$

(4) $e^{-4\nu t}$

Thus $\rho(\underline{k},\omega)$ may be written explicitly in terms of the following set of dimensionless "modified Gordeyev" integrals:-

$$\left.\begin{aligned}
&(1) \quad \int_0^\infty \exp\{-\nu'\tau - \Phi(\tau) + i(\tfrac{\omega}{\Omega} \pm \ell)\tau\}\, d\tau \\
&(2) \quad \int_0^\infty \exp\{-2\nu'\tau - \Phi(\tau) + i(\tfrac{\omega}{\Omega} \pm m)\tau\}\, d\tau \\
&(3) \quad \int_0^\infty \exp\{-3\nu'\tau - \Phi(\tau) + i(\tfrac{\omega}{\Omega} \pm 1)\tau\}\, d\tau \\
&(4) \quad \int_0^\infty \exp\{-4\nu'\tau - \Phi(\tau) + i\,\tfrac{\omega}{\Omega}\tau\}\, d\tau
\end{aligned}\right\} \qquad \text{(II.16)}$$

where $\ell = 0, 1$

$m = 0, 1, 2$

$\tau = \Omega t$

and $\nu' = \frac{\nu}{\Omega}$

We now define a general dimensionless "modified Gordeyev" integral:-

$$G_M = \int_0^\infty \exp\{i\omega^*\tau - r\nu'\tau - \Phi(\tau)\}\, d\tau \qquad \text{(II.17)}$$

where $\omega^* = \frac{\omega}{\Omega} \pm s$; $s = 0, 1, 2$ and $r = 1, 2, 3, 4$. Any of the integrals in (II.15) may be obtained by using the appropriate values of r and s in (II.16).

The dimensionless expression $\Phi(\tau)$ is given by:-

$$\Phi(\tau) = \frac{k_{\parallel}^2\rho^2}{(\nu')^2}\left(\nu'\tau - 1 + e^{-\nu'\tau}\right) + \frac{k_{\perp}^2\rho^2}{[(\nu')^2+1]}\left(\cos\chi + \nu'\tau - e^{-\nu'\tau}\cos(\tau-\chi)\right)$$

Chapter 4

In Appendix IV we show that for $k\rho \gg 1$ and $(k\rho)^{-2} \lesssim \nu' \ll 1$ the integral G_M given by equation (II.16) may be approximated thus:-

$$G_M \simeq -\frac{i}{\sqrt{2}k\rho} \; Z \left(\frac{\omega^*}{\sqrt{2}k\rho} \right) \tag{II.18}$$

where $Z(\zeta)$ is the Fried-Conte plasma dispersion function. When evaluating $\rho(\underline{k},\omega)$, integrals of the form

$$\int_0^\infty e^{-n\nu'\tau} \cos m\tau \;\; \exp \{i\omega'\tau - \Phi(\tau)\} \, d\tau$$

(where $\omega' = \frac{\omega}{\Omega}$) are approximated by

$$-\frac{i}{\sqrt{2}k\rho} \left[Z \left(\frac{\omega'+m}{\sqrt{2}k\rho} \right) + Z \left(\frac{\omega'-m}{\sqrt{2}k\rho} \right) \right]$$

and integrals of the form

$$\int_0^\infty e^{-n\nu'\tau} \sin m\tau \;\; \exp \{i\omega'\tau - \Phi(\tau)\} \, d\tau$$

are approximated by

$$\frac{1}{2\sqrt{2}k\rho} \left[Z \left(\frac{\omega'+m}{\sqrt{2}k\rho} \right) - Z \left(\frac{\omega'-m}{\sqrt{2}k\rho} \right) \right]$$

It is straightforward, though lengthy, to write down expressions in terms of Fried-Conte functions for $k^\lambda C_\lambda$, $k^\lambda H_\lambda$, ${}^\lambda H_\lambda$, D, and $\int_0^\infty \mathcal{J}_{\pm 1} \exp \{i\omega t - \Phi(t)\} d\tau$.

For example, the expression for ${}^\lambda H_\lambda$ is

$$\begin{aligned} {}^\lambda H_\lambda \simeq \frac{n_0 v_T{}^2}{\Omega} \cdot \frac{-i}{\sqrt{2}k\rho} \Big\{ & Z \left(\frac{\omega'}{\sqrt{2}k\rho} \right) + [Z \left(\frac{\omega'+1}{\sqrt{2}k\rho} \right) + Z \left(\frac{\omega'-1}{\sqrt{2}k\rho} \right) \\ & + k^2\rho^2 [(Z \left(\frac{\omega'}{\sqrt{2}k\rho} \right) - [Z \left(\frac{\omega'+1}{\sqrt{2}k\rho}\right) + Z \left(\frac{\omega'-1}{\sqrt{2}k\rho} \right)] + \tfrac{1}{2} [Z \left(\frac{\omega'+2}{\sqrt{2}k\rho}\right) + Z \left(\frac{\omega'-2}{\sqrt{2}k\rho}\right. \\ & -2i\nu' ([- 2i\nu'([Z \left(\frac{\omega'+1}{\sqrt{2}k\rho} \right) - Z \left(\frac{\omega'-1}{\sqrt{2}k\rho} \right)] - \tfrac{1}{2} [Z \left(\frac{\omega'+2}{\sqrt{2}k\rho} \right) + Z \left(\frac{\omega'-2}{\sqrt{2}k\rho} \right)])] \Big\} \end{aligned} \tag{II.19}$$

The other expressions have a similar form.

Taking $\zeta = \frac{\omega' \pm m}{k\rho} << 1$, we may expand the Fried-Conte functions in a power series as follows:-

$$\left.\begin{aligned} Z(\zeta) &= i\pi^{\frac{1}{2}} \exp(-\zeta^2) - 2\zeta\,[\,1 - \tfrac{2}{3}\zeta^2 \ \ldots .\,] \\ &\simeq i\pi^{\frac{1}{2}} - 2\zeta \text{ to order } \zeta \\ &\simeq i\pi^{\frac{1}{2}}(1-\zeta^2) - 2\zeta \text{ to order } \zeta^2 \\ &\quad \text{etcetera} \end{aligned}\right\} \qquad \text{(II.20)}$$

Substituting (II.20) to order ζ in (II.19)

$$\lambda_{H_\lambda} \simeq \frac{n_0 v_T^{\,2}}{\Omega} \frac{-i}{\sqrt{2}k\rho} \left\{3i\pi^{\frac{1}{2}} - \frac{3\sqrt{2}\omega'}{k\rho} + k^2\rho^2\,[\text{ term } 0\,([\tfrac{\omega'}{k\rho}]^2) \text{ at most} + 2i\nu'(i\pi^{\frac{1}{2}} - \frac{\sqrt{2}(\omega'-2)}{k\rho})]\right\}$$

$$\sim \frac{n_0 v_T^{\,2}}{\Omega} \ [\,0(\nu' k\rho) + 0(\,[k\rho\,]^{-1}) + 0(\nu') + 0(\,[k\rho\,]^{-2}\,)\,]$$

The leading order term is $0(\nu' k\rho)$, since $(k\rho)^{-2} \lesssim \nu' < (k\rho)^{-1}$ [see Appendix IV]

$$\text{Thus} \quad \lambda_{H_\lambda} \sim 0\left(\frac{n_0 v_T^{\,2}}{\Omega}\,\nu' k\rho\right) \qquad \text{(II.21)}$$

where we use the sign "~" to denote that the leading order term of the expression is given on the right-hand side.

Carrying out similar expansions for the other required expressions, we find that

$$\left.\begin{aligned} \int_0^\infty \mathcal{J}_1 \exp\{i\omega t - \Phi(t)\}dt &\simeq \frac{\rho^2}{3\sqrt{2}(k\rho)} \left\{ 2\sqrt{2}\,.\,\frac{\nu'+i}{k\rho} - \frac{3\pi(2\omega'-1)}{(k\rho)^2} \right\} \\ \int_0^\infty \mathcal{J}_{-1} \exp\{i\omega t - \Phi(t)\}dt &\simeq \frac{\rho^2}{3\sqrt{2}(k\rho)} \left\{ 2\sqrt{2}\,.\,\frac{\nu'-i}{k\rho} - \frac{3\pi(2\omega'+1)}{(k\rho)^2} \right\} \end{aligned}\right\} \qquad \text{(II.22)}$$

Thus from (II.15)

$$\lambda_{CC_\lambda} \simeq -\frac{4}{9}\frac{n_0{}^2\rho^2}{(k\rho)^2}\left\{1+\frac{3\sqrt{2\pi}}{2\ k\rho}\ [\ 2\omega'\nu' - i\ ]\ \right\}$$

and so $\lambda_{CC_\lambda} \sim O(n_0{}^2\rho^2(k\rho)^{-2})$ (II.23)

Similarly for D

$$D \simeq \frac{n_0}{\Omega}\frac{-i}{\sqrt{2}k\rho}\left\{\frac{1}{9}\ [6i\pi^{\frac{1}{2}} + O([k\rho]^{-2})] - \frac{4}{9}(k\rho)^2\ [\ O([k\rho]^{-2})\text{ or less }] + \frac{1}{9}(k\rho)^4\ [\ O([k\rho]^{-4})\text{ or less }]\right\}$$

That is

$$D \sim O(\frac{n_0}{\Omega}(k\rho)^{-1})$$

and therefore

$$\begin{aligned} 3\nu D - n_0 &= 3\nu'\Omega D - n_0 \\ &\simeq -n_0(1 + O(\nu'(k\rho)^{-1})) \\ &\sim O(n_0) \end{aligned} \qquad \text{(II.24)}$$

$$\begin{aligned} \text{Now}\quad \lambda_{CA_\lambda} &= \Omega\ \frac{\nu'\ \lambda_{CC_\lambda}}{3\nu D - n_0} \\ &\sim O\left(\frac{\Omega\nu' n_0{}^2\rho^2(k\rho)^{-2}}{n_0}\right) \\ &\sim O\left(\Omega\nu' n_0\rho^2(k\rho)^{-2}\right) \end{aligned} \qquad \text{(II.25)}$$

using (II.23) and (II.24).

$$M_\lambda^\lambda = \frac{1}{v_T{}^2}\ \left\{\ {}^\lambda H_\lambda - 3\ {}^\lambda C\ A_\lambda\right\}$$

$$\sim O(\frac{n_0}{\Omega}\nu' k\rho)$$ using (II.24) and (II.25).

Therefore $\nu M_\lambda^\lambda \sim O(n_0(\nu')^2 k\rho)$ (II.26)

and so $\dfrac{\nu M_\lambda^\lambda}{n_0-\nu M_\lambda^\lambda} \sim O(\ (\nu')^2 k\rho)$ since $(\nu')^2 < (k\rho)^{-2}$.

Thus $1 + \dfrac{\nu M_\lambda^\lambda}{n_0-\nu M_\lambda^\lambda} \simeq 1$ for $(k\rho)^{-2} \leq \nu' < (k\rho)^{-1}$ (II.27)

From (II.11) using (II.27)

$$\rho(\underline{k},\omega) \simeq - \frac{iq^2\phi}{mv_T^2} \left[H_\lambda + \left(\frac{1}{v_T^2} H_\delta^\delta - 3H \right) A_\lambda \right] k^\lambda \tag{II.28}$$

We may write $k^\lambda A_\lambda$ as

$$k^\lambda A_\lambda = \frac{\nu k^\lambda C_\lambda}{3\nu D - n_0} \simeq - \frac{\nu}{n_0} k^\lambda C_\lambda$$

using (II.24)

Now $k^\lambda C_\lambda$ may be written as

$$k^\lambda C_\lambda = n_0 \left[k^1 k_1 \int_0^\infty \mathcal{J}_1 \exp\{i\omega t - \Phi(t)\} dt \right.$$

$$\left. + k^{-1}k_{-1} \int_0^\infty \mathcal{J}_{-1} \exp\{i\omega t - \Phi(t)\} dt \right]$$

$$\simeq \frac{n_0}{3\sqrt{2}} \left[2\sqrt{2}\,\nu' - \frac{6\pi\omega'}{k\rho} \right] \quad \text{using} \quad \text{(II.22)}$$

$$\text{Thus } k^\lambda A_\lambda \simeq - \frac{\Omega\nu'}{3\sqrt{2}} \left[2\sqrt{2}\,\nu' - \frac{6\pi\omega'}{k\rho} \right]$$

$$\text{and } k^\lambda A_\lambda \sim O\left(\Omega\omega' \frac{\nu'}{k\rho}\right) \tag{II.29}$$

for $\nu' < (k\rho)^{-1}$.

Using the F-function derived earlier for H_δ^δ it can be shown that

$$\frac{1}{v_T^2} H_\delta^\delta - 3H \sim O\left(\frac{n_0}{\Omega} \cdot \frac{1}{k\rho}\right) \tag{II.30}$$

and using the F-function derived for $k^\lambda H_\lambda$

$$k^\lambda H_\lambda \sim O(n_0) \text{ for } \nu' < (k\rho)^{-1} \tag{II.31}$$

Thus the expression

$$\left[\frac{1}{v_T^2} H_\delta^\delta - 3H \right] A_\lambda k^\lambda \sim O\left(n_0\omega' \frac{\nu'}{(k\rho)^2}\right) \tag{II.32}$$

from (II.29) and (II.30).

Comparing (II.32) with (II.31), we see that

$[\frac{1}{v_T{}^2} H^\delta_\delta - 3H] A_\lambda k^\lambda$ may be neglected when compared with $k^\lambda H_\lambda$.

Now the expression for $k^\lambda H_\lambda$ in terms of Fried-Conte functions is as follows:-

$$k^\lambda H_\lambda \simeq - \frac{in_0}{2\sqrt{2}} k\rho [Z (\frac{\omega'+1}{\sqrt{2}k\rho}) - Z (\frac{\omega'-1}{\sqrt{2}k\rho})] \qquad \text{(II.33)}$$

Consider the functions

$$X = -\frac{k\rho}{2\sqrt{2}} [Z (\frac{\omega'+1}{\sqrt{2}k\rho}) - Z (\frac{\omega'-1}{\sqrt{2}k\rho})]$$

$$\text{and} \quad Y = -\tfrac{1}{2} Z' (\frac{\omega'}{\sqrt{2}k\rho})$$

In terms of small argument expansions

$$X \simeq 1 + \frac{i\pi^{\frac{1}{2}}\omega'}{\sqrt{2}k\rho} - \frac{(\omega')^2 + \frac{1}{3}}{(k\rho)^2} + \ldots\ldots$$

$$Y \simeq 1 + \frac{i\pi^{\frac{1}{2}}\omega'}{\sqrt{2}k\rho} - \frac{(\omega')^2}{(k\rho)^2} + \ldots\ldots$$

Thus $X - Y \sim O([k\rho]^{-2}) << 1$ in our approximation and so $X \simeq Y$; that is we may write (II.33) as

$$k^\lambda H_\lambda \simeq - \frac{in_0}{2} Z' (\frac{\omega}{\sqrt{2}kv_T})$$

Equation (II.28) now reduces to

$$\rho(\underline{k},\omega) \simeq - \frac{n_0 q^2\phi}{2mv_T{}^2} Z' (\frac{\omega}{\sqrt{2}kv_T}) \qquad \text{(II.34)}$$

As in Chapter 1 of this Section, we use Poisson's equation and the electrostatic approximation, with a harmonic variation in the electrostatic potential ϕ of the form $\exp\{i (\underline{k} \cdot \underline{r} - \omega t)\}$. The resulting equation is

$$k^2\phi = 4\pi \sum_{\text{species}} \rho(\underline{k},\omega) \qquad \text{(II.35)}$$

Thus the dispersion relation for waves propagating perpendicular to a constant magnetic field $\underline{B}_0$ in an electron-ion plasma in which $k\rho >> 1$ and $(k\rho)^{-2} \lesssim \nu' < (k\rho)^{-1}$ is given by substituting the appropriate values of $\rho(\underline{k},\omega)$ from (II.34) into (II.35).
The dispersion relation is then

$$1 + \frac{k_i^2}{2k^2} Z' \left(\frac{\omega}{\sqrt{2}kv_i}\right) + \frac{k_e^2}{2k^2} Z'\left(\frac{\omega}{\sqrt{2}kv_e}\right) = 0 \qquad \text{(II.36)}$$

where $v_{i,e}$ is the mean thermal velocity of the given species.

Equation (II.36) is the dispersion relation for ion-acoustic waves travelling in an unmagnetized plasma with warm ions, and is the same dispersion relation that holds in the region of Figure (4) marked $k_{\parallel}^2\rho^2 \gtrsim 1$ in the collisionless case. Thus it seems that for $k\rho >> 1$ the effect of introducing a collision frequency ν' (such that $(k\rho)^{-2} \lesssim \nu' < (k\rho)^{-1}$) is to disrupt the gyroresonances that allow the existence of undamped Bernstein modes for $k_{\parallel} = 0$. If this holds for undamped Bernstein modes, it must hold all the more strongly for the region of damped Bernstein modes (marked $k_{\parallel}^2\rho^2 << 1$ in Figure (4)).

The statement of our result is as follows:-
For the regime $k\rho >> 1$ in a magnetized electron-ion plasma, the introduction of a collision frequency ν (such that $(k\rho)^{-2} \lesssim \frac{\nu}{\Omega} < (k\rho)^{-1}$) results in the replacement of the Bernstein mode dispersion relation by the normal ion-acoustic mode dispersion relation even for propagation perpendicular to the magnetic field.

Certain recently - published work lends support to the idea that Bernstein modes and the Bernstein instability are unlikely to be important in practise. Lampe et al [13b] showed that in the linear theory there is a smooth transition from the Bernstein instability to the ion acoustic instability as the magnetic field $\underline{B}$ tends to zero, and that when non-linear turbulent fields are introduced, the electron gyroresonances for modes with $k\rho_e > 1$ are smeared out. Two-dimensional computer simulations [14] have also shown the finite amplitude stabilization of the Bernstein instability.

SECTION III :- Temperature gradient driven ion acoustic instability.

Chapter 1

It is well known that ion acoustic waves may be driven unstable by introducing a relative drift velocity between ions and electrons. The problem was originally investigated by Fried and Gould [15], who solved the dispersion relation numerically and gave a neutral stability curve showing the critical drift velocity as a function of electron to ion temperature ratio, $^{T}e/T_i$. Although this analysis was for an unmagnetized plasma the same general conclusions may be applied to ion acoustic waves in a magnetized plasma provided that (a) the drift velocity is parallel to the magnetic field [16] or (b) the drift velocity is perpendicular to the magnetic field but the waves are effectively unmagnetized, that is $k_\perp >> \rho_e^{-1}$, where ρ_e is the electron gyroradius [17]. In the opposite limit of $k_\perp\rho_e < 1$ either the ion acoustic instability or the modified two-stream instability may arise depending on the ratio $^{k_\parallel}/_k$ [18]. A large number of recent papers have examined this latter limit ($k_\perp\rho_e < 1$); in particular the effects of drifts caused by plasma inhomogeneities have been examined [19], [20], [21], [22].

In this Section we examine case (b), that is a perpendicular drift and $k_\perp \rho_e >> 1$. The result of Appendix (III c) supports the idea that the maximum effect of a gradient drift within the local approximation occurs when $k_\perp\rho_e >> 1$. The significance of inhomogeneity drifts in this limit was examined analytically by Priest and Sanderson [23] for $^{T}e/T_i >> 1$. In that paper, hereafter referred to as I, it was shown that weak inhomogeneities in electron density and magnetic field (see also [17]) have a negligible effect on the ion acoustic instability, but that an electron temperature gradient could have a very significant destabilizing effect. Since the physical explanation of this lies in the distortion of the electron distribution function (see Chapter 2 of this Section), it was suggested that a sufficiently large temperature gradient might drive ion acoustic waves unstable even in an equal temperature plasma ($T_e \sim T_i$) and for very small or zero net drift velocities. (note that it is only necessary to consider small gradients when density and temperature gradients occur together; see Appendix (III a)). The numerical investigation of this prediction is the chief aim of this Section.

We restrict attention to the physically significant temperature gradient, choosing the density constant and the magnetic field inhomogeneity vanishingly small. The latter is achieved in a similar manner to that detailed in Appendix (III d). Equation (AIII.9) of Appendix III is

$$v_d = \frac{2\bar{v}_B}{\beta_\perp} \qquad \text{(III.1)}$$

where v_d is the net drift between ions and electrons, and $\bar{v}_B$ is the average $\underline{\nabla}B$ drift velocity. In this Section we consider equal temperatures parallel and perpendicular to $\underline{B}$, so that $\beta_\perp = \beta_\parallel = \beta$. Considering the electrons (III.1) becomes

$$v_d = \frac{2\bar{v}_B}{\beta_e} \qquad \text{(III.2)}$$

where $\beta_e = \dfrac{8\pi P_e}{B^2}$, the electron pressure being P_e. Thus if $\beta_e \ll 1$

we may neglect the magnetic field inhomogeneity, and choose $\underline{B} = B_0\,\hat{\underline{z}}$, where B_0 is constant and x,y,z are Cartesian axes. The temperature gradient and the steady state electric field are in the x-direction. By integrating the first velocity moment of the electron distribution function (given in Chapter 2) over velocity space, a value for v_d may be obtained. Combining this with (III.2) gives the equation

$$\underline{v}_d = \underline{v}_0 - \underline{v}_T \qquad \text{(III.3)}$$

where $\underline{v}_0 = \frac{cE_0}{B_0}\underline{\hat{y}}$ is the $\underline{E} \wedge \underline{B}$ drift velocity, (E_0 and B_0 being the steady state electric and magnetic fields respectively)

and $\underline{v}_T = \frac{\delta v_e^2}{\Omega_e}\underline{\hat{y}}$

where δ is the temperature gradient, $v_e = \left(\frac{T_e}{m_e}\right)^{\frac{1}{2}}$ the mean electron thermal velocity, and Ω_e the electron gyrofrequency. The electric field is necessary to establish equilibrium; although this could also be achieved by an opposing magnetic pressure gradient, the resulting configuration would be less unstable than the one we consider.

The results presented in this Section consist of a series of graphs showing the frequency $\omega = \omega_R + i\gamma$ plotted against the wave-number k for various values of the following parameters:-

the ratio of ion temperature to electron temperature T_i/T_e

the normalized temperature gradient drift velocity v_T/v_e

the normalized net drift velocity v_d/v_e

the sine of the angle between $\underline{k}$ and $\underline{B}$ $k_\perp/k$

We take the ion-electron mass ration to be 1836, thus considering a hydrogen plasma throughout. We also show how critical values of v_T for neutral stability may be obtained for a given temperature ratio, and we provide stability diagrams for different values of v_d.

Chapter 2

We consider a Maxwellian ion distribution, and an electron distribution function f_0 as follows:-

$$f_0 = \frac{n_0}{(2\pi v_e^2)^{3/2}} \left\{ 1 + \frac{\delta}{2\Omega_e} \cdot \left[\frac{1}{v_e^2} (\underline{v} - \underline{v}_0)^2 - 3 \right]\left[x - \frac{(v_y - v_0)}{\Omega_e} \right] \right\}$$

$$\cdot \exp\left[- \frac{1}{2v_e^2} (\underline{v} - \underline{v}_0)^2 \right] \qquad \text{(III.4)}$$

This expression comes from (I.11) and (I.12) of Section I with $\varepsilon = 0$, $\delta_\perp = \delta_\parallel$ and $T_0^\perp = T_0^\parallel$. An $\underline{E} \wedge \underline{B}$ drift $v_0\hat{\underline{y}}$ has been introduced into the distribution. Figure (5) shows how the temperature gradient δ distorts the electron distribution away from the Maxwellian and introduces a positive slope, thus making instability more likely (from I).

We may now carry through an argument essentially similar to that given in the first part of Chapter 1 in Section II. A dispersion relation is derived of the form

$$1 + K_i + K_e = 0 \qquad \text{(III.5)}$$

where K_i is the ion contribution and K_e the electron contribution. Following through the argument with the modifications introduced by warm ions and drifts in the electron distribution, it is found that K_i and K_e have the following forms:-

$$K_i = \frac{k_i^2}{k^2} \left[1 + i \frac{\omega}{\Omega_i} G_i \right] \qquad \text{(III.6)}$$

where G_i is the dimensionless Gordeyev integral given by equation (II.2).

$$K_e = \frac{k_e^2}{k^2} \left[1 + \frac{i(\omega - k_\perp v_0)}{\Omega_e} G(\mu) + \frac{ik_\perp v_T}{\Omega_e} \frac{\partial}{\partial\mu} G(\mu) \right]_{\mu=1} \qquad \text{(III.7)}$$

where $G(\mu)$ is as follows:-

$$G(\mu) = \int_0^\infty d\tau \exp\{i\omega'\tau - \mu k_\perp{}^2\rho_e^2 (1-\cos\tau) - \tfrac{1}{2}\mu k_\parallel{}^2\rho_e^2 \tau^2\}$$

Here $\omega' = \dfrac{\omega - k_\perp v_0}{\Omega_e}$, assuming a wave vector of the form $\underline{k} = (0, k_\perp, k_\parallel)$.

We now take $k_\perp{}^2\rho_i{}^2 \gg 1$, and to exclude the Bernstein modes in the collisionless case, we must restrict attention to the regime $k_\parallel{}^2\rho_i{}^2 \gtrsim 1$. Equation (III.6) reduces to

$$K_i \approx - \frac{k_i{}^2}{2k^2} Z' \left(\frac{\omega}{\sqrt{2}kv_i}\right) \qquad \text{(III.8)}$$

using the same method as in the final part of Chapter 2 in Section I.

Similarly, when $k_\perp{}^2\rho_e{}^2 \gg 1$ and we exclude Bernstein modes in the collisionless case by taking $k_\parallel{}^2\rho_e{}^2 \gtrsim 1$, K_e becomes

$$K_e \simeq \frac{k_e{}^2}{k^2} \left[1 + \frac{(\omega - k_\perp v_d - \frac{3}{2} k_\perp v_T)}{\sqrt{2}kv_e} Z(\zeta_e) - \frac{k_\perp v_T}{2\sqrt{2}kv_e} \zeta_e Z'(\zeta_e) \right] = 0 \qquad \text{(III.9)}$$

where $\zeta_e = (\omega - k_\perp v_0)/\sqrt{2}kv_e$

Substitution of (III.8) and (III.9) into (III.5) results in the following dispersion relation:-

$$1 - \frac{k_i{}^2}{2k^2} Z' \left(\frac{\omega}{\sqrt{2}kv_i}\right) + \frac{k_e{}^2}{k^2} \left[1 + \frac{\omega - k_\perp v_d - \frac{3}{2} k_\perp v_T}{\sqrt{2}kv_e} Z(\zeta_e) - \frac{k_\perp v_T}{2\sqrt{2}kv_e} \zeta_e Z'(\zeta_e)\right] = 0 \quad \text{(III.10)}$$

Note that, as in the calculation of reference [13a], the magnetic field now appears in the dispersion relation only through the drift velocities.

Equation (III.10) describes ion acoustic waves in the region $k_{\parallel}^2\rho^2 \gtrsim 1$ in the collisionless case. However, we see from (III.6) and (III.7) that K_i and K_e involve Gordeyev integrals and modified Gordeyev integrals which are independent of the drift velocities. It was shown in Chapter 2 of Section II that cyclotron resonances resulting in Bernstein modes arise within the Gordeyev integral itself, and in Chapters 3 and 4 of Section II that the introduction of a small collision frequency ν (such that $(k\rho)^{-2} \lesssim \frac{\nu}{\Omega} < (k\rho)^{-1}$) is sufficient to disrupt these resonances within the Gordeyev integrals resulting in the ion acoustic wave dispersion relation even for $k_{\parallel} = 0$. We carry over this result into the present Section, and assume the existence of a collision frequency ν (as above) which destroys the cyclotron resonances within the Gordeyev integrals in K_i and K_e. This allows us to relax the collisionless restriction that $k_{\parallel}^2\rho^2 \gtrsim 1$ for exclusion of the Bernstein modes, so that from now on we may use the dispersion relation (III.10) to describe waves propagating perpendicular and near-perpendicular to the magnetic field. That is, we take the ion acoustic wave dispersion relation to be valid even for $k_{\parallel} = 0$.

Computational solutions of (III.10) under various parameter changes were obtained on the U.K.A.E.A. Culham Laboratory I C L 4-70 computer using the interactive root-finding programme of Martin [24]. We give the results of these computations under two main headings, namely:-

(a) The effect of a temperature gradient drift velocity

and (b) Stability diagrams.

(a) <u>The effect of a temperature gradient drift velocity</u>

As mentioned previously, a Cartesian reference frame has been chosen such that the magnetic field lies in the z-direction; the y-axis lies along the net drift velocity direction and along the direction of $k_{\perp}$.

Thus the ratio $k_\perp/k$ is equal to sin θ for ion acoustic waves propagating at an angle θ to the magnetic field. As θ increases, $\frac{k_\perp}{k}$ also increases to the value 1.

Figure (6) shows the normalized growth rate γ/ω_e plotted against $\frac{\sqrt{2}k}{k_e}$ for $\frac{k_\perp}{k} = 0.5,\ 0.866,\ 1$ and for $\frac{v_d}{v_e} = 0$, $\frac{v_T}{v_e} = 0.5$, $\frac{T_i}{T_e} = 0.3$ (where $\omega_e^{\,2} = \frac{4\pi n_0 e^2}{m_e}$ and $k_e^{\,2} = \frac{\omega_e^{\,2}}{v_e^{\,2}}$). For $\frac{k_\perp}{k} = 0.5$ ion acoustic waves are stable (that is γ is always negative and the waves decay). As $\frac{k_\perp}{k}$ increases, the waves go unstable over a range of wave-numbers, and the maximum positive growth-rate occurs for $\frac{k_\perp}{k} = 1$. Other values of the parameters v_T/v_e, v_d/v_e, T_i/T_e were investigated, and the same effect was found to occur in each case. The explanation for this is that ion acoustic waves extract the maximum amount of energy from the drift motion when propagating parallel to the net drift velocity [15]. In figures (7) to (15) we consider only maximum growth rates, and so we choose $k_\perp/k = 1$ in each case.

Figure (7) shows how a plasma with no net drift ($v_d = 0$) and $T_i/T_e = 0.3$ is driven unstable over a range of wave-numbers by increasing the value of v_T/v_e. For $\frac{v_T}{v_e} = 0.1$, ion acoustic waves decay; but for $\frac{v_T}{v_e} = 0.5$ an unstable wave-band appears. Figure (8) shows a similar situation. A net drift of $\frac{v_d}{v_e} = 0.5$ is not sufficient to counteract the stabilizing effect of a temperature ratio of order unity; even with $\frac{v_T}{v_e} = 0.5$ ion acoustic waves still decay. However, taking v_T equal to v_e (that is, the gradient $\delta = \Omega_e/v_e$) we find a considerable positive growth-rate. Figures (7) and (8) are typical examples of a number of sets of results obtained, all of which show the same effect occurring for a variety of parameter values.

Figures (9), (10) and (11) show how the real part of the frequency ω_R is affected by increasing $^{v}T/v_e$; Figures (9) and (11) are for the situations described in Figures (7) and (8). In Figures (9) and (10), over the chosen range of k, the curves of ω_R against k for $v_T = 0$ are convex relative to the k-axis [due to computing difficulties, no $v_T = 0$ curve was found for Figure (9)]. As v_T increases, the convexity is seen to decrease in all three Figures, until the curves become concave. For $^{v}T/v_e = 1$, the concavity is very obvious, though it may be seen in Figure (7) that over the final part of the $^{v}T/v_e = 1$ curve, a convexity is again becoming apparent. In a normal ion acoustic wave, the phase velocity decreases with increasing wave-number. We note from the above results that the effect of a large electron temperature gradient is to modify the wave in such a manner as to give increasing phase velocity with increasing wave number.

(b) Stability diagrams

In figures (12) to (14) we give a detailed picture of the transition from complete stability to complete instability over the chosen range of wave-number, for $\frac{v_d}{v_e} = 0$ and $^{T}i/T_e = 1.5$

Figure (12) :- at $\frac{v_T}{v_e} = 1.25$ we have complete stability, while at $\frac{v_T}{v_e} = 1.41$ we have complete instability.

Figure (13) :- expanding the growth-rate scale and including more values of $\frac{v_T}{v_e}$, we see how the γ against k curves peak more and more sharply as we increase the value of $\frac{v_T}{v_e}$.

Figure (14) :- finally we expand the growth-rate scale still further, and plot a series of curves over a small range of $\frac{v_T}{v_e}$. At $\frac{v_T}{v_e} = 1.283$, the curve just crosses the k-axis, giving a small unstable wave-band. As $\frac{v_T}{v_e}$ increases, the unstable wave-band covers a wider range of wave-number and has a greater maximum positive growth-rate. Ultimately the unstable wave-band will stretch from k = 0 past $\frac{\sqrt{2}k}{k_e} = 1$, as demonstrated by the $\frac{v_T}{v_e} = 1.41$ curve in Figure (12). From Figure (15) we see that the same peaking effect occurs for non-zero net drift velocity. It may be noted that the peaking behaviour of the γ against k curves only becomes obvious when $T_i/T_e > 1$. Below this value, the transition from a completely stable state with no appreciable peak to an unstable state occurs over a very small range of v_T/v_e. For example, the transition occurs between $\frac{v_T}{v_e} = 0.932$ and $\frac{v_T}{v_e} = 0.933$ for $\frac{T_i}{T_e} = 1$, $\frac{v_d}{v_e} = 0.1$.

The point of neutral stability for a given temperature ratio occurs where ion acoustic waves are on the point of instability; that is just before the transition for $\frac{T_i}{T_e} \leq 1$, and where the peak of the γ against k curve just touches the k-axis for $\frac{T_i}{T_e} > 1$. Fried and Gould [15] showed that for a simple drift ($v_d = v_0$ in our case) the points of neutral stability for the range $\frac{T_i}{T_e} = 0.05$ to 20 occur at k = 0. We see here that the peaking of the γ against k curve due to the introduction of a large electron temperature gradient leads to the occurrence of a neutral point at some k greater than zero. We may note that these neutral points occur where the phase velocity $v_p \approx v_i$ (where $v_p = \frac{\omega_R}{k}$ and $v_i = \left(\frac{T_i}{m_i}\right)^{\frac{1}{2}}$). Obviously we now have a straightforward computational method for finding neutral points.

We vary ${}^{v_T}/_{v_e}$ until, depending on the value of ${}^{T_i}/_{T_e}$, we either find a sudden transition from stability to instability, or we find the curve whose peak touches the k-axis; the accuracy in finding a neutral point depends only on how much computing time the operator wishes to spend on each point.

Neutral points were obtained for various values of ${}^{T_i}/_{T_e}$ in the cases $\frac{v_d}{v_e} = 0$ and $\frac{v_d}{v_e} = 0.1$. The results are shown in Figure (16). For a plasma with a given net drift, electron temperature gradient and temperature ratio, ion acoustic waves are damped if the plasma lies below the relevant curve, and grow if the plasma lies above the curve. It may also be noted that the formula $v_d + \frac{3}{2} v_T$ given in I for the "effective drift velocity" is verified by the diagram over the whole range of v_T investigated, with a relative error of modulus less than 3% ; this error is about the same as the sum of the computational and plotting errors. Thus for small values of $\frac{v_d}{v_e}$ at least, the formula holds remarkably well when $\frac{T_i}{T_e}$ is of order unity, even though it was proposed for $\frac{T_i}{T_e} << 1$. We see from this formula that an increase in v_T has a greater destabilizing effect on ion acoustic waves than the same increase in v_d.

Chapter 3

In this Chapter we give explanations of the results outlined in Chapter 2, from (a) the analytic point of view and (b) the physical point of view.

(a) Analytic approximation to γ

An analytic approximation to γ for $v_d = 0$ and $\frac{k_\perp}{k} = 1$ may be obtained from the dispersion relation (III.10) by making the following assumptions:-

(1) $\gamma \ll \omega_R$ (2) $v_p \ll v_e$ (3) $v_T \sim v_e$.

The first implies an examination of the region close to neutral stability; the second is supported by values of $\frac{v_p}{v_e}$ derived from the computational results of Chapter 2. The third is required for the peaking effect to be obvious. Although the approximation is not accurate enough to rival the computed results, it provides an insight into the reasons for the peaking effect.

Consider the Fried-Conte function $Z(z)$ where $z = x + iy$ and $y \ll x$. Using a Taylor expansion

$$Z(z) = Z(x) + iyZ'(x) - \frac{y^2}{2!} Z''(x) + \ldots.$$

$$\simeq Z(x) + iyZ'(x)$$

Now
$$\frac{\omega}{\sqrt{2}kv_i} = \frac{\omega_R}{\sqrt{2}kv_i} + \frac{i\gamma}{\sqrt{2}kv_i}$$

$$= \frac{1}{\sqrt{2}} \frac{v_p}{v_i} + \frac{i\gamma}{\sqrt{2}kv_i}$$

$$\therefore \; Z\left(\frac{\omega}{\sqrt{2}kv_i}\right) \simeq Z\left(\frac{1}{\sqrt{2}} \frac{v_p}{v_i}\right) + \frac{i\gamma}{\sqrt{2}kv_i} Z'\left(\frac{1}{\sqrt{2}} \frac{v_p}{v_i}\right) \qquad \text{(III.11)}$$

using assumption (1)

Also $$\frac{\omega - k_\perp v_T}{\sqrt{2}kv_e} = \frac{1}{\sqrt{2}}\frac{v_p}{v_e} - \frac{1}{\sqrt{2}}\frac{v_T}{v_e} + \frac{i\gamma}{\sqrt{2}kv_e}$$

$$\simeq -\frac{1}{\sqrt{2}}\frac{v_T}{v_e} + \frac{i\gamma}{\sqrt{2}kv_e}$$

using assumptions (2) and (3)

Therefore

$$Z\left(\frac{\omega - k_\perp v_T}{\sqrt{2}kv_e}\right) \simeq Z\left(-\frac{1}{\sqrt{2}}\frac{v_T}{v_e}\right) + \frac{i\gamma}{\sqrt{2}kv_e} Z'\left(-\frac{1}{\sqrt{2}}\frac{v_T}{v_e}\right) \qquad \text{(III.12)}$$

using all three assumptions.

The standard expression for $Z'(z)$ is

$$Z'(z) = -2(1 + z.Z(z))$$

and substituting the approximations (III.11) and (III.12) into (III.10), the following expression for γ/ω_e may be obtained from the imaginary part of (III.10):-

$$\frac{\gamma}{\omega_e} \simeq \frac{A(x)}{B(x)} \qquad \text{(III.13)}$$

where

$$A(x) = x.\text{Im } Z(x) - C$$

$$B(x) = \frac{k_e}{k}\frac{T_i}{T_e}\left\{\sqrt{2}\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}\left(\frac{T_e}{T_i}\right)^{\frac{1}{2}}\left[x - \tfrac{1}{2}\text{ Re}Z(x).(1-2x^2)\right]\right.$$

$$\left. - \frac{1}{\sqrt{2}}\text{ Re } Z(b)\left[1 - \left(\frac{3}{\sqrt{2}} + 1\right) . 2b^2 + \sqrt{2}\, b^3\right]\right\}$$

$$C = \tfrac{1}{2}\frac{T_i}{T_e}\text{ Im } Z(b).\left[2b^3 - 3b\right]$$

$$x = \frac{1}{\sqrt{2}}\frac{v_p}{v_i}$$

$$b = -\frac{1}{\sqrt{2}}\frac{v_T}{v_e}$$

In regions where the peaking effect of the γ against k curve is obvious, B(x) is positive over the range of k investigated. For small $^{k}/k_e$ it is large, and it decreases as k increases. The sign of γ is therefore determined by the sign of A(x), and for given $^{T}i/T_e$ and $^{v}T/v_e$ in the regions investigated, C is fixed and positive (note that x . Im Z(x) is positive, so that A(x) and therefore $^{\gamma}/\omega_e$ should always be positive when C is negative, that is when $^{v}T/v_e \geq \sqrt{3}$; a set of computed results for $\frac{\sqrt{2}k}{k_e} = 0.1$ and $\frac{T_i}{T_e} = 2$ was obtained in which $\frac{\gamma}{\omega_e}$ changes from negative to positive at $\frac{v_T}{v_e} \approx 1.73$). For $\frac{T_i}{T_e} = 1.5$ and $\frac{v_T}{v_e} = 1.283$, C = 0.717 and the function A(x) takes the form shown in Figure (17). As noted previously, for these values of $\frac{T_i}{T_e}$ and $\frac{v_T}{v_e}$ the phase velocity v_p, and therefore x, increases with increasing k; thus the peaking behaviour of the γ against k curves is shown to be a result of increasing phase velocity. In this approximation $|\gamma| \to 0$ as $k \to 0$ since B(x) involves a factor $\frac{k_e}{k}$. No computed results were obtained for $\frac{\sqrt{2}k}{k_e} < 0.065$ because of computing difficulties, but the trend of the γ against k curves up to this point agrees with the approximation.

The general variation in $\frac{v_p}{v_i}$ with k is shown in Table (3), for values of $\frac{T_i}{T_e}$ and $\frac{v_T}{v_e}$ close to neutral stability ($\frac{v_d}{v_e} = 0$ in all cases). The subscripts on the variables in Table (3) represent the following parameters values:-

{1} => $\frac{T_i}{T_e} = 0.05$; $\frac{v_T}{v_e} = 0.017$ {2} => $\frac{T_i}{T_e} = 1.5$; $\frac{v_T}{v_e} = 0.229$

{3} => $\frac{T_i}{T_e} = 1.0$; $\frac{v_T}{v_e} = 1.025$ {4} => $\frac{T_i}{T_e} = 1.5$; $\frac{v_T}{v_e} = 1.283$

For large values of $\frac{T_i}{T_e}$ and $\frac{v_T}{v_e}$, $\frac{v_p}{v_i}$ is sensitive to changes in k, and is increasingly less sensitive as $\frac{T_i}{T_e}$ and $\frac{v_T}{v_e}$ decrease. The approximation (III.13) depends heavily on $\frac{v_p}{v_i}$. In case {4} of Table (3) the rapid change in $\frac{v_p}{v_i}$ results in a well-defined peak in the γ against k curve (see Figures (13) and (14)), while for lower values of $\frac{T_i}{T_e}$ and $\frac{v_T}{v_e}$ smaller changes in $\frac{v_p}{v_i}$, allied with a decrease in C, result in the peak being flattened and spread out over a large range of k. In fact, in regions such as {1} of Table (3), where normal ion acoustic behaviour is apparent, the peak no longer exists.

b) Physical approach

The physical explanation of the peaking effect lies in the distortion of the electron distribution function. In a simple small displacement of the peaks of the ion and electron distribution functions, ion acoustic waves have decreasing phase velocity v_p with increasing k. In the region of low v_p $(k \simeq k_e)$, the growth due to the small positive slope of the electron distribution is cancelled out by ion Landau damping; instability appears at larger $v_p (k \to 0)$. However, as mentioned previously, the effect of a large temperature gradient drift velocity v_T is to modify the ion acoustic wave in such a manner that v_p increases with increasing k. This effect enables us to explain the peaking effect of the γ against k curves for large v_T directly from diagrams of the ion and electron distributions as follows.

Consider firstly Figure (18a). This shows the qualitative forms of the distribution functions when $\frac{T_i}{T_e}$ is $\leq$ 1, and both $\frac{v_d}{v_e}$ and $\frac{v_T}{v_e}$

are zero. In this case, v_p decreases as k increases. Therefore, near k = 0, v_p is large and the gradient of the electron distribution function is small and negative, so that electron Landau damping is small. As k increases, v_p decreases; the electron gradient becomes more negative, and electron damping increases. Finally, the ion function takes over and damping increases rapidly. It is therefore easily seen that the γ against k curve will have the general form shown in Figure (18b); this form is very similar to the computational curve shown in Figure (7) for $\frac{v_d}{v_e} = 0$ and $\frac{v_T}{v_e} = 0.1$, with $\frac{T_i}{T_e} = 0.3$ in this case; Figure (8) for $\frac{v_T}{v_e} = 0$ and $\frac{v_d}{v_e} = 0.5$ shows a similar form. In the case of small $\frac{v_T}{v_e}$, therefore, we have a monotonic damping curve until large k is reached, and no peak appears on the (k,γ) curve in the regions investigated.

Figure (19a) shows the forms of the distribution functions for $\frac{v_d}{v_e} = 0$, $\frac{v_T}{v_e} \sim 1$ and $\frac{T_i}{T_e} \sim 1$; Figure (19b) shows the resulting (k,γ) curve. The effect of taking $\frac{v_T}{v_e}$ of order unity is to reverse the variation of v_p with k. We may divide the positive v-axis into four regions (a, b, c, d) as shown, and examine each separately.

Region (a):- For k near zero, v_p is small. There is low ion damping, and it can be shown (by differentiating the distribution function with respect to v) that the electron distribution slope is zero or very small, though positive, in this region. The net result is small damping.

Region (b):- Increasing k increases v_p, so that strong ion damping occurs, though the positive electron slope is still small. The net result is strong damping.

Region (c) :- The increasing positive slope of the electron distribution function overcomes the negative trend of the ion slope, so that the net damping decreases until a minimum damping point is reached when v_p is near the peak of the electron distribution function.

Region (d) :- Ion damping is now small, but the slope of the electron distribution function has become negative, so that damping increases rapidly.

Thus the form of the (k,γ) curve given in Figure (19b) results directly from the shapes of the distribution functions given in Figure (19a); the computational curve plotted in Figure (14) for $\frac{v_d}{v_e} = 0$, $\frac{T_i}{T_e} = 1.5$ and $\frac{v_T}{v_e} = 1.28$ is obviously of the form given by Figure (19b). As $\frac{v_T}{v_e}$ is increased, the peak of the electron distribution function is moved further to the right in Figure (19a), into a region of smaller ion damping, so that the peak rises until it touches the k-axis (as happens in Figure (14)), giving a point of neutral stability; finally a region of instability is produced (again see Figure (14)). Points of neutral stability occur at $v_p \simeq 0.98\ v_i$ for $\frac{T_i}{T_e} = 1.25$, and at $v_p \simeq 0.97\ v_i$ for $\frac{T_i}{T_e} = 1.5$. A similar explanation holds for $v_d \neq 0$, except that a smaller gradient is sufficient to achieve the same effect.

We may note that the boundary between the regions in which v_p increases and decreases with k is determined by the curve of constant v_p in Figures (9), (10) and (11). This is a straight line between the concave and convex curves, and must also be the boundary between the regions in which a peak in the (k,γ) curve exists and does not exist.

The geometry of plasma inhomogeneities and electric and magnetic fields assumed in these calculations has wider applicability to experimental situations than just the collisionless shock experiments referred to in I; see, for example, references [25] and [26]. Observations in these experiments, however, appear to be restricted to waves with $k_\perp \rho_e \lesssim 1$, so that the theory of Liu [21] is perhaps more appropriate.

SUMMARY OF RESULTS

a) In Section I we extended the "polarized" co-ordinate system for use in describing the propagation of waves in a collisionless plasma with general density gradients, temperature gradients both parallel and perpendicular to the applied magnetic field, and an $\underline{E} \wedge \underline{B}$ drift. The use of polarized tensors enabled us to derive a dispersion relation which was compact and unambiguous. In this dispersion relation the gradients appeared as coefficients multiplying moment tensors whose components could be evaluated separately from a given problem, and listed for easy reference. The dispersion relation for a simpler situation than the general case could then be found merely by substituting the appropriate moment tensor elements. Also the conductivity tensors derived in Section I are in tensor equation form, and so may be evaluated in any other co-ordinate system merely by transforming the necessary tensors according to the appropriate transformation law and substituting into the conductivity tensor equation.

The full dispersion relation was derived within the local approximation, and under the condition $\beta_{\perp} << 1$ in order that gradients in magnetic field might be neglected, to simplify the situation. It was found that no gradients in density or temperature can exist in the direction of the magnetic field, to ensure that the equilibrium particle distribution f_0 is a function of constants of the motion. It was also found that within the local approximation the effect of parameter gradients was strongest in the high-frequency regime $k_{\perp}\rho > 1$. For $k_{\perp} \rho \lesssim 1$ the local approximation is valid only for very small gradients.

b) In Section II we used the results of J.P. Dougherty to show that for a wave-band within the high-frequency regime $k\rho >> 1$ in a magnetized electron-ion plasma including a small collision frequency ν such that

$(k\rho)^{-2} \lesssim \frac{\nu}{\Omega} < (k\rho)^{-1}$, the cyclotron resonances which generate the Bernstein modes propagating perpendicular and near-perpendicular to the magnetic field are destroyed; as a result of this the dispersion relation involving Bernstein modes may be replaced by the ion-acoustic dispersion relation even for propagation perpendicular to the magnetic field.

c) In Section III we extended the result of Section II to a plasma with unmagnetized ions, and with electrons subject to $\underline{E} \wedge \underline{B}$ and temperature gradient drifts. We then verified that the inclusion of the electron temperature gradient increases the likelihood of ion acoustic instability, and showed that, given a net drift of $\frac{1}{2}v_e$, a gradient of magnitude Ω_e/v_e can drive ion-acoustic waves unstable even when $T_i = T_e$. Given a greater net drift, a smaller gradient produces instability, though the gradient drift is a more effective destabilizing agent than the net drift velocity. Also, a large gradient reverses the variation of phase velocity with wave number for ion acoustic waves.

Next we showed that the existence of a large gradient changes the behaviour of the growth-rate curves in such a manner as to create easily-calculated points of neutral stability at wave-numbers greater than zero, and we provided analytic and physical explanations of this effect, which also results in the creation of isolated unstable wave-bands at values of the gradient greater than the critical value. Finally we provided diagrams from which the stability of ion-acoustic waves in a given plasma may be determined. These verified the formula $v_d + \frac{3}{2} v_T$ for the "effective drift velocity" over a range of T_i/T_e greater than could have been anticipated from the analytic calculations of I.

APPENDIX I

a) In dealing with problems in magnetized plasmas, the zero-order field $\underline{B}_0$ defines a preferred direction in the plasma, since charged particles are constrained to move in helices aligned along the direction of $\underline{B}_0$. Orthogonal Cartesian co-ordinates and their associated tensor system have no preferred direction, that is no cylindrical symmetry in this case; this makes compactness of notation and the retention of tensor form very difficult to achieve when these Cartesians are used to describe magnetized plasmas.

The first possibility to come to mind when discussing cylindrical symmetry is the familiar one of cyclindrical polar co-ordinates (ρ,θ,z) defined in terms of the orthogonal Cartesians (x,y,z) by

$$\rho^2 = x^2 + y^2$$

$$\theta = \tan^{-1} {}^{y}/_{x}$$

$$z = z.$$

In this case the z-direction defines the preferred direction.

The definition of the element of length ds in tensor notation is

$$(ds)^2 = g_{mn}\, dx^m\, dx^n \qquad \text{(AI.1)}$$

where lower indices are covariant, upper indices are contravariant and we use the Einstein summation convention. The tensor g_{mn} is the metric tensor for the system of variables x^r.

For orthogonal Cartesians the metric tensor is

$$(g_{mn})_{o.c.} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Contravariant and covariant components are identical in this system. For cylindrical polars the metric tensor is

$$\left(g_{mn}\right)_{c.p.} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this case we must distinguish between contravariant and covariant components.

Suppose we require to differentiate a vector X_r with respect to the components of the position system x^s, say. In general we must define a <u>covariant derivative</u> in order that the result of the differentiation is itself a tensor. This covariant derivative is given by

$$X_{r,s} = \frac{\partial X_r}{\partial x^s} - \left\{ {m \atop r\ s} \right\} X_m \qquad \text{(AI.2)}$$

where $\left\{ {m \atop r\ s} \right\}$ is the Christoffel symbol of the second kind

$$\left\{ {m \atop r\ s} \right\} = \tfrac{1}{2}\, g^{mp} \left[\frac{\partial g_{sp}}{\partial x^r} + \frac{\partial g_{pr}}{\partial x^s} - \frac{\partial g_{rs}}{\partial x^p} \right]$$

and g^{mp} is the metric tensor for covariant differentials. For orthogonal Cartesians g_{mn} is constant so that $\left\{ {m \atop r\ s} \right\} = 0$ and $X_{r,s} = \dfrac{\partial X_r}{\partial x^s}$.

For cylindrical polars $\left\{ {m \atop r\ s} \right\} \neq 0$, and so the full covariant derivative (AI.2) must be used.

We would like to have a co-ordinate system which includes a preferred direction like cylindrical polars, but which also retains the simple derivatives of the orthogonal Cartesian system; this is achieved by the <u>polarized co-ordinate system</u> introduced by Buneman [6], and extended by Dougherty [7]. The elements of the position system (x^1, x^0, x^{-1}) are given in terms of (x,y,z) by the following:-

$$\left.\begin{aligned} x^1 &= 2^{-\frac{1}{2}}\,(x + iy) \\ x^0 &= z \\ x^{-1} &= 2^{-\frac{1}{2}}\,(x - iy) \end{aligned}\right\} \qquad \text{(AI.3)}$$

Note that the component labels are (1, 0, -1) instead of the usual (1, 2, 3).

From now on we use Greek indices to denote polarized co-ordinates and Roman indices to denote orthogonal Cartesians. We define the metric tensor in polarized co-ordinates to be $g_{\lambda\mu}$, and the metric tensor in orthogonal Cartesians to be $g_{mn} = \delta_{mn}$. To find an expression for $g_{\lambda\mu}$, we look at the invariant length element ds.

$$\left.\begin{aligned} (ds)^2 &= g_{mn}\, dx^m\, dx^n \\ &= g_{\lambda\mu}\, dx^\lambda\, dx^\mu \end{aligned}\right\} \qquad \text{(AI.4)}$$

Now $g_{mn}\, dx^m\, dx^n = dx^2 + dy^2 + dz^2$ and from (AI.3)

$$dx = 2^{-\frac{1}{2}}\,(dx^1 + dx^{-1})$$

$$dy = -\,i2^{-\frac{1}{2}}\,(dx^1 - dx^{-1})$$

$$dz = dx^0$$

Therefore

$$dx^2 + dy^2 + dz^2 = dx^1\, dx^{-1} + dx^{-1}\, dx^1 + dx^0\, dx^0$$

$$= g_{\lambda\mu}\, dx^\lambda\, dx^\mu \qquad \text{from (AI.4)}$$

Comparing coefficients of the differentials we find that

$$g_{1,-1} = 1 \;;\; g_{-1,1} = 1 \;;\; g_{0,0} = 1$$

and all the other coefficients are zero.

Therefore

$$\left.\begin{aligned} g_{\lambda\mu} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \delta_{\lambda,-\mu} \end{aligned}\right\} \qquad \text{(AI.5)}$$

If we have a vector A^{λ}, the associated vector A_{μ} is given by

$$A_{\mu} = \delta_{\lambda,-\mu} \; A^{\lambda}$$

Therefore $$A_1 = \delta_{1,-1} A^1 + \delta_{0,-1} A^0 + \delta_{-1,-1} A^{-1}$$
$$= A^{-1}$$

Similarly $$A_0 = A^0 \text{ and } A_{-1} = A^1$$

Thus indices are raised or lowered merely by changing sign. We may note that the invariant $(ds)^2$ in polarized co-ordinates contains products of differentials with different indices, so that the system is non-orthogonal.

By using polarized co-ordinates, we now have the x^0-direction as a preferred direction, while the (x^1, x^{-1}) co-ordinates together play the same role as the (ρ, θ) co-ordinates in cylindrical polars; they define the plane perpendicular to the x^0-direction and retain cylindrical symmetry. The advantages of this system are that contravariant and covariant components are very simply related (by a change of sign), and $g_{\lambda\mu}$ has constant elements. The Christoffel symbols are therefore always zero, and covariant derivatives reduce to ordinary partial derivatives.

We note here the general tensor result that the position system x^r is not a vector, but that covariant differentiation of a tensor with respect to x^r does result in a tensor with an extra covariant index. Therefore in the case of polarized co-ordinates partial differentiation of a tensor with respect to x^ν results in a tensor with one more covariant index than before.

One further point to be noted concerns the question of choosing the contravariance or covariance of vectors and tensors when transforming from (say) Cartesian tensor notation to polarized tensor notation. Our procedure is to arbitrarily choose basic physical vectors to be contravariant, and then all other quantities are chosen according to the Einstein summation convention.

For example, suppose we have the equation

$$\underline{j} = \underline{\underline{\sigma}} \cdot \underline{E}$$

We let $\underline{j}$ and $\underline{E}$ be represented by j^μ and E^ν in polarized co-ordinates; then the equation takes the form

$$j^\mu = \sigma^\mu_\nu \; E^\nu$$

Thus the conductivity tensor σ^μ_ν is chosen for us by the summation convention.

b) We now derive some properties of polarized tensors, defining certain useful scalars and tensors. Firstly, we define $g = \det [\, g_{\lambda\mu} \,] = -1$ from (AI.5). Next we define the permutation tensors $e^{\lambda\mu\nu}$ and $e_{\lambda\mu\nu}$ to be such that

$$\left.\begin{array}{l} e^{\lambda\mu\nu} \\ \text{or} \\ e_{\lambda\mu\nu} \end{array}\right\} \begin{array}{l} = +1 \text{ if } (\lambda,\mu,\nu) \text{ is an \underline{even} permutation of } (1,0,-1) \\ = -1 \text{ if } (\lambda,\mu,\nu) \text{ is an \underline{odd} permutation of } (1,0,-1) \\ = \;\; 0 \;\; \text{otherwise} \end{array}$$

Now $e^{\lambda\mu\nu}$ is a relative (or pseudo-) tensor of weight 1, while $e_{\lambda\mu\nu}$ is a relative tensor of weight -1. We now define corresponding absolute tensors, the ε-systems, such that

$$\varepsilon^{\lambda\mu\nu} = \frac{1}{\sqrt{g}} e^{\lambda\mu\nu} = -i\, e^{\lambda\mu\nu}$$

and
$$\varepsilon_{\lambda\mu\nu} = \sqrt{g}\, e_{\lambda\mu\nu} = i\, e_{\lambda\mu\nu}$$

Raising and lowering of the indices of permutation tensors is achieved through multiplication by the appropriate metric tensor, for example

$$e^{\lambda.\nu}_{\ \ \mu} = \delta_{\rho,-\mu}\, e^{\lambda\rho\nu}$$

$$e_{\lambda.\nu}^{\ \ \mu} = \delta^{\rho,-\mu}\, e_{\lambda\rho\nu}$$

Since the metric tensor is an absolute tensor, $e^{\lambda.\nu}_{\ \ \mu}$ and $e_{\lambda.\nu}^{\ \ \mu}$ are relative tensors of weights 1 and -1 respectively, and are such that

$$\left.\begin{array}{c} e^{\lambda.\nu}_{\ \ \mu} \\ \text{or} \\ e_{\lambda.\nu}^{\ \ \mu} \end{array}\right\} \begin{array}{l} = +1 \text{ if } (\lambda,-\mu,\nu) \text{ is an \underline{even} permutation of } (1,0,-1) \\ = -1 \text{ if } (\lambda,-\mu,\nu) \text{ is an \underline{odd} permutation of } (1,0,-1) \\ = 0 \text{ otherwise} \end{array}$$

The vector product of two tensors a^λ and b^μ is defined as

$$\left.\begin{array}{l} (\underline{a} \wedge \underline{b})^\rho = \varepsilon^{\rho\lambda\mu} a_\lambda b_\mu = -i\, e^{\rho\lambda\mu} a_\lambda b_\mu \\ (\underline{a} \wedge \underline{b})_\rho = \varepsilon_{\rho\lambda\mu} a^\lambda b^\mu = i\, e_{\rho\lambda\mu} a^\lambda b^\mu \end{array}\right\} \qquad (AI.6)$$

As an example of the use of these expressions, we derive the following result:-

$$[\ \underline{k} \wedge (\underline{k} \wedge \underline{E}_1)\ ]^\alpha = [\ k^\alpha k_\beta - k^2 \delta^\alpha_\beta\ ]\ (E_1)^\beta$$

Now

$$\begin{aligned} [\underline{k} \wedge (\underline{k} \wedge \underline{E}_1)\ ]^\alpha &= -i\, e^{\alpha\rho\mu} k_\rho\ (\underline{k} \wedge \underline{E}_1)_\mu \\ &= -i\, e^{\alpha\rho\mu} k_\rho\ [\, i\, e_{\mu\gamma\beta}\, k^\gamma (E_1)^\beta\ ] \\ &= e^{\alpha\rho\mu}\, e_{\mu\gamma\beta}\, k_\rho\, k^\gamma (E_1)^\beta \\ &= \delta^{\alpha\rho}_{\gamma\beta}\, k_\rho\, k^\gamma (E_1)^\beta \end{aligned}$$

where

$$\delta^{\alpha\rho}_{\gamma\beta} = \begin{cases} 0 \text{ when two of the subscripts or superscripts are the same, or when the superscripts do not have the same two values as the subscripts.} \\ +1 \text{ when } (\alpha,\rho) \text{ and } (\gamma,\beta) \text{ are the same permutation of the same two numbers.} \\ -1 \text{ when } (\alpha,\rho) \text{ and } (\gamma,\beta) \text{ are opposite permutations of the same two numbers.} \end{cases}$$

Comparison of elements verifies the result:-

$$\delta^{\alpha\rho}_{\gamma\beta} = \delta^{\alpha}_{\gamma}\,\delta^{\rho}_{\beta} - \delta^{\alpha}_{\beta}\,\delta^{\rho}_{\gamma}$$

Thus

$$[\ \underline{k} \wedge (\underline{k} \wedge \underline{E}_1)\ ]^{\alpha} = [\ \delta^{\alpha}_{\gamma}\,\delta^{\rho}_{\beta} - \delta^{\alpha}_{\beta}\,\delta^{\rho}_{\gamma}\ ]\ k_{\rho}k^{\gamma}(E_1)^{\beta}$$

$$= [\ k^{\alpha}\,k_{\beta} - k^2\,\delta^{\alpha}_{\beta}\ ]\ (E_1)^{\beta}$$

as required.

One further useful property is that in certain circumstances the tensor indices may be used as algebraic quantities, condensing the notation even further. For example, suppose we look at the unperturbed orbit of a charged particle in a constant magnetic field $\underline{B}_0$:-

$$\dot{\underline{V}} = \frac{q}{mc}\ \underline{V} \wedge \underline{B}_0 \quad \text{and} \quad \dot{\underline{X}} = \underline{V}$$

with $\underline{V} = \underline{v}$ and $\underline{X} = 0$ at $t = 0$.

Writing these in terms of orthogonal Cartesians, and then transforming to polarized co-ordinates, we get the equations

$$\dot{V}^1 = -i\Omega V^1 \qquad \dot{X}^1 = V^1$$

$$\dot{V}^0 = 0 \qquad \dot{X}^0 = V^0$$

$$\dot{V}^{-1} = i\Omega V^{-1} \qquad \dot{X}^{-1} = V^{-1}$$

with corresponding initial conditions ($\Omega = \dfrac{qB_0}{mc}$).

In polarized vector notation this is

$$\dot{V}^{\lambda} = - i\Omega\lambda V^{\lambda} \quad \text{and} \quad \dot{X}^{\lambda} = V^{\lambda}$$

with $V^{\lambda} = v^{\lambda}$ and $X^{\lambda} = 0$ at $t = 0$.

Solving these we find that

$$\left.\begin{aligned} V^{\lambda} &= v^{\lambda}\, e^{-i\Omega\lambda t} \\ \text{and} \qquad X^{\lambda} &= \frac{v^{\lambda}}{i\Omega\lambda}\,(1 - e^{-i\Omega\lambda t}) \end{aligned}\right\} \qquad \text{(AI.7)}$$

where the summation convention would only apply to the vector quantities. Note that when $\lambda = 0$, to find X^{λ} we must use

$$\lim_{\lambda \to 0} \left[\frac{1}{\lambda} (1 - e^{-i\Omega\lambda t}) \right] = i\Omega t$$

Equations (AI.7) are much more compact than the corresponding solution in terms of Cartesian vectors.

APPENDIX II

The tensor moment integrals defined in Chapter 3 of Section I are as follows:-

$A^{\alpha} = I^{\alpha}\{1\}$ $\qquad C^{\alpha} = I^{\alpha}\{v_{\perp}^{2}\}$ $\qquad D^{\alpha} = I^{\alpha}\{v_{\parallel}^{2}\}$

$F^{\alpha}_{\beta} = I^{\alpha}\{w_{\beta}\}$ $\qquad G^{\alpha}_{\rho} = I^{\alpha}\{v_{\rho}\}$

$K^{\alpha}_{\rho} = I^{\alpha}\{v_{\perp}^{2}v_{\rho}\}$ $\qquad L^{\alpha}_{\rho} = I^{\alpha}\{v_{\parallel}^{2}v_{\rho}\}$

$M^{\alpha}_{\mu\rho} = I^{\alpha}\{w_{\mu}v_{\rho}\}$

$$\text{where } I^{\alpha}\{\quad\} = \frac{iq^2}{m\Omega}\int_{\underline{v}}\int_{t=o}^{\infty} V^{\alpha} f_M e^{i(\omega t-\underline{k}.\underline{X})}\{\quad\}\,\underline{dv}\,dt$$

As an example of how the integrations involved in the use of $I^{\alpha}\{\quad\}$ are carried out, we now derive the element G^{1}_{-1}. The elements of A^{α} and G^{α}_{ρ} are then listed. Finally a method is given for deriving the elements of C^{α} and D^{α} from those of A^{α}, and the elements of F^{α}_{ρ}, K^{α}_{ρ} and L^{α}_{ρ} from those of G^{α}_{ρ}.

The unperturbed trajectory of a charged particle travelling in a constant magnetic field $B_0\hat{\underline{z}}$ and passing through the point $\underline{r} = 0$ at time t is

$$\left.\begin{aligned} X(t) &= \frac{v_{\perp}}{\Omega}\left[\sin(\Omega t+\phi) - \sin\phi\right] \\ Y(t) &= \frac{v_{\perp}}{\Omega}\left[\cos(\Omega t+\phi) - \cos\phi\right] \\ Z(t) &= v_{\parallel}t \end{aligned}\right\} \qquad \text{(AII.1)}$$

where ϕ is a phase angle.

Thus

$$\left.\begin{aligned} V_x(t) &= v_{\perp}\cos(\Omega t+\phi) \\ V_y(t) &= -v_{\perp}\sin(\Omega t+\phi) \\ V_z(t) &= v_{\parallel} \end{aligned}\right\} \qquad \text{(AII.2)}$$

We choose the wave vector $\underline{k} = (k_\perp, 0, k_\parallel)$. That is the x-direction is chosen to lie along the direction of $k_\perp$; there is no loss of generality since the x and y axes are arbitrary at this stage.

It is easily seen that the polarized $\underline{V}$ components are

$$\left.\begin{aligned} V^1 &= \frac{1}{\sqrt{2}} v_\perp e^{-i(\Omega t+\phi)} \\ V^0 &= v_\parallel \\ V^{-1} &= \frac{1}{\sqrt{2}} v_\perp e^{-i(\Omega t+\phi)} \end{aligned}\right\} \qquad \text{(AII.3)}$$

Similarly the components of $\underline{v}$ are

$$\begin{aligned} v^1 &= \frac{1}{\sqrt{2}} v_\perp e^{-i\phi} = v_{-1} \\ v^0 &= v_\parallel \\ v^{-1} &= \frac{1}{\sqrt{2}} v_\perp e^{i\phi} = v_1 \end{aligned}$$

Therefore $V^1 v_{-1} = \frac{1}{2} v_\perp^2 e^{-i(\Omega t+2\phi)}$

Now

$$\omega t - \underline{k}.\underline{X} = (\omega - k_\parallel v_\parallel)t - \frac{k_\perp v_\perp}{\Omega} [\sin(\Omega t+\phi) - \sin\phi]$$

and

$$f_M = n_0 \left(\frac{a_\perp}{\pi}\right)\left(\frac{a_\parallel}{\pi}\right)^{\frac{1}{2}} \exp\{-(a_\perp v_\perp^2 + a_\parallel v_\parallel^2)\}$$

from Equation (I.12).

So

$$G^1_{-1} = \int_{v_\perp=0}^{\infty}\int_{v_\parallel=-\infty}^{\infty}\int_{\phi=0}^{2\pi}\int_{t=0}^{\infty} \frac{1}{2} v_\perp^2 e^{-i(\Omega t+2\phi)} f_M$$

$$. \exp\{i([\omega - k_\parallel v_\parallel]t - \zeta[\sin(\Omega t+\phi) - \sin\phi])\} v_\perp dv_\perp dv_\parallel d\phi dt$$

$$\text{where } \zeta = \frac{k_\perp v_\perp}{\Omega}$$

The t-integral in this expression is

$$I_t = \int_0^{\infty} e^{-i\Omega t} \cdot e^{i[\omega - k_{\parallel} v_{\parallel}]t} \exp\{-i\zeta \sin(\Omega t+\phi)\} \, dt$$

We now make use of the following Bessel function identity:-

$$e^{i\zeta \sin\theta} = \sum_{n=-\infty}^{\infty} J_n(\zeta) \, e^{in\theta} \qquad \text{(AII.4)}$$

where $J_n(\zeta)$ is the Bessel function of the first kind of order n (see reference [27]). Using this we may write

$$I_t = \sum_{n=-\infty}^{\infty} J_n(\zeta) e^{-in\phi} \int_0^{\infty} \exp[\, i\, \{\omega - k_{\parallel} v_{\parallel} - (n+1)\Omega\} t \,] \, dt$$

$$= \sum_{n=-\infty}^{\infty} J_n(\zeta) e^{-in\phi} \left[\frac{\exp[\,\{\omega - k_{\parallel} v_{\parallel} - (n+1)\Omega\} t\,]}{i\{\omega - k_{\parallel} v_{\parallel} - (n+1)\Omega\}} \right]_{t=o}^{\infty} \qquad \text{(AII.5)}$$

The limit $t = \infty$ in the above expression is equivalent to a time in the infinite past according to our original formulation of the problem. The principle of causality asserts that "effect" must follow "cause"; in this case the "cause" is a perturbation in the electromagnetic field in the infinite past, while the "effect" is the perturbation distribution function f_1. With reference to the principle of causality then, we assume that f_1 grows from zero in the infinite past to its value at $t = o$, and therefore that all quantities derived from f_1 must also have zero values in the infinite past, since they must "follow" the perturbation in the electromagnetic field. In (AII.5) this implies that the exponential in square brackets gives a zero result when evaluated at $t = \infty$; note that this is equivalent to saying that ω has a small positive imaginary part. With this assumption (AII.5) becomes

$$I_t = i \sum_{n=-\infty}^{\infty} J_n(\zeta) e^{-in\phi} \left(\omega - k_{\parallel} v_{\parallel} - (n+1)\Omega\right)^{-1} \qquad \text{(AII.6)}$$

Now consider the ϕ-integral:-

$$I_\phi = \int_{\phi=o}^{2\pi} e^{-i(n+2)\phi} \, . \, e^{i\zeta \sin \phi} \, d\phi$$

$$= \sum_{m=-\infty}^{\infty} J_m(\zeta) \int_{\phi=o}^{2\pi} e^{i(m-n-2)\phi} \, d\phi$$

$$= \sum_{m=-\infty}^{\infty} J_m(\zeta) \, . \, 2\pi \, \delta_{m,n+2}$$

(where $\delta_{m,n+2}$ is the Kronecker delta)

$$= 2\pi \, J_{n+2}(\zeta) \qquad \text{(AII.7)}$$

Thus the $v_\perp$-integration becomes

$$I_{v_\perp} = \int_{\zeta=o}^{\infty} \frac{\Omega^4}{k_\perp^4} \zeta^3 \exp\left(- \frac{a_\perp \Omega^2}{k_\perp^2} \zeta^2 \right) J_n(\zeta) J_{n+2}(\zeta) d\zeta$$

Standard Bessel function identities [28] give the following expressions:-

$$J_{n+2}(\zeta) = \frac{2(n+1)}{\zeta} J_{n+1}(\zeta) - J_n(\zeta)$$

and $$J_{n+1}(\zeta) = \frac{n}{\zeta} J_n(\zeta) - J_n{}'(\zeta)$$

Therefore

$$J_n(\zeta) \, J_{n+2}(\zeta) = \frac{2(n+1)n}{\zeta^2} J_n{}^2(\zeta) - \frac{n+1}{\zeta} \frac{d}{d\zeta} \{J_n{}^2(\zeta)\} - J_n{}^2(\zeta) \}$$

We may now write $I_{v_\perp}$ as the sum of the following three integrals:-

$$I_1 = \int_{\zeta=o}^{\infty} \frac{\Omega^4}{k_\perp^4} \zeta^3 \, . \, \frac{2(n+1)n}{\zeta^2} J_n{}^2(\zeta) \exp\left(- \frac{a_\perp \Omega^2}{k_\perp^2} \zeta^2 \right) d\zeta$$

$$= (a_\perp s)^{-2} . \, 2(n+1)n \int_{\rho=o}^{\infty} \rho e^{-\rho^2} J_n{}^2(s\rho) d\rho$$

where $\zeta = s\rho$ and $s = \left(\frac{k_\perp^2}{\Omega^2 a_\perp} \right)^{\frac{1}{2}}$

$$I_2 = -(a_\perp s)^{-2}\,(n+1)\int_{\rho=0}^{\infty}\rho^2 e^{-\rho^2}\,\frac{d}{d\rho}\{J_n^{\,2}(s\rho)\}\,d\rho$$

$$I_3 = -(a_\perp)^{-2}\int_{\rho=0}^{\infty}\rho^3 e^{-\rho^2}\,J_n^{\,2}(s\rho)\,d\rho$$

$$= -(a_\perp)^{-2}\left[\int_{\rho=0}^{\infty}\rho e^{-\rho^2}\,J_n^{\,2}(s\rho)\,d\rho + \tfrac{1}{2}\int_{\rho=0}^{\infty}\rho^2 e^{-\rho^2}\,\frac{d}{d\rho}\{J_n^{\,2}(s\rho)\}\,d\rho\right]$$

on integration by parts

Thus

$$I_{v_\perp} = I_1 + I_2 + I_3$$

$$= (a_\perp)^{-2}\left[\frac{2(n+1)n}{s^2} - 1\right]\int_{\rho=0}^{\infty}\rho^2 e^{-\rho^2}\,J_n^{\,2}(s\rho)\,d\rho$$

$$-(a_\perp)^{-2}\left[\frac{n+1}{s^2} + \tfrac{1}{2}\right]\int_{\rho=0}^{\infty}\rho^2 e^{-\rho^2}\,\frac{d}{d\rho}\{J_n^{\,2}(s\rho)\}\,d\rho$$

We now make use of the following identities (see [27]) :-

$$\int_0^{\infty}\rho e^{-\rho^2}\,J_n^{\,2}(s\rho)\,d\rho = \tfrac{1}{2}\,e^{-\frac{s^2}{2}}\,I_n\left(\frac{s^2}{2}\right)$$

and
$$\int_0^{\infty}\rho^2 e^{-\rho^2}\,\frac{d}{d\rho}\{J_n^{\,2}(s\rho)\}\,d\rho = \frac{s^2}{2}\,e^{-\frac{s^2}{2}}\left[I_n'\left(\frac{s^2}{2}\right) - I_n\left(\frac{s^2}{2}\right)\right]$$

where $I_n(x)$ is the Bessel function of the second kind of order n.

Therefore

$$I_{v_\perp} = (a_\perp)^{-2}\,e^{-\frac{s^2}{2}}\left[\left\{\tfrac{1}{4}s^2 + \tfrac{1}{2}n + n(n+1)s^{-2}\right\}I_n\left(\frac{s^2}{2}\right) - \left\{\tfrac{1}{4}s^2 + \tfrac{1}{2}(n+1)\right\}I_n'\left(\frac{s^2}{2}\right)\right] \qquad \text{(AII.8)}$$

Finally the $v_{\parallel}$ integral is:-

$$I_{v_{\parallel}} = \int_{v_{\parallel}=-\infty}^{\infty} e^{-a_{\parallel} v_{\parallel}^2} (\omega - k_{\parallel} v_{\parallel} - (n+1)\Omega)^{-1} \, dv_{\parallel}$$

$$= -\frac{1}{k_{\parallel}} \int_{p=-\infty}^{\infty} e^{-p^2} \left(p - \frac{\omega + (n+1)\Omega}{k_{\parallel} \, a_{\parallel}^{-\frac{1}{2}}} \right)^{-1} dp$$

where $p = a_{\parallel}^{\frac{1}{2}} v_{\parallel}$

Now the plasma dispersion function $Z(z)$ is defined by

$$Z(z) = \pi^{-\frac{1}{2}} \int_{p=-\infty}^{\infty} e^{-p^2} (p-z)^{-1} \, dp$$

Thus

$$I_{v_{\parallel}} = -\frac{\pi^{\frac{1}{2}}}{k_{\parallel}} Z(z_{n+1}) \qquad \text{(AII.9)}$$

$$\text{where } z_{n+1} = \frac{\omega + (n+1)\Omega}{k_{\parallel} \, a_{\parallel}^{-\frac{1}{2}}}$$

Combining (AII.6), (AII.7), (AII.8), (AII.9) and collecting together all the numerical factors gives the following expression for G^1_{-1} :-

$$G^1_{-1} = \sum_{n=-\infty}^{\infty} \frac{-in_0}{k_{\parallel}} \frac{a_{\parallel}^{\frac{1}{2}}}{a_{\perp}} e^{-\frac{s^2}{2}} \left[\{\tfrac{1}{4}s^2 + \tfrac{1}{2}n + n(n+1)s^{-2}\} I_n\left(\frac{s^2}{2}\right) - \{\tfrac{1}{4}s^2 + \tfrac{1}{2}(n+1)\} I_n'\left(\frac{s^2}{2}\right) \right] Z(z_{n+1})$$

We evaluate the other elements of G^{α}_{ρ} , and those of A^{α}, by similar procedures. The results are

$$A^1 = \sum_{n=-\infty}^{\infty} -\frac{in_0}{\sqrt{2} k_{\parallel} s} \left(\frac{a_{\parallel}}{a_{\perp}}\right)^{\frac{1}{2}} e^{-\frac{s^2}{2}} \left[(n+\tfrac{s^2}{2}) I_n\left(\frac{s^2}{2}\right) - \frac{s^2}{2} I_n'\left(\frac{s^2}{2}\right) \right] Z(z_{n+1})$$

$$A^0 = \sum_{n=-\infty}^{\infty} -\frac{in_0}{k_{\parallel}} e^{-\frac{s^2}{2}} I_n\left(\frac{s^2}{2}\right) \{ 1 + z_n Z(z_n) \}$$

$$A^{-1} = \sum_{n=-\infty}^{\infty} - \frac{in_0}{\sqrt{2}k_\parallel s} \left(\frac{a_\parallel}{a_\perp}\right)^{\frac{1}{2}} e^{-\frac{s^2}{2}} \left[(n - \frac{s^2}{2})\, I_n(\frac{s^2}{2}) + \frac{s^2}{2}\, I_n'(\frac{s^2}{2}) \right] Z(z_{n-1})$$

and

$$G^1_1 = \sum_{n=-\infty}^{\infty} - \frac{in_0}{2k_\parallel} \frac{a_\parallel^{\frac{1}{2}}}{a_\perp} e^{-\frac{s^2}{2}} \left\{ (1 - \frac{s^2}{2})\, I_n + \frac{s^2}{2}\, I_n' \right\} Z(z_{n+1})$$

$$G^1_0 = \sum_{n=-\infty}^{\infty} - \frac{in_0}{\sqrt{2}k_\parallel s} (a_\perp)^{-\frac{1}{2}} e^{-\frac{s^2}{2}} \left\{ (n + \frac{s^2}{2})\, I_n - \frac{s^2}{2} I_n' \right\} [1 + z_{n+1} Z(z_{n+1})]$$

$$G^1_{-1} = \sum_{n=-\infty}^{\infty} - \frac{in_0}{k_\parallel} \frac{a_\parallel^{\frac{1}{2}}}{a_\perp} e^{-\frac{s^2}{2}} \left[\left\{ \tfrac{1}{4}s^2 + \tfrac{1}{2}n + n(n+1)s^{-2} \right\} I_n - \left\{ \tfrac{1}{4}s^2 + \tfrac{1}{2}(n+1) \right\} I_n' \right] Z(z_{n+1})$$

$$G^0_1 = \sum_{n=-\infty}^{\infty} - \frac{in_0}{\sqrt{2}k_\parallel s} (a_\perp)^{-\frac{1}{2}} e^{-\frac{s^2}{2}} \left\{ (n - \frac{s^2}{2})\, I_n + \frac{s^2}{2} I_n' \right\} [1 + z_n Z(z_n)]$$

$$G^0_0 = \sum_{n=-\infty}^{\infty} - \frac{in_0}{k_\parallel} (a_\parallel)^{-\frac{1}{2}} e^{-\frac{s^2}{2}} I_n \cdot z_n [1 + z_n Z(z_n)]$$

$$G^0_{-1} = \sum_{n=-\infty}^{\infty} - \frac{in_0}{\sqrt{2}k_\parallel s} (a_\perp)^{-\frac{1}{2}} e^{-\frac{s^2}{2}} \left\{ (n + \frac{s^2}{2})\, I_n - \frac{s^2}{2} I_n' \right\} [1 + z_n Z(z_n)]$$

$$G^{-1}_1 = \sum_{n=-\infty}^{\infty} - \frac{in_0}{k_\parallel} \frac{a_\parallel^{\frac{1}{2}}}{a_\perp} e^{-\frac{s^2}{2}} \left[\left\{ \tfrac{1}{4}s^2 - \tfrac{1}{2}n + n(n-1)s^{-2} \right\} I_n - \left\{ \tfrac{1}{4}s^2 - \tfrac{1}{2}(n-1) \right\} I_n' \right] Z(z_{n-1})$$

$$G^{-1}_0 = \sum_{n=-\infty}^{\infty} - \frac{in_0}{\sqrt{2}k_\parallel s} (a_\perp)^{-\frac{1}{2}} e^{-\frac{s^2}{2}} \left\{ (n - \frac{s^2}{2})\, I_n + \frac{s^2}{2}\, I_n' \right\} [1 + z_{n-1} Z(z_{n-1})]$$

$$G^{-1}_{-1} = \sum_{n=-\infty}^{\infty} - \frac{in_0}{2k_\parallel} \frac{a_\parallel^{\frac{1}{2}}}{a_\perp} e^{-\frac{s^2}{2}} \left\{ (1 - \frac{s^2}{2}) I_n + \frac{s^2}{2}\, I_n' \right\} Z(z_{n-1})$$

where $s = \left(\frac{k_\perp^2}{\Omega^2 a_\perp}\right)^{\frac{1}{2}}$ $\qquad z_m = \frac{\omega + m\Omega}{k_\parallel\, a_\parallel^{-\frac{1}{2}}}$

and the argument of all Bessel functions is $\frac{s^2}{2}$

Note that, since $F_{\rho}^{\alpha} = I^{\alpha}\{w_{\rho}\}$

$$\text{and} \quad G_{\rho}^{\alpha} = I^{\alpha}\{v_{\rho}\}$$

the elements of F_{ρ}^{α} are easily found in terms of those of G_{ρ}^{α} :-

$$F_{1}^{\alpha} = a_{\perp} G_{1}^{\alpha} \quad ; \quad F_{0}^{\alpha} = a_{\parallel} G_{0}^{\alpha} \quad ; \quad F_{-1}^{\alpha} = a_{\perp} G_{-1}^{\alpha}$$

Now

$$f_M = n_0 \left(\frac{a_{\perp}}{\pi}\right) \left(\frac{a_{\parallel}}{\pi}\right)^{\frac{1}{2}} \exp\left[-(a_{\perp} v_{\perp}^2 + a_{\parallel} v_{\parallel}^2)\right]$$

and

$$-\frac{\partial}{\partial a_{\perp}} \left\{\exp\left[-(a_{\perp} v_{\perp}^2 + a_{\parallel} v_{\parallel}^2)\right]\right\} = v_{\perp}^2 \exp\left[-(a_{\perp} v_{\perp}^2 + a_{\parallel} v_{\parallel}^2)\right]$$

$$-\frac{\partial}{\partial a_{\parallel}} \left\{\exp\left[-(a_{\perp} v_{\perp}^2 + a_{\parallel} v_{\parallel}^2)\right]\right\} = v_{\parallel}^2 \exp\left[-(a_{\perp} v_{\perp}^2 + a_{\parallel} v_{\parallel}^2)\right]$$

Hence

$$C^{\alpha} = \frac{iq^2}{m\Omega} n_0 \left(\frac{a_{\perp}}{\pi}\right) \left(\frac{a_{\parallel}}{\pi}\right)^{\frac{1}{2}} \int_{\underline{v}} \int_{t=o}^{\infty} V^{\alpha} v_{\perp}^2 \exp\left[-(a_{\perp} v_{\perp}^2 + a_{\parallel} v_{\parallel}^2)\right] \exp\left[i(\omega t - \underline{k} \cdot \underline{X})\right] \underline{dv}\, dt$$

$$= \frac{iq^2}{m\Omega} n_0 \left(\frac{a_{\perp}}{\pi}\right) \left(\frac{a_{\parallel}}{\pi}\right)^{\frac{1}{2}} \left(-\frac{\partial}{\partial a_{\perp}}\right) \int_{\underline{v}} \int_{t=o}^{\infty} V^{\alpha} \exp\left[-(a_{\perp} v_{\perp}^2 + a_{\parallel} v_{\parallel}^2)\right] \exp\left[i(\omega t - \underline{k} \cdot \underline{X})\right] \underline{dv}\, dt$$

$$= -a_{\perp} \frac{\partial}{\partial a_{\perp}} \left[a_{\perp}^{-1} A^{\alpha}\right]$$

$$\text{or} \quad C^{\alpha} = \left[\frac{A^{\alpha}}{a_{\perp}} - \frac{\partial A^{\alpha}}{\partial a_{\perp}}\right] \qquad \text{(AII.10)}$$

Similarly

$$D^{\alpha} = \left[\tfrac{1}{2} \frac{A^{\alpha}}{a_{\parallel}} - \frac{\partial A^{\alpha}}{\partial a_{\parallel}}\right]$$

$$K_{\rho}^{\alpha} = \left[\frac{G_{\rho}^{\alpha}}{a_{\perp}} - \frac{\partial G_{\rho}^{\alpha}}{\partial a_{\perp}}\right]$$

$$L_{\rho}^{\alpha} = \left[\tfrac{1}{2} \frac{G_{\rho}^{\alpha}}{a_{\parallel}} - \frac{\partial G_{\rho}^{\alpha}}{\partial a_{\parallel}}\right]$$

Hence the elements of all the tensor moment integrals except $M^{\alpha}_{\mu\rho}$ may be derived from the elements of A^{α} and G^{α}_{ρ} merely by differentiating them with respect to $a_{\perp}$ or $a_{\parallel}$, and then substituting in the appropriate formula. Unfortunately the third order tensor $M^{\alpha}_{\mu\rho}$ must be evaluated in a similar way to A^{α} and G^{α}_{ρ}. More complicated Bessel function identities are necessary, but in principle the method of evaluation is the same.

APPENDIX III

(a) The zero-order Vlasov equation in this case is

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{q}{mc} (\underline{v} \wedge \underline{B}_0) \cdot \frac{\partial f}{\partial \underline{v}} = 0 \qquad \text{(AIII.1)}$$

and $f_0 = [1 + A(x + \frac{v_y}{\Omega})] f_M$

where $A = (\varepsilon + \delta_\perp [a_\perp v_\perp^2 - 1] + \delta_\parallel [a_\parallel v_\parallel^2 - \frac{1}{2}])$

In order to substitute this f_0 into the left-hand side of (AIII.1) we need the following expressions:-

$$\frac{\partial f_0}{\partial t}, \quad \frac{\partial f_0}{\partial \underline{r}}, \quad \frac{\partial f_0}{\partial \underline{v}}$$

Now
$$\frac{\partial f_0}{\partial t} = 0$$

while
$$\frac{\partial f_0}{\partial \underline{r}} = \left(\frac{\partial f_0}{\partial x}, 0, 0 \right)$$

and
$$\frac{\partial f_0}{\partial x} = A f_M$$

With $\underline{B}_0 = B_0 \hat{\underline{z}}$, $\quad \frac{q}{mc} (\underline{v} \wedge \underline{B}_0) \cdot \frac{\partial f_0}{\partial \underline{v}} = \Omega [v_y \frac{\partial f_0}{\partial v_x} - v_x \frac{\partial f_0}{\partial v_y}]$

$$\frac{\partial f_0}{\partial v_x} = \left\{ -2a_\perp v_x [1 + A(x + \frac{v_y}{\Omega})] + 2v_x \delta_\perp a_\perp (x + \frac{v_y}{\Omega}) \right\} f_M$$

$$\frac{\partial f_0}{\partial v_y} = \left\{ -2a_\perp v_y [1 + A(x + \frac{v_y}{\Omega})] + \frac{1}{\Omega} A + 2v_y \delta_\perp a_\perp (x + \frac{v_y}{\Omega}) \right\} f_M$$

Therefore

$$\Omega [v_y \frac{\partial f_0}{\partial v_x} - v_x \frac{\partial f_0}{\partial v_y}] = - v_x A f_M$$

and so

$$\frac{\partial f_0}{\partial t} + \underline{v} \cdot \frac{\partial f_0}{\partial \underline{r}} + \frac{q}{mc} (\underline{v} \wedge \underline{B}_0) \cdot \frac{\partial f_0}{\partial \underline{v}} = v_x A f_M - v_x A f_M$$
$$= 0$$

Thus our f_0 satisfies (AIII.1).

To derive expressions for $n(x)$, $T^{\perp}(x)$ and $T^{\parallel}(x)$ from f_0, we use the following definitions:-

$$n(x) = \int_{\underline{v}} f_0 \, \underline{dv} \qquad \text{(A III.2)}$$

$$n(x) \, \kappa \, T^{\perp}(x) = \int_{\underline{v}} \tfrac{1}{2} m v_{\perp}^2 f_0 \, \underline{dv} \qquad \text{(A III.3)}$$

$$\tfrac{1}{2} n(x) \, \kappa \, T^{\parallel}(x) = \int_{\underline{v}} \tfrac{1}{2} m v_{\parallel}^2 f_0 \, \underline{dv} \qquad \text{(A III.4)}$$

Consider (A III.2):-

$$n(x) = \int_{\underline{v}} f_0 \, \underline{dv}$$

$$= \int_{v_{\perp}=0}^{\infty} \int_{v_{\parallel}=-\infty}^{\infty} \int_{\phi=0}^{2\pi} \Big[1 + (\varepsilon - \delta_{\perp} - \tfrac{1}{2}\delta_{\parallel})x + (\varepsilon - \delta_{\perp} - \tfrac{1}{2}\delta_{\parallel}) \frac{v_y}{\Omega}$$

$$+ \delta_{\perp} a_{\perp} x v_{\perp}^2 + \delta_{\parallel} a_{\parallel} x v_{\parallel}^2$$

$$+ \frac{1}{\Omega} \delta_{\perp} a_{\perp} v_{\perp}^2 v_y + \frac{1}{\Omega} \delta_{\parallel} a_{\parallel} v_{\parallel}^2 v_y \Big] f_M v_{\perp} dv_{\perp} dv_{\parallel} d\phi$$

Hence we require the following integrals:-

$$I_1 = \int_{\underline{v}} f_M \, \underline{dv} \qquad I_4 = \int_{\underline{v}} v_{\parallel}^2 f_M \, \underline{dv}$$

$$I_2 = \int_{\underline{v}} v_y f_M \, \underline{dv} \qquad I_5 = \int_{\underline{v}} v_{\perp}^2 v_y f_M \, \underline{dv}$$

$$I_3 = \int_{\underline{v}} v_{\perp}^2 f_M \, \underline{dv} \qquad I_6 = \int_{\underline{v}} v_{\parallel}^2 v_y f_M \, \underline{dv}$$

Now $v_y = - v_{\perp} \sin \phi$, so that integrals involving v_y contain

$\int_0^{2\pi} \sin \phi \, d\phi$, which is zero. Therefore I_2, I_5 and I_6 are all zero.

$$I_1 = n_0 \left(\frac{a_\perp}{\pi}\right)\left(\frac{a_{\shortparallel}}{\pi}\right)^{\frac{1}{2}} \int_{v_\perp=0}^{\infty} v_\perp \exp\{-a_\perp v_\perp^2\} dv_\perp \int_{v_{\shortparallel}=-\infty}^{\infty} \exp\{-a_{\shortparallel} v_{\shortparallel}^2\} dv_{\shortparallel} \int_{\phi=0}^{2\pi} d\phi$$

$$= 2\pi\, n_0 \left(\frac{a_\perp}{\pi}\right)\left(\frac{a_{\shortparallel}}{\pi}\right)^{\frac{1}{2}} \left(\frac{\pi}{a_{\shortparallel}}\right)^{\frac{1}{2}} \cdot -\frac{1}{2a_\perp} \int_{v_\perp=0}^{\infty} \frac{d}{dv_\perp}\left(e^{-a_\perp v_\perp^2}\right) dv_\perp$$

$$\left(\text{since} \int_{-\infty}^{\infty} \exp\{-a_{\shortparallel} v_{\shortparallel}^2\} dv_{\shortparallel} = \left(\frac{\pi}{a_{\shortparallel}}\right)^{\frac{1}{2}}\right)$$

$$= -n_0 \left[e^{-a_\perp v_\perp^2} \right]_0^{\infty}$$

$$= n_0$$

By the same argument as that used to derive equation (AII.10) of Appendix II, we may write

$$I_3 = \left[\frac{I_1}{a_\perp} - \frac{\partial I_1}{\partial a_\perp} \right] = \frac{n_0}{a_\perp}$$

$$I_4 = \left[\frac{1}{2} \frac{I_1}{a_{\shortparallel}} - \frac{\partial I_1}{\partial a_{\shortparallel}} \right] = \frac{1}{2} \frac{n_0}{a_{\shortparallel}}$$

Thus

$$n(x) = [1 + (\varepsilon - \delta_\perp - \tfrac{1}{2}\delta_{\shortparallel})x]\, I_1 + \delta_\perp a_\perp x I_3 + \delta_{\shortparallel} a_{\shortparallel} x I_4$$

$$= n_0 [1 + (\varepsilon - \delta_\perp - \tfrac{1}{2}\delta_{\shortparallel})x] + \delta_\perp x n_0 + \tfrac{1}{2}\delta_{\shortparallel} x n_0$$

That is $n(x) = n_0 [1 + \varepsilon x]$

Equation (AIII.3) is

$$\frac{2n(x)\kappa}{m} T^\perp(x) = \int_{\underline{v}} v_\perp^2 f_0 \underline{dv}$$

In this case integrals involving v_y are again zero, and we must evaluate the following:-

$$I_7 = \int_{\underline{v}} v_\perp^2 f_M \underline{dv} = I_3 = \frac{n_0}{a_\perp}$$

$$I_8 = \int_{\underline{v}} v_\perp^4 f_M \underline{dv} = \left[\frac{I_7}{a_\perp} - \frac{\partial I_7}{\partial a_\perp} \right] = \frac{2n_0}{a_\perp^2}$$

$$I_9 = \int_{\underline{v}} v_{\shortparallel}^2 v_{\perp}^2 \; f_M \, \underline{dv} = [\tfrac{1}{2} \frac{I_7}{a_{\shortparallel}} - \frac{\partial I_7}{\partial a_{\shortparallel}}] = \frac{n_0}{2a_{\perp} a_{\shortparallel}}$$

Then

$$\frac{2n(x)\kappa}{m} T^{\perp}(x) = [1 + (\varepsilon - \delta_{\perp} - \tfrac{1}{2} \delta_{\shortparallel})x] \; I_7 + \delta_{\perp} \, a_{\perp} \, x \, I_8 + \delta_{\shortparallel} \, a_{\shortparallel} \, x \, I_9$$

$$= \frac{n_0}{a_{\perp}} \; [1 + (\varepsilon - \delta_{\perp} - \tfrac{1}{2} \delta_{\shortparallel})x] \; + \frac{2n_0}{a_{\perp}} \delta_{\perp} \, x + \frac{n_0}{2a_{\perp}} \; \delta_{\shortparallel} \, x$$

$$= \frac{n_0}{a_{\perp}} \; [1 + (\varepsilon + \delta_{\perp})x]$$

That is $T^{\perp}(x) = T_0^{\perp} \; [1 + (\varepsilon + \delta_{\perp})x] \; [1 + \varepsilon x]^{-1}$

with $a_{\perp} = \dfrac{m}{2\kappa T_0^{\perp}}$ and $n(x) = n_0 \, [1 + \varepsilon x]$

We may expand $[1 + \varepsilon x]^{-1}$ as follows:-

$$[1 + \varepsilon x]^{-1} = 1 - \varepsilon x + (\varepsilon x)^2 - (\varepsilon x)^3 \; . \; . \; . \; .$$

Then

$$[1 + (\varepsilon + \delta_{\perp})x][1 + \varepsilon x]^{-1} = 1 + \delta_{\perp} \, x + (\varepsilon^2 - \delta_{\perp}\varepsilon)x^2 + 0(\varepsilon^3 \, x^3)$$

Thus if we assume that ε and $\delta_{\perp}$ are small, we may neglect terms of order $(\varepsilon x)^2$ or $(\delta_{\perp} \, x)^2$ provided $(\varepsilon x, \; \delta_{\perp} \, x) << 1$, giving

$$T^{\perp}(x) \simeq T_0^{\perp} \, [1 + \delta_{\perp} \, x]$$

Similarly, fron (AIII.4) we find that

$$T^{\shortparallel}(x) \simeq T_0^{\shortparallel} \, [1 + \delta_{\shortparallel} \, x]$$

Note that if $\varepsilon = 0$ no approximation is necessary, and we may write

$$T^{\perp,\shortparallel}(x) = T_0^{\perp,\shortparallel} \; [1 + \delta_{\perp,\shortparallel} \, x]$$

(b) We have a frame of reference in which the zero-order Vlasov equation has the following form:-

$$\frac{\partial f_0}{\partial t} + \underline{v} \cdot \frac{\partial f_0}{\partial \underline{r}} + \frac{q}{m} \left(\underline{E}_0 + \frac{\underline{v} \wedge \underline{B}_0}{c} \right) \cdot \frac{\partial f_0}{\partial \underline{v}} = 0 \qquad \text{(AIII.5)}$$

This frame of reference is now transformed to a second frame moving with a velocity $\underline{v}_0 = c\frac{(\underline{E}_0 \wedge \underline{B}_0)}{B_0^2}$ with respect to the first frame.

The transformed values of the fields $\underline{E}$ and $\underline{B}$ in the original frame are denoted now by $\underline{E}'$ and $\underline{B}'$, and they are related to $\underline{E}$ and $\underline{B}$ as follows:-

$$\left.\begin{aligned} \underline{E}'_{\parallel} &= \underline{E}_{\parallel} \quad ; \quad \underline{B}'_{\parallel} = \underline{B}_{\parallel} \\ \underline{E}'_{\perp} &= \underline{E}_{\perp} + \frac{\underline{v}_0}{c} \wedge \underline{B} \; ; \; \underline{B}'_{\perp} = \underline{B}_{\perp} - \frac{\underline{v}_0}{c} \wedge \underline{E} \end{aligned}\right\} \qquad \text{(AIII.6)}$$

where $\parallel$ and $\perp$ mean parallel and perpendicular to $\underline{v}_0$ (see reference [29] ; $\gamma = 1$ here)

Now, since $\underline{v}_0 = \dfrac{c(\underline{E}_0 \wedge \underline{B}_0)}{B_0{}^2}$, and we are interested in the effect of the transformation on $\underline{E}_0$ and $\underline{B}_0$, we take $\underline{E}_{\parallel} = 0 = \underline{B}_{\parallel}$. Then $\underline{E}_{\perp} = \underline{E}_0$ and $\underline{B}_{\perp} = \underline{B}_0$. Substituting these values in (AIII.6) and noting that $\underline{E}_0 \cdot \underline{B}_0 = 0$, the following relations hold:-

$$\underline{E}'_{\parallel} = 0 \qquad \underline{B}'_{\parallel} = 0$$

$$\underline{E}'_{\perp} = (\underline{E}_0 \cdot \underline{B}_0)\underline{B}_0 = 0$$

$$\underline{B}'_{\perp} = \left(1 - \frac{E_0{}^2}{B_0{}^2}\right)\underline{B}_0 + \frac{\underline{E}_0 \cdot \underline{B}_0}{B_0{}^2}\,\underline{E}_0$$

$$= \left[1 - \left(\frac{E_0}{B_0}\right)^2 \right]\underline{B}_0$$

Now $\qquad \underline{v}_0 = \dfrac{c(\underline{E}_0 \wedge \underline{B}_0)}{B_0{}^2}$

Therefore $\frac{v_0}{c} = \frac{|\underline{E}_0 \wedge \underline{B}_0|}{B_0{}^2}$

$$= \frac{E_0}{B_0}$$

and $\underline{B}'_\perp = \left(1 - \left(\frac{v_0}{c}\right)^2 \right)\underline{B}_0$

In our non-relativistic approximation $\frac{v_0}{c} << 1$, implying that $\underline{B}'_\perp \approx \underline{B}_0$

Hence the zero-order fields in the frame travelling with velocity $\underline{v}_0$ relative to the original frame are

$$\underline{E}' = 0 \quad \text{and} \quad \underline{B}' \approx \underline{B}_0$$

In this transformed frame (AIII.5) becomes

$$\frac{\partial f_0{}'}{\partial t} + \underline{v} \cdot \frac{\partial f_0{}'}{\partial \underline{r}} + \frac{q}{mc} (\underline{v} \wedge \underline{B}_0) \cdot \frac{\partial f_0{}'}{\partial \underline{v}} \approx 0$$

where $f_0{}'$ is independent of $\underline{v}_0$.

(c) Equation (I.26) gives the following expression for $\frac{\partial f_0}{\partial v^\mu}$:-

$$\frac{\partial f_0}{\partial v^\mu} = -\left\{2w_\mu + \frac{i}{\Omega}\gamma_\nu e^{\nu.o}_{\ \ \mu} + \frac{2i}{\Omega} e^{\nu\rho o} v_\rho (\delta^{\perp}_{\nu} w^{\perp}_{\mu} + \delta^{\parallel}_{\nu} w^{\parallel}_{\mu} - \gamma_\nu w_\mu)\right\} f_M$$

The term in w_μ alone [term (1), say] is the term that would occur if no gradients existed in the plasma. We would like to compare the term $\frac{i}{\Omega}\gamma_\nu e^{\nu.o}_{\ \ \mu}$ [term (2)] with term (1), and with

$\frac{2i}{\Omega} e^{\nu\rho o} v_\rho (\delta^{\perp}_{\nu} w^{\perp}_{\mu} + \delta^{\parallel}_{\nu} w^{\parallel}_{\mu} - \gamma_\nu w_\mu)$ [term (3)] in the hope that some simplification is possible. However, terms (2) and (3) as they stand are very similar, and there is no obvious way of comparing their magnitudes in a meaningful sense. We must therefore resort to examining all three terms after integration along unperturbed trajectories has been carried out. This integration is a fourfold one involving $v_\perp$, $v_\parallel$, ϕ and t.

There are no gradients in the $v_\parallel$-direction so that the $v_\parallel$ integration does not involve terms (2) and (3). The ϕ-integration is dimensionless, and $\frac{\partial f_0}{\partial v_\mu}$ is independent of t, so that the ϕ-integral gives no contribution to the dimensional properties of the full integral, while the term involving $\frac{\partial f_0}{\partial v^\mu}$ may be extracted from the t-integral.

Thus we need only investigate the $v_\perp$-integral.

Suppose we represent terms (1), (2) and (3) by the following simplified forms containing the essential properties of the respective terms:-

$$(1) \longrightarrow T_1 = \frac{v_\perp}{v_T^{\ 2}}$$

$$(2) \longrightarrow T_2 = \frac{\tau_1}{\Omega}$$

$$(3) \longrightarrow T_3 = \frac{v_\perp}{\Omega} \cdot \frac{\tau_2\, v_\perp}{v_T^{\ 2}}$$

where τ_1 and τ_2 represent the appropriate parameter gradients. Now the $v_\perp$-integration takes the following form (see Appendix II):-

$$\int_{v_\perp=0}^{\infty} v_\perp^{\,n}\, e^{-a_\perp v_\perp^2}\, B_q\left(\frac{k_\perp v_\perp}{\Omega}\right) . T_q \, dv_\perp$$

where T_q $(q = 1, 2, 3)$ is one of the terms T_1, T_2, T_3.

$B_q \left(\frac{k_\perp v_\perp}{\Omega}\right)$ is the appropriate dimensionless combination of Bessel functions associated with T_q

$v_\perp^{\,n}$ is whatever power of $v_\perp$ is necessary for the integral.

We now change the variable, setting $\zeta = \frac{k_\perp v_\perp}{\Omega}$. The resulting integrals are as follows:-

T_1 integral :-

$$\frac{1}{k_\perp^2 \rho^2} \cdot \frac{\Omega^n}{k_\perp^{\,n}} \int_{\zeta=0}^{\infty} \zeta^{n+1} \exp\left\{-\frac{\zeta^2}{2k_\perp^2\rho^2}\right\} B_1(\zeta)\, d\zeta$$

where $\rho = \left[\frac{1}{2a_\perp \Omega^2}\right]^{\frac{1}{2}}$ is the Larmor radius for the particle species concerned.

T_2 integral :-

$$\frac{\tau_1}{k_\perp} \cdot \frac{\Omega^n}{k_\perp^{\,n}} \int_{\zeta=0}^{\infty} \zeta^{n} \exp\left\{-\frac{\zeta^2}{2k_\perp^2\rho^2}\right\} B_2(\zeta)\, d\zeta$$

T_3 integral :-

$$\frac{\tau_2}{k_\perp} \cdot \frac{1}{k_\perp^2\rho^2} \frac{\Omega^n}{k_\perp^{\,n}} \int_{\zeta=0}^{\infty} \zeta^{n+2} \exp\left\{-\frac{\zeta^2}{2k_\perp^2\rho^2}\right\} B_3(\zeta)\, d\zeta$$

Since the Bessel function integrals are independent of plasma parameters apart from the factor $\exp\left\{-\frac{\zeta^2}{2k_\perp^2\rho^2}\right\}$ which is common to all of them, we assume that they have the same order of magnitude, so that the magnitudes of the terms investigated depend on the coefficients of the integrals.

We order the terms in the following way:-

Term (1)	:	Term (2)	:	Term (3)
1	:	$\frac{\tau_1}{k_\perp} k_\perp^2\rho^2$	:	$\frac{\tau_2}{k_\perp}$

The condition that the local approximation (as used to derive (I.26)) holds is that $\left(\frac{\tau_1}{k_\perp}, \frac{\tau_2}{k_\perp}\right) << 1$. We must therefore examine different regimes of $k_\perp\rho$.

(a) $\underline{k_\perp\rho > 1}$:- in this regime term (2) is the dominant gradient term, and term (3) may be neglected when compared with terms (1) and (2).

(b) $\underline{k_\perp\rho \lesssim 1}$:- here the validity of the local approximation is only certain if we take small enough gradients that $\frac{\tau_1}{k_\perp}, \frac{\tau_2}{k_\perp} << 1$; thus as we decrease $k_\perp$, we must consider smaller and smaller gradients in order that the local approximation is valid.

Suppose we neglect term (3), and approximate $\frac{\partial f_0}{\partial v^\mu}$ by

$$\frac{\partial f_0}{\partial v^\mu} = - \left\{ 2w_\mu + \frac{i}{\Omega}\gamma_\nu e^\nu{}_{\mu}{}^{o} \right\} f_M$$

This is a good approximation in region (a); in region (b) we must consider small gradients anyway, in order that the local approximation holds. Thus the above equation is a good approximation in region (b) within the limits of the local approximation.

(d) The introduction of temperature and density gradients into a magnetized plasma results in particle drifts perpendicular to the magnetic field $\underline{B}$ and to the gradient directions (these drifts may be evaluated by performing the integral $\frac{1}{n_0}\int \underline{v}\, f_0\, \underline{dv}$, where f_0 is given by (I.24)). If $\underline{v}_d$ is the net drift velocity, there is an equilibrium current given by

$$\underline{j}_0 = q\, n_0\, \underline{v}_d$$

The Maxwell equation involving current is

$$\underline{\nabla} \wedge \underline{B} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c}\underline{j}$$

Therefore in the equilibrium situation we have

$$\underline{j}_0 = \frac{c}{4\pi}\,\underline{\nabla} \wedge \underline{B}$$

or

$$\underline{v}_d = \frac{c}{4\pi\, q\, n_0}\,\underline{\nabla} \wedge \underline{B} \qquad \text{(AIII.7)}$$

This implies that $\underline{\nabla} \wedge \underline{B}$ is non-zero for non-zero net drift; that is $\underline{B}$ must have an $\underline{r}$-dependence.

Suppose we choose a linear $\underline{r}$-dependence for $\underline{B}$, say

$$\underline{B} = \underline{B}_0\,(1 + \varepsilon_i{}'\, r_i)$$

where $\underline{B}_0 = B_0\,\hat{\underline{z}}$ (with B_0 constant)

$\varepsilon_i{}'$ is the Cartesian gradient vector $(\varepsilon_1{}', \varepsilon_2{}', \varepsilon_3{}')$

$\varepsilon_i{}'\, r_i$ is a Cartesian scalar product.

Then $\underline{\nabla} \wedge \underline{B} = B_0(\varepsilon_2{}', -\varepsilon_1{}', 0)$

Thus no magnetic field gradient is necessary in the field direction, since all drifts are perpendicular to this direction.

Using a representative magnetic field gradient ε', the x and y components of (AIII.7) have the form

$$v_d = \frac{c\varepsilon'\, B_0}{4\pi\, q\, n_0} \qquad \text{(AIII.8)}$$

The $\underline{\nabla}B$ drift is given by

$$v_B = \frac{\varepsilon' v_\perp^2}{\Omega}$$

(see reference [30])

We define an average $\underline{\nabla}B$ drift as follows:-

$$\bar{v}_B = \frac{1}{n_0} \int_{\underline{v}} v_B f_0 \, \underline{dv}$$

$$= \frac{\varepsilon' (v_T)_\perp^2}{\Omega}$$

$$= \frac{\varepsilon' \kappa T_\perp}{m \Omega}$$

Thus (AIII.8) may be written as

$$v_d = \frac{2\bar{v}_B}{\beta_\perp} \qquad \text{(AIII.9)}$$

where $\beta_\perp = \dfrac{8\pi n_0 \kappa T_\perp}{B_0^2}$ is the ratio of perpendicular plasma pressure to magnetic field pressure.

We see from (AIII.9) that if we assume $\bar{v}_B$ to be small compared with v_d, this assumption is justified provided $\beta_\perp << 1$; so the condition for neglecting the effects of a magnetic field gradient in the integrations along particle orbits is that $\beta_\perp$ must be much less than unity.

To determine the types of waves described by our final dispersion relation and the possible couplings between them, we must remember that we did not invoke the electrostatic approximation in deriving the conductivity tensor. That is, it was not assumed that $\underline{E}_1$ could be replaced by $- \underline{\nabla}\phi$ where ϕ is a scalar potential. This means that the dispersion relation describes both longitudinal waves ($\underline{E}_1$ parallel to $\underline{k}$) and transverse waves ($\underline{E}_1$ perpendicular to $\underline{k}$).

Lashmore-Davies [31] and Callen and Guest [32] have shown that the necessary condition for non-negligible coupling between longitudinal and transverse waves is that $\omega_{pe} \gtrsim ck$, where $\omega_{pe} = \left(\frac{4\pi n_0 e^2}{m_e} \right)^{\frac{1}{2}}$ is the electron plasma frequency.

Now

$$\frac{\omega_{pe}^2}{c^2 k^2} = \frac{4\pi n_0 e^2}{m_e c^2} \cdot \frac{1}{(k\rho_e)^2} \cdot \frac{(v_\perp)_e^2}{\Omega_e^2}$$

$$= \frac{1}{2} \frac{8\pi n_0 \kappa (T_\perp)_e}{B_0^2} \cdot \frac{1}{(k\rho_e)^2}$$

That is

$$\frac{\omega_{pe}^2}{c^2 k^2} = \frac{1}{2} \frac{(\beta_\perp)_e}{(k\rho_e)^2} \qquad \text{(AIII.10)}$$

where $\rho_e = \frac{(v_\perp)_e}{\Omega_e}$, $(v_\perp)_e = \left(\frac{\kappa(T_\perp)_e}{m_e} \right)^{\frac{1}{2}}$

$(T_\perp)_e$ is the perpendicular electron temperature

and $\Omega_e = \frac{eB_0}{mc}$

In order that magnetic field gradients may be neglected, we must have $(\beta_\perp)_e << 1$. Also, for density and temperature gradient effects to be significant, $k_\perp \rho_e$ must be > 1, which implies $k\rho_e > 1$. Thus from (AIII.10) we have

$$\frac{\omega_{pe}^2}{c^2 k^2} << 1$$

$$\text{or} \quad \frac{\omega_{pe}}{c\, k} < 1$$

In the region $k\rho_e < 1$, non-negligible coupling is possible, but the temperature and density gradient effects are no longer significantly large within the local approximation. The situation is as follows:-

$(\beta_\perp)_e \ll 1$ and $k\rho_e < 1$	$(\beta_\perp)_e \ll 1$ and $k_\perp\rho_e > 1$
Possible non-negligible coupling, but insignificant gradient effects within the local approximation.	Significant gradient effects within the local approximation, but negligible coupling; the dispersion relation separates into independent electrostatic and electromagnetic modes.

APPENDIX IV

In this Appendix we apply to the integrals in (II.15) a similar approximation process to that given in Chapter 2 of Section II.

Let $\tau = \Omega t$ and $\nu' = \frac{\nu}{\Omega}$ where ν' is small compared with unity.

Also let $\omega^* = \frac{\omega}{\Omega} \pm s$ where $s = 0, 1, 2$

Now define the general dimensionless "Gordeyev" integral

$$G_M = \int_0^{\infty} \exp\left\{ -r\nu'\tau - \Phi(\tau) + i\omega^*\tau \right\} d\tau \qquad \text{(AIV.1)}$$

where $r = 1, 2, 3, 4$

(r and s take the values required to give any of the integrals in (II.15)).

From (II.14)

$$\Phi(t) = \frac{1}{v_T^{\,2}} \cdot \frac{k^2}{\nu^2 + \Omega^2} \left[\cos\chi + \nu t - e^{-\nu t} \cos(\Omega t - \chi) \right]$$

Setting $\tau = \Omega t$ and defining $\rho = \frac{v_T}{\Omega}$ to be the Larmor radius this gives

$$\Phi(\tau) = \frac{k^2\rho^2}{1+(\nu')^2} \left[\cos\chi + \nu'\tau - e^{-\nu'\tau} \cos(\tau - \chi) \right]$$

Our basic assumption in this case is that when $k^2\rho^2 >> 1$, the integrand in G_M contributes significantly to the integral only in regions where

$$f(\tau) = \cos\chi + \nu'\tau - e^{-\nu'\tau}\cos(\tau - \chi) \approx 0$$

The equivalent function in Chapter 2 was $1 - \cos\tau$, and it can be seen that for small ν' and small τ, $f(\tau)$ is close to $1 - \cos\tau$. As τ tends to infinity, $f(\tau)$ tends to the straight line $\cos\chi + \nu'\tau$. The general form of $f(\tau)$ is shown in Figure (3b). Note that $f(\tau)$ has only one zero for $\tau \geq 0$, at $\tau = o$, whereas $1 - \cos\tau$ has an infinite number of zeros at $\tau = 2m\pi$ ($m = 0, 1, 2, \ldots$). However, $f(\tau)$ has minima at $\tau \simeq 2m\pi$ ($m = 0, 1, 2, \ldots$) with only the $m = 0$ minimum actually touching the τ-axis. Computed results for τ_m($m = 1, 2, \ldots$), the values of τ at the minima of $f(\tau)$, indicate that τ_m does not deviate from $2\pi m$ by more than one part in 10^4 until m is about 2,000.

As in the case with no collisions, G_M may be represented by an infinite sum of integrals over small domains of significance. However, as τ increases the minima of $f(\tau)$ occur at steadily increasing values of $f(\tau)$. Qualitatively, we would expect a large contribution to the integral G_M to occur in the region near $\tau = 0$, and any other significant contributions to come from the regions around a finite number of successive minima. The number of these minima required depends on the values of ν' and $k\rho$; we would like as small a number of significant minima as possible consistent with small ν' and large $k\rho$, in order to simplify the final result. A lower bound on the value of ν' in terms of $k\rho$ can be derived as follows:-

$$
\begin{aligned}
\text{Define} \quad f^*(\tau) &= f(\tau) - \nu'\tau \\
&= \cos\chi - e^{-\nu'\tau}\cos(\tau - \chi) \\
&= \cos\chi - e^{-\nu'\tau}(\cos\tau\cos\chi + \sin\tau\sin\chi)
\end{aligned}
$$

For small ν', $\cos\chi \simeq 1$ and $\sin\chi \ll 1$.

Thus $$f^*(\tau) \simeq 1 - e^{-\nu'\tau}\cos\tau$$

and minimum values of this occur when $\cos\tau \simeq 1$.

Now $1 - e^{-\nu'\tau} > 0$ for $\tau \geq \theta$ where θ is some number greater than zero, implying that $f^*(\tau) > 0$ for $\tau \geq \theta$ and thus

$$f(\tau) > \nu'\tau \text{ for } \tau \geq \theta \tag{AIV.2}$$

$$\text{Define } g(\tau) = \exp\left\{ i\omega^*\tau - r\nu'\tau - \frac{k^2\rho^2}{1+(\nu')^2} f(\tau) \right\} \tag{AIV.3}$$

$$\text{so that} \quad G_M = \int_0^\infty d\tau\, g(\tau)$$

If we replace $f(\tau)$ by $\nu'\tau$ in (AIV.3), the inequality (AIV.2) enables us to state the following:-

$$
\begin{aligned}
J &= \int_\theta^\infty d\tau \exp\left\{ i\omega^*\tau - r\nu'\tau - \frac{k^2\rho^2}{1+(\nu')^2}\nu'\tau \right\} \\
&> \int_\theta^\infty d\tau\, g(\tau) = I_\theta \qquad \text{(say)}
\end{aligned}
$$

Define $$K = \int_{\theta}^{\infty} d\tau \ \exp\left\{ - \frac{k^2\rho^2}{1+(\nu')^2} \nu'\tau \right\}$$

$$\simeq \frac{1}{k^2\rho^2\nu'} \exp\left\{ - k^2\rho^2\nu'\theta \right\} \text{ for } 0 < \nu' << 1$$

Now $$K \gtrsim \max\left[\mathrm{Re}(J), \mathrm{Im}(J) \right]$$

$$> \max\left[\mathrm{Re}(I_\theta), \mathrm{Im}(I_\theta) \right]$$

Thus G_M may now be approximated by the finite integral

$$L = \int_{o}^{\theta} d\tau \ g(\tau)$$

provided $K << \min\left[\mathrm{Re}(L), \mathrm{Im}(L) \right]$ (AIV.4)

Obviously if $\nu' << (k\rho)^{-2}$, a large value of θ might be necessary to ensure that (AIV.4) would be satisfied for all possible L. Thus to make certain that only a small number of minima need be included, we choose the following lower bound for ν'

$$\nu' \gtrsim (k\rho)^{-2} \qquad \text{(AIV.5)}$$

[Note that in Section III we choose $\nu' < (k\rho)^{-1}$ in order to satisfy the condition $\nu' << 1$. A definite upper bound on ν' is necessary there so that we may have a clear ordering of the expansion terms]

We have

$$G_M \simeq \int_{o}^{\theta} d\tau \ g(\tau) \qquad \text{(AIV.6)}$$

In a similar way to that given in Chapter 2 of Section II, we define a domain of significance δ_m around the m-th minimum τ_m. The integral G_M may now be written as

$$G_M \simeq I_0 + \sum_{m=1}^{n} I_m$$

where $$I_0 = \int_{o}^{\delta_o} d\tau \ g(\tau)$$

and $$I_m = \int_{\tau_m-\delta_m}^{\tau_m+\delta_m} d\tau \ g(\tau)$$

In this case $\theta = \tau_n + \delta_n$, where n is an integer ≥ 1.

Within the m-th domain of significance we make the change of variable

$$\tau = \tau_m + \phi \qquad (m = 0, 1, 2, \ldots.)$$

$$\text{where} \quad |\phi| << 1$$

Then

$$f(\tau) = \cos\chi + \nu'(\tau_m+\phi) - e^{-\nu'(\tau_m+\phi)}\cos(\tau_m + \phi - \chi)$$

$$\approx 1 - \frac{\chi^2}{2} + \nu'(\tau_m+\phi) - e^{-\nu'\tau_m}(1 - \nu'\phi - \frac{\phi^2}{2} + \phi\chi - \frac{\chi^2}{2}) \qquad \text{(AIV.7)}$$

taking $\tau_m \simeq 2\pi m$ and neglecting terms involving products of greater than second order in ϕ, χ, ν'.

For the case $m = 0$, $\tau_0 = 0$ and

$$f(\phi) \simeq \frac{\phi^2}{2} - [\ \chi - 2\nu'\ ]\phi$$

Now $\nu' = \tan\frac{\chi}{2} \simeq \frac{\chi}{2} - \frac{\chi^3}{6}$

Thus $\chi - 2\nu' \simeq -\frac{\chi^3}{3}$

and $f(\phi) \simeq \frac{\phi^2}{2} + \frac{\chi^3\phi}{3}$.

If we now neglect terms of second order in ν', χ and ϕ compared with unity, we get

$$f(\phi) \simeq \frac{\phi^2}{2}$$

The expression $g(\tau)$ becomes $g(\phi)$ where

$$g(\phi) \simeq \exp\left\{ i\omega^*\phi - r\nu'\phi - k^2\rho^2\frac{\phi^2}{2} \right\}$$

$$\simeq \exp\left\{ i\omega^*\phi - k^2\rho^2\frac{\phi^2}{2} \right\}$$

neglecting $\nu'\phi$ compared with unity; we retain the term in ϕ^2 since $k^2\rho^2\phi^2 >> \phi^2$.

Thus

$$I_0 \simeq \int_0^{\delta_0} d\phi \exp\{i\omega^*\phi - \tfrac{1}{2}k^2\rho^2\phi^2\} \qquad \text{(AIV.8)}$$

Considering now the case $m \neq 0$, if we replace the factor $e^{-\nu' \tau_m}$ in (AIV.7) by its maximum value, unity, and carry out the same procedure used to derive (AIV.8), the following can be stated:-

$$I_m < \exp\{-(k^2\rho^2+r)\nu'\tau_m\} \int_{-\delta_m}^{\delta_m} d\phi \exp\{i\omega^*\phi - \tfrac{1}{2}k^2\rho^2\phi^2\} \qquad \text{(AIV.9)}$$

Since the integrand of the integral in (AIV.9) is the same as that in (AIV.8), δ_0 and δ_m must be of the same order. Therefore the integrals in (AIV.8) and (AIV.9) must also be of the same order.

Thus the ratio $\frac{I_m}{I_0} \sim e^{-[k^2\rho^2+r]\nu'\tau_m}$

$$\text{That is} \quad \frac{I_m}{I_0} \sim e^{-\tau_m} \approx (0.002)^m \qquad \text{(AIV.10)}$$

if we take the lower bound value for ν', namely $\nu' \sim (k\rho)^{-2}$ [remember $r \ll k^2\rho^2$], and take $\tau_m \approx 2\pi m$.

The approximation (AIV.10) suggests that the term I_0 is the only significant contribution to G_M, and the computed results given in Table (2) for $\nu' = 0.001$, $\omega^* = 0.1$ and $r = 1$ support this. Similar results were obtained for $\nu' = 0.01$ and 0.0001.

Therefore we have

$$G_M \approx \int_0^{\delta_0} d\phi \exp\{i\omega^*\phi - \tfrac{1}{2}k^2\rho^2\phi^2\} \quad . \qquad \text{(AIV.11)}$$

Since δ_0 is a domain of significance for a less convergent integrand than the one in (AIV.11), the upper limit may be replaced by infinity, so that

$$G_M \approx \int_0^{\infty} d\phi \exp\{i\omega^*\phi - \tfrac{1}{2}k^2\rho^2\phi^2\}$$

$$= -\frac{i}{\sqrt{2}k\rho} Z\left(\frac{\omega^*}{\sqrt{2}k\rho}\right) \qquad \text{(AIV.12)}$$

as in Chapter 2 of Section II.

The computed results for $\nu' = 0.001$, $\omega' = 0.1$ and $r = 1$ given in Table (3) support the approximation given in (AIV.12). We may note that the values $\nu' = 0.0001$ and 0.01 were tried, with $\omega^* = 0.1, 2.1$ and with $r = 4$; this did not affect the accuracy of the approximation significantly. In Table (3), we begin with the lower bound value of $k^2\rho^2$ for the given ν', and as is expected the approximation is more accurate the larger the value of $k^2\rho^2$.

{NOTE :- All computed results mentioned in this Appendix were obtained using the full integrand in (AIV.1). The integrals were evaluated using Simpson's Rule with step length ~ 0.0003. It was found that $\delta_m \sim 0.1$, and that the values of the portions of the integrals between $\tau_m + \delta_m$ and $\tau_{m+1} - \delta_{m+1}$ were so small as to cause computer underflow, implying that the values were $< 10^{-75}$, verifying our initial assumption. The value $\frac{\omega}{\Omega} = 0.1$ was used as a test value since the results of Section III give an ion acoustic wave frequency of this order. The Fried-Conte function was evaluated using a routine given by Ferguson [33] }.

TABLE 1. VARIATION IN v_p/v_i WITH $\sqrt{2}k/k_e$ FOR VARIOUS VALUES OF T_i/T_e AND v_T/v_e

$\frac{\sqrt{2}k}{k_e}$	$\left(\frac{v_p}{v_i}\right)_{\{1\}}$	$\left(\frac{v_p}{v_i}\right)_{\{2\}}$	$\left(\frac{v_p}{v_i}\right)_{\{3\}}$	$\left(\frac{v_p}{v_i}\right)_{\{4\}}$
0·065	4·83	2·67	1·04	0·62
0·6	4·52	2·58	1·22	0·83
1·0	4·13	2·45	1·48	1·29

TABLE 2. I_1 AND I_2 COMPARED WITH I_0 FOR DIFFERENT VALUES OF $(k\rho_e)^2$

$(k\rho_e)^2$	$\frac{\text{Re}\, I_1}{\text{Re}\, I_0}$	$\frac{\text{Im}\, I_1}{\text{Im}\, I_0}$	$\frac{\text{Re}\, I_2}{\text{Re}\, I_0}$	$\frac{\text{Im}\, I_2}{\text{Im}\, I_0}$
10^3	10^{-5}	10^{-3}	10^{-11}	10^{-8}
$2{\cdot}5 \times 10^3$	10^{-13}	10^{-11}	10^{-27}	10^{-24}
10^4	10^{-55}	10^{-52}	$<10^{-75}$	$<10^{-75}$

TABLE 3. COMPARISON OF I_0 AND $-(i/\sqrt{2}k\rho_e)Z(\omega^*/\sqrt{2}k\rho_e)$ FOR DIFFERENT VALUES OF $(k\rho_e)^2$

$(k\rho_e)^2$	I_0	$-(i/\sqrt{2}k\rho_e)\,Z(\omega^*/\sqrt{2}k\rho_e)$
10^3	$0{\cdot}396384 \times 10^{-1}$	$0{\cdot}396331 \times 10^{-1}$
	$+ 0{\cdot}100035 \times 10^{-3}\, i$	$+ 0{\cdot}999997 \times 10^{-3}\, i$
$2{\cdot}5 \times 10^3$	$0{\cdot}250676 \times 10^{-1}$	$0{\cdot}250662 \times 10^{-1}$
	$+ 0{\cdot}400057 \times 10^{-4}\, i$	$+ 0{\cdot}399999 \times 10^{-4}\, i$
10^4	$0{\cdot}125333 \times 10^{-1}$	$0{\cdot}125331 \times 10^{-1}$
	$+ 0{\cdot}100004 \times 10^{-4}\, i$	$+ 0{\cdot}100000 \times 10^{-4}\, i$

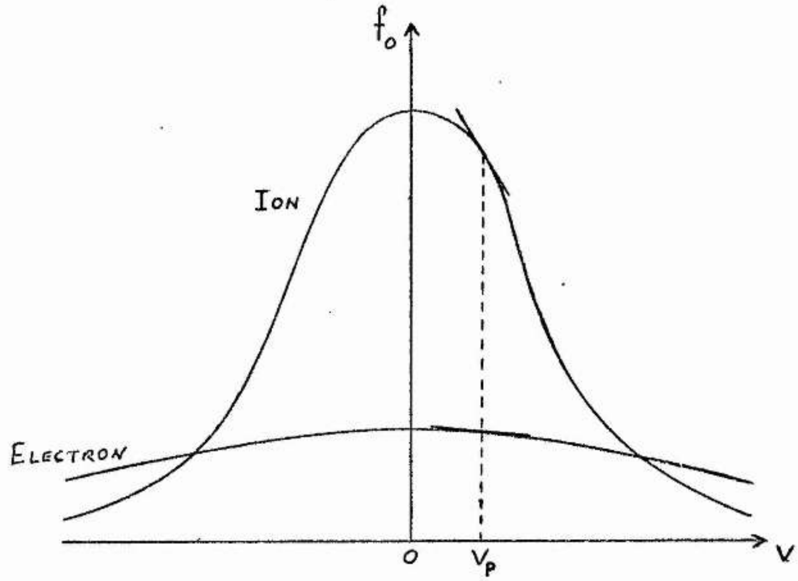

FIG. 1a :- Ion and electron equilibrium distribution functions.

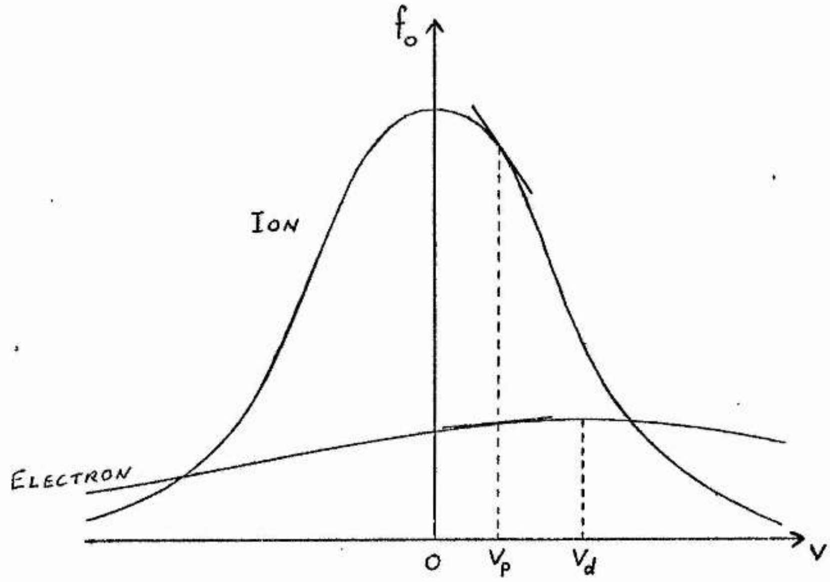

FIG. 1b :- Equilibrium distribution functions with electron net drift v_d.

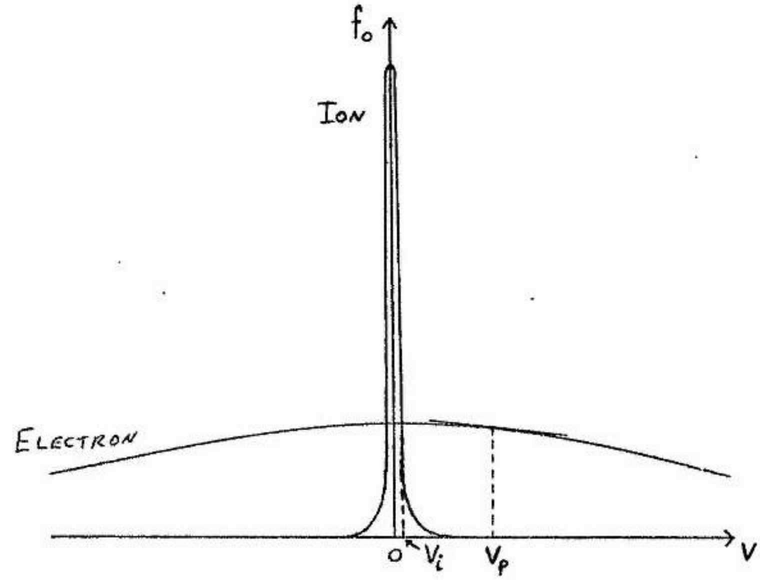

FIG. 2a :- Equilibrium distribution functions for $T_i \ll T_e$

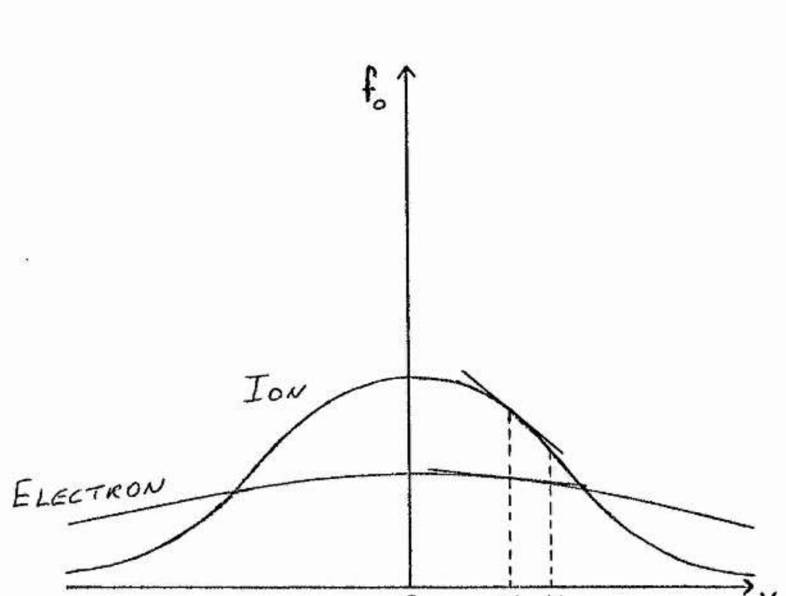

FIG. 2b :- Equilibrium distribution functions for $T_i \sim T_e$

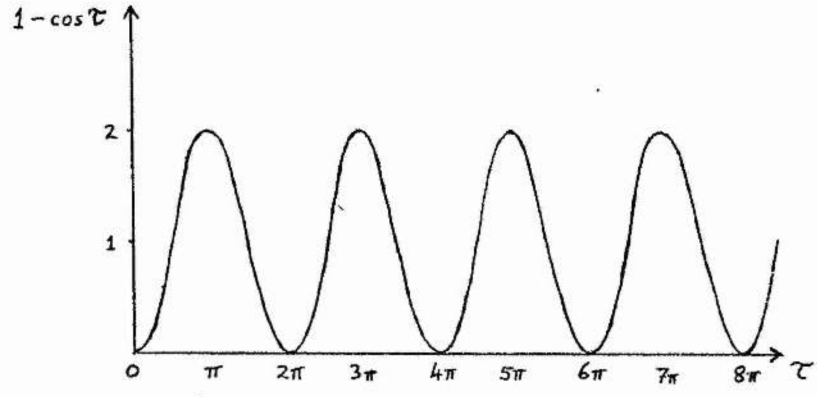

FIG. 3a :- General form of the function 1 - cos τ

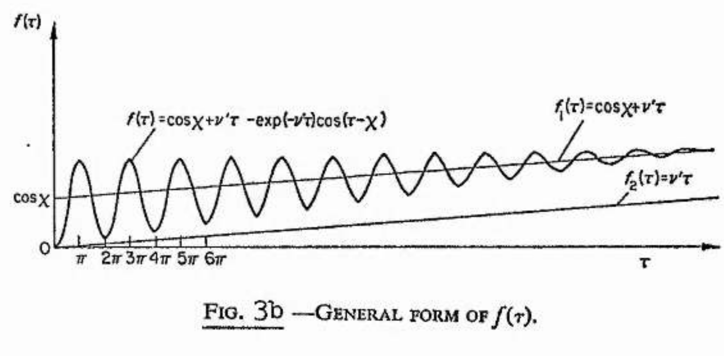

FIG. 3b —GENERAL FORM OF $f(\tau)$.

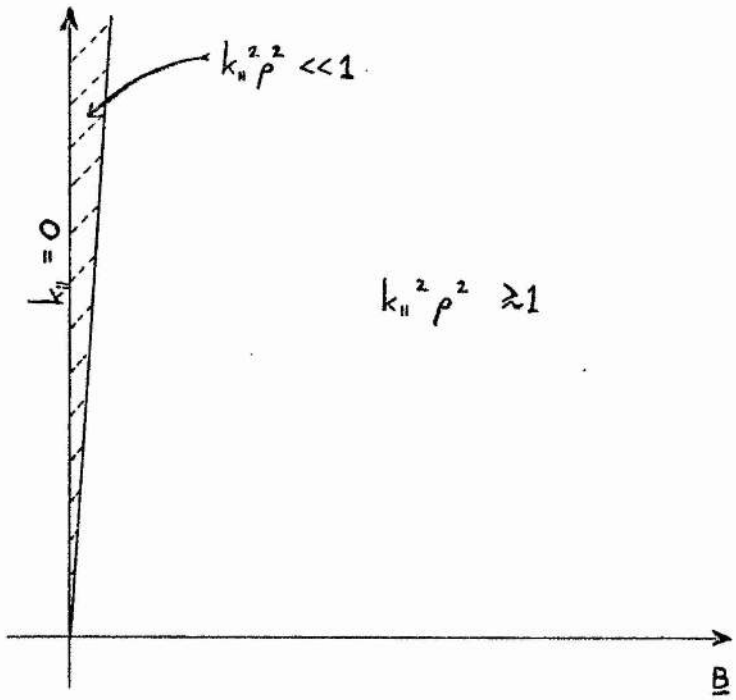

FIG. 4 :- Regions of Bernstein wave dominance ($k_\parallel=0$ and $k_\parallel^2\rho^2 << 1$) and of ion acoustic wave dominance ($k_\parallel^2\rho^2 \gtrsim 1$).

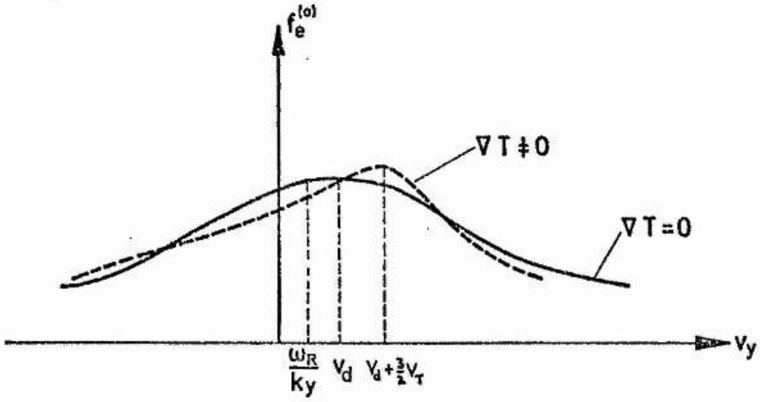

FIG. 5—Temperature gradient distortion of the electron distribution function.

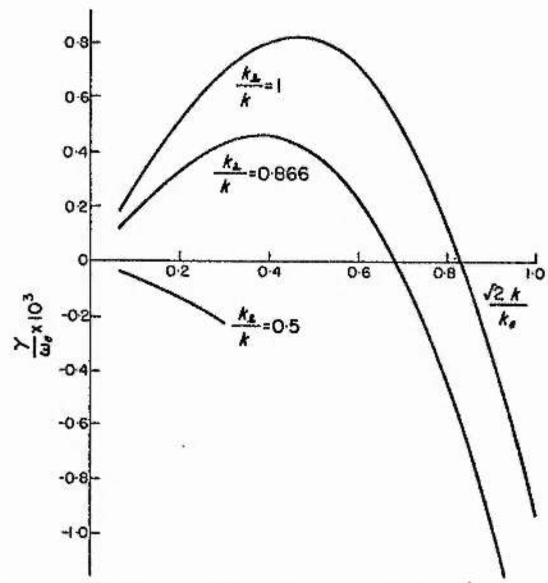

FIG. 6—γ/ω_e against $\sqrt{2}k/k_e$ for $v_T/v_e = 0{\cdot}5$; $v_d/v_e = 0$; $T_i/T_e = 0{\cdot}3$, and for $k_\perp/k =$ 0·5, 0·866, 1.

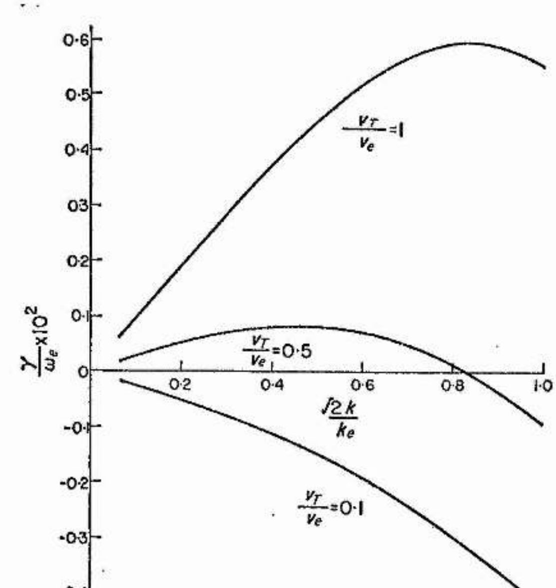

FIG. 7—γ/ω_e against $\sqrt{2}k/k_e$ for $v_d/v_e = 0$; $T_i/T_e = 0{\cdot}3$; $k_\perp/k = 1$, and for $v_T/v_e =$ 0·1, 0·5, 1.

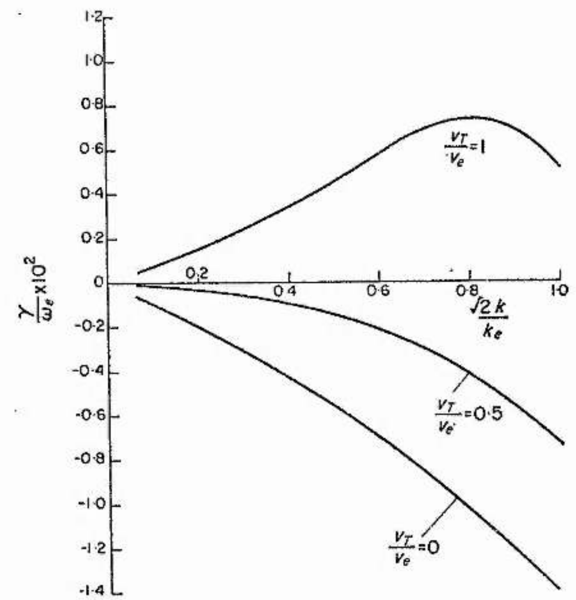

FIG. 8—γ/ω_e against $\sqrt{2}k/k_e$ for $v_d/v_e = 0{\cdot}5$; $T_i/T_e = 1$; $k_y/k = 1$, and for $v_T/v_e =$ 0, 0·5, 1.

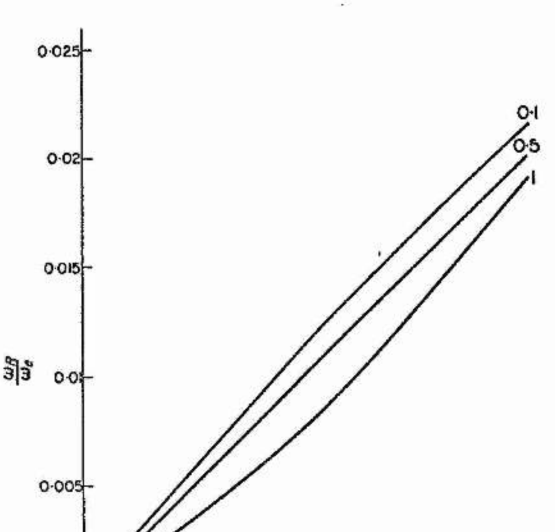

FIG. 9—ω_R/ω_e against $\sqrt{2}k/k_e$ for $v_d/v_e = 0$; $T_i/T_e = 0{\cdot}3$; $k_y/k = 1$, and for $v_T/v_e = 0{\cdot}1$, 0·5, 1.

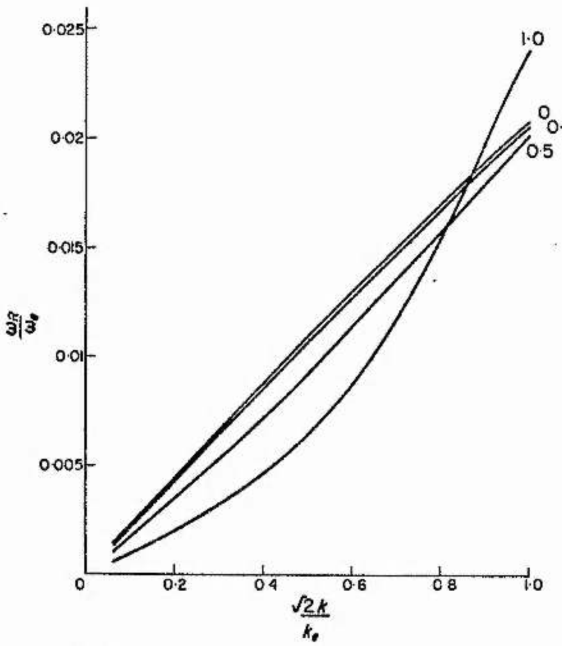

FIG.10—ω_R/ω_e against $\sqrt{2k/k_e}$ for $v_d/v_e = 0{\cdot}5$; $T_i/T_e = 0{\cdot}3$; $k_y/k = 1$, and for $v_T/v_e = 0$, 0·1, 0·5, 1.

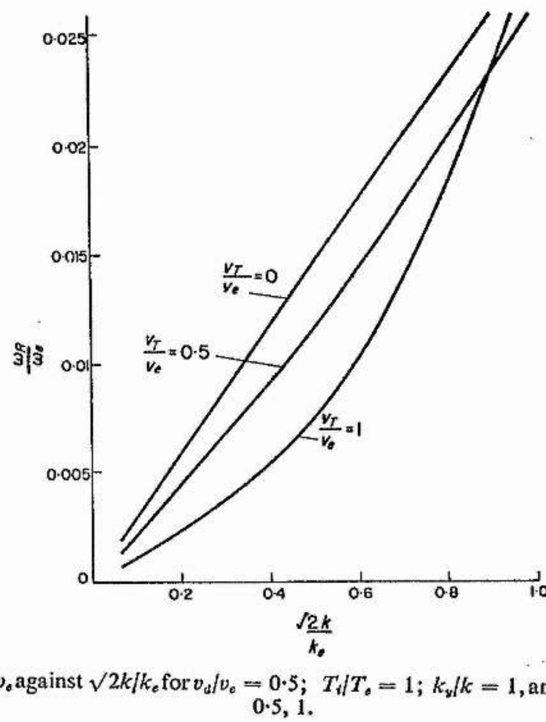

FIG.11—ω_R/ω_e against $\sqrt{2k/k_e}$ for $v_d/v_e = 0{\cdot}5$; $T_i/T_e = 1$; $k_y/k = 1$, and for $v_T/v_e = 0$, 0·5, 1.

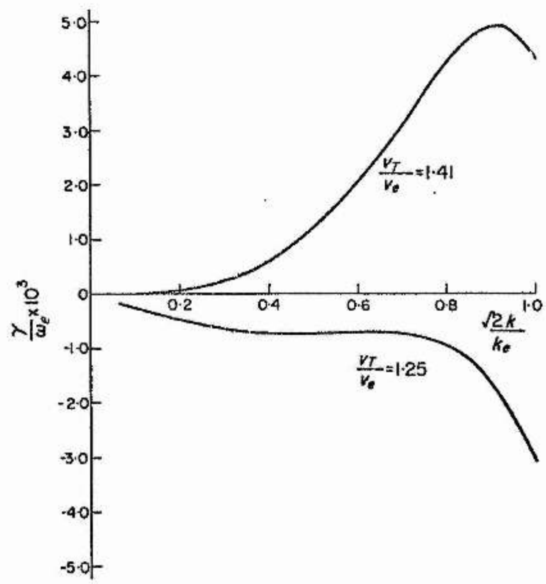

FIG. 12—γ/ω_e against $\sqrt{2}k/k_e$ for $v_d/v_e = 0$; $T_i/T_e = 1{\cdot}5$; $k_y/k = 1$, and for $v_T/v_e = 1{\cdot}25$, 1·41.

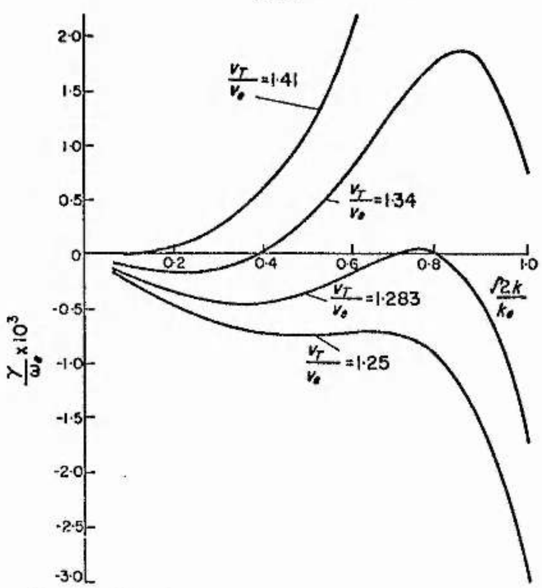

FIG. 13—γ/ω_e against $\sqrt{2}k/k_e$ for $v_d/v_e = 0$; $T_i/T_e = 1{\cdot}5$; $k_y/k = 1$, and for $v_T/v_e =$ 1·25, 1·283, 1·34, 1·41.

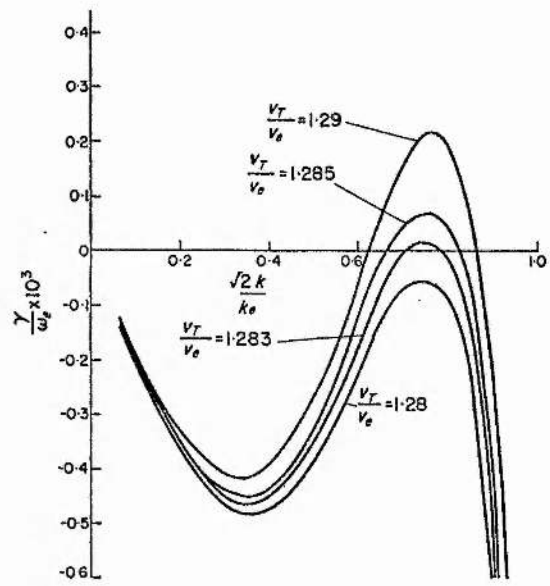

FIG. 14—γ/ω_e against $\sqrt{2}k/k_e$ for $v_d/v_e = 0$; $T_i/T_e = 1{\cdot}5$; $k_y/k = 1$, and for $v_T/v_e =$ 1·28, 1·283, 1·285, 1·29.

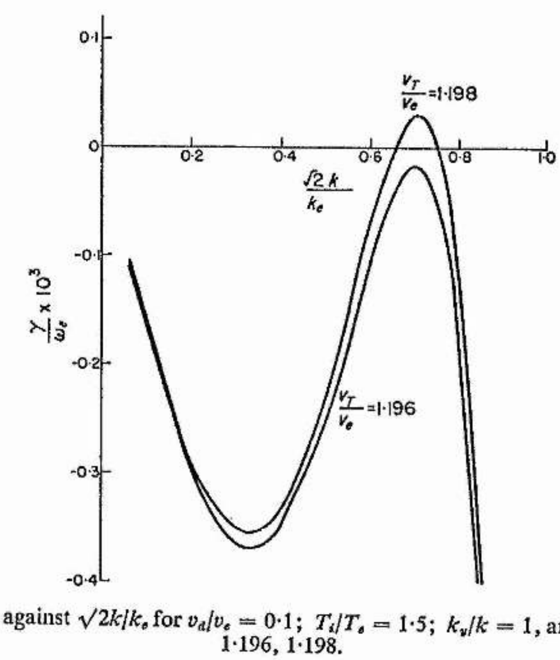

FIG. 15 —γ/ω_e against $\sqrt{2}k/k_e$ for $v_d/v_e = 0{\cdot}1$; $T_i/T_e = 1{\cdot}5$; $k_y/k = 1$, and for $v_T/v_e =$ 1·196, 1·198.

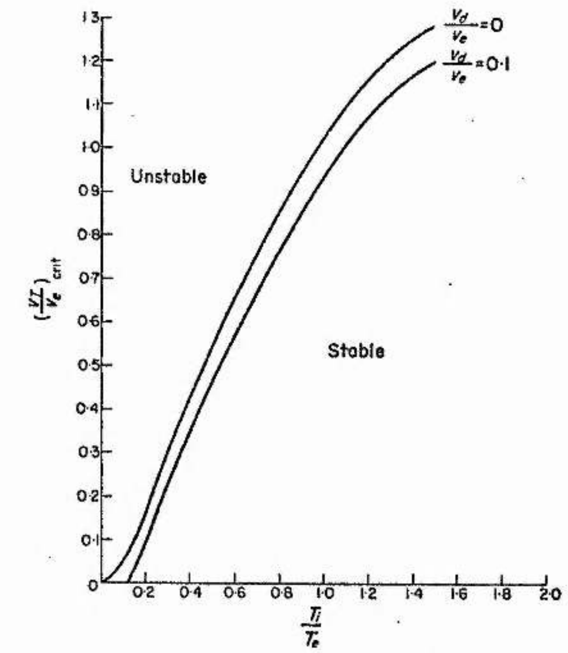

FIG. 16 —Critical value of v_T/v_e against T_i/T_e.

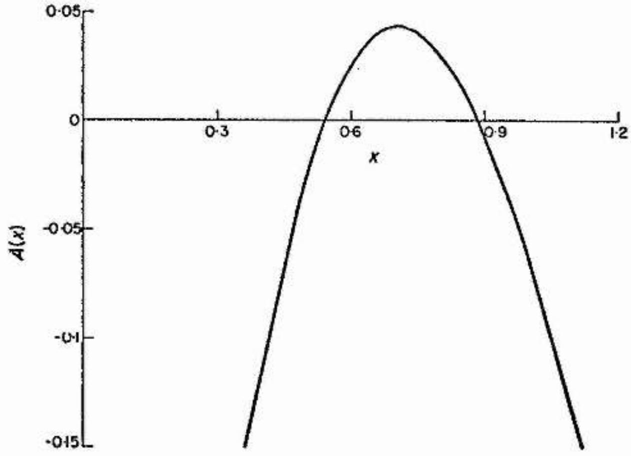

FIG 17 —$A(x)$ against x for $C = 0{\cdot}717$.

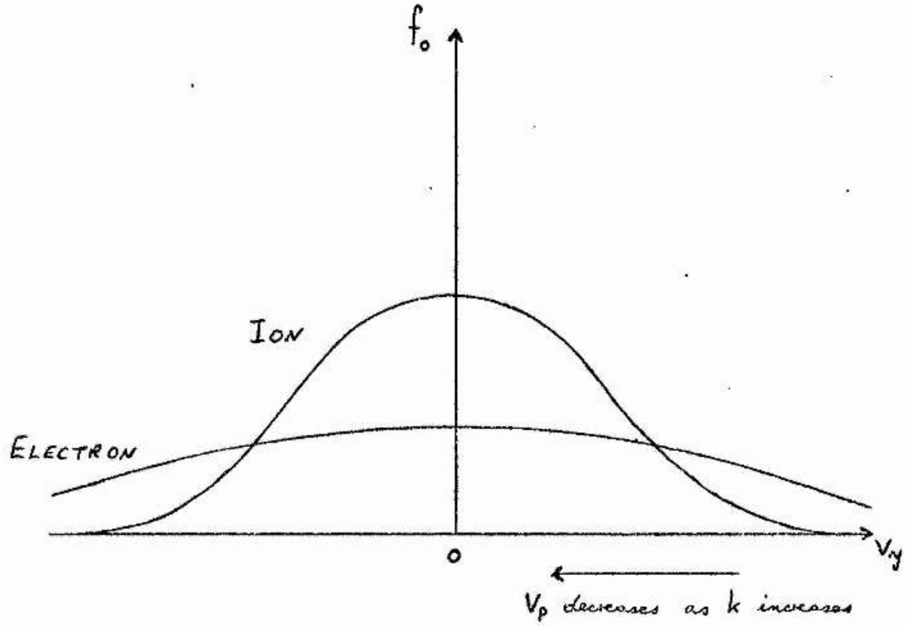

FIG. 18a :- Equilibrium distribution functions with $v_T = 0$

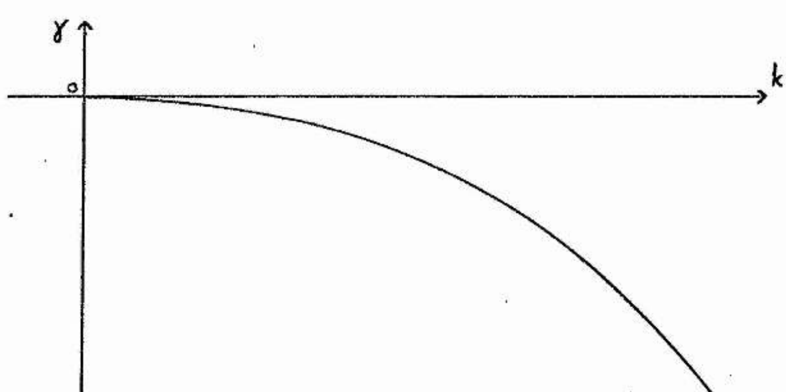

FIG. 18b :- γ against k curve for $v_T = 0$

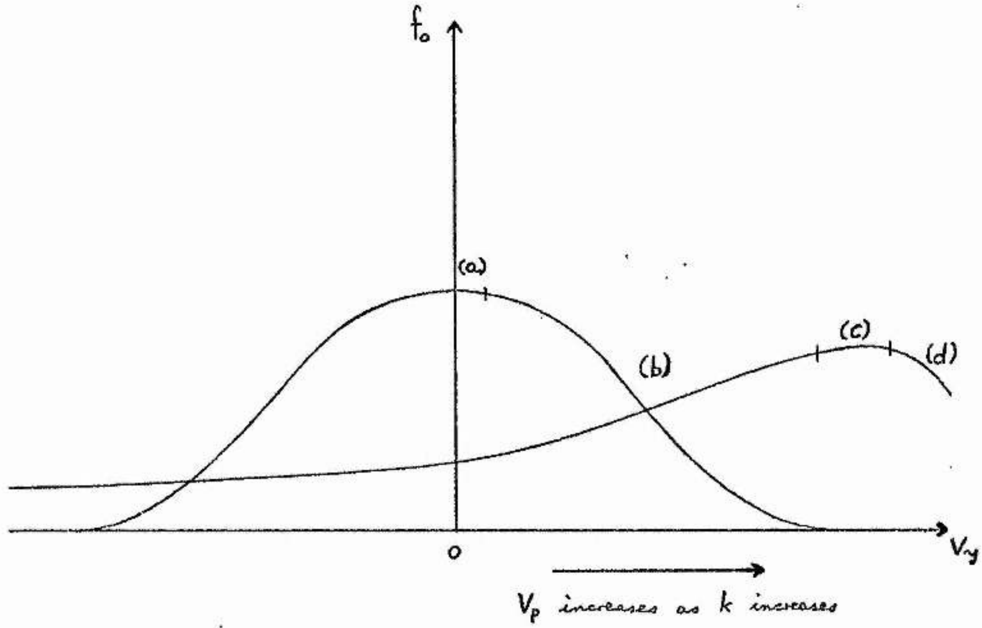

FIG. 19a :- Equilibrium distribution functions for $\frac{v_T}{v_e} \sim 1$

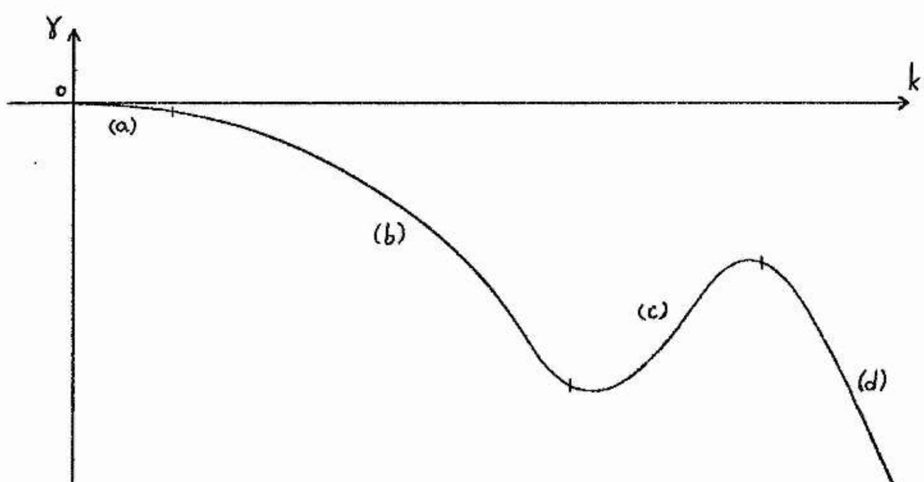

FIG. 19b :- γ against k curve for $\frac{v_T}{v_e} \sim 1$

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