Affine Rigidity and Conics at Infinity

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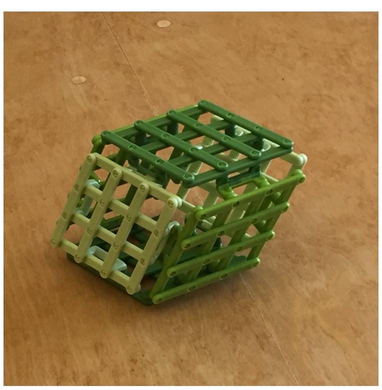
Abstract

We prove that if a framework of a graph is neighborhood affine rigid in d-dimensions (or has the stronger property of having an equilibrium stress matrix of rank $n-d-1$) then it has an affine flex (an affine, but non Euclidean, transform of space that preserves all of the edge lengths) if and only if the framework is ruled on a single quadric. This strengthens and also simplifies a related result by Alfakih. It also allows us to prove that the property of super stability is invariant with respect to projective transforms and also to the coning and slicing operations. Finally this allows us to unify some previous results on the Strong Arnold Property of matrices.

1 Introduction

Let G be a connected graph with n labeled vertices and m edges and $\mathbf{p}$ be a configuration of the n vertices in $\mathbb{E}^d$. Throughout, we will assume that $\mathbf{p}$ has a full d dimensional affine span. The pair $(G, \mathbf{p})$ is called a framework in $\mathbb{E}^d$. Two frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are called (Euclidean) equivalent if they share the same m lengths measured along the edges in G. Two frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are called (Euclidean) congruent if they are related through a d-dimensional Euclidean transform.

In rigidity theory, one may be interested in knowing if there is a second framework $(G, \mathbf{q})$ in $\mathbb{E}^d$ that is equivalent to, but not congruent to $(G, \mathbf{p})$. If there is no such $(G, \mathbf{q})$ in $\mathbb{E}^d$, we say that $(G, \mathbf{p})$ is *globally rigid* in $\mathbb{E}^d$. If there is no such $(G, \mathbf{q})$ in *any* dimension, we say that $(G, \mathbf{p})$ is *universally rigid*.



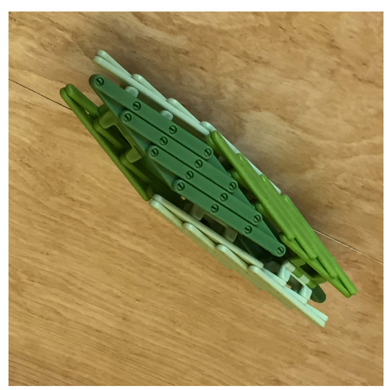


Figure 1: Two Euclidean equivalent "frameworks" related by an affine flex. The frameworks are not ruled, and they are not neigborhood affine rigid.

When trying to establish global or universal rigidity, we can often use a certificate, called an "equilibrium stress matrix", to rule out the existence of any equivalent framework to $(G, \mathbf{p})$ *except* *for those that arise through d-dimensional affine transforms* [5, 7]. Then an extra argument is needed to establish that any affine transform that preserves the m lengths of $(G, \mathbf{p})$ must actually be a Euclidean transform. This last step is equivalent to proving that the edge directions of $(G, \mathbf{p})$ do not lie on a "conic at infinity".

Figure 1 shows a case where there exists an affine motion applied to a framework that does preserve all of the "edge" lengths.

We will use the following conventions. Differences between pairs of points in $\mathbb{E}^d$ give vectors in a d-dimensional linear space, which we will identify with $\mathbb{R}^d$. Then after fixing an origin point in $\mathbb{E}^d$ we will also identify each point with its affine coordinates in $\mathbb{R}^d$.

Definition 1.1. *Let $(G, \mathbf{p})$ be a framework in $\mathbb{E}^d$. For any edge $\{ij\}$ in G, let its edge vector in $\mathbb{R}^d$ be $\mathbf{e}_{ij} := \mathbf{p}_j - \mathbf{p}_i$. We say that the edge directions of $(G, \mathbf{p})$ lie on a conic at infinity of $\mathbb{E}^d$ if there exists a non-zero symmetric $d \times d$ matrix $\mathbf{Q}$ such that for all of the edge vectors, we have $\mathbf{e}_{ij}^t \mathbf{Q} \mathbf{e}_{ij} = 0$.*

(This property does not depend on the choice affine coordinates for $\mathbb{E}^d$.)

In $\mathbb{E}^2$, the conic condition means that all of the edges are in, at most, two directions.

Remark 1.2. *The terminology "conic at infinity" comes from the following projective setup. Identify $\mathbb{E}^d$ with the affine patch of $\mathbb{P}^d$ that has coordinates $(\cdots : 1)$. For computations, we give a point $\mathbf{x} \in \mathbb{E}^d$ homogeneous coordinates $\hat{\mathbf{x}}$, which is the vector in $\mathbb{R}^{d+1}$ obtained by appending a 1 to $\mathbf{x}$. With this choice, difference vectors, such as the edge vectors $\mathbf{e}_{ij}$, have homogeneous coordinates (not identically zero) of the form $(\cdots : 0)$, so they correspond to the point at infinity on the line through $\mathbf{p}_i$ and $\mathbf{p}_j$. In this view, the edge directions of $(G, \mathbf{p})$ are on a conic at infinity if the $\mathbf{e}_{ij}$ are contained in a quadric that lies in the hyperplane at infinity.*

Definition 1.3. *Let A be an affine map A on $\mathbb{E}^d$, and define $A(\mathbf{p})$ by $A(\mathbf{p})_i := A(\mathbf{p}_i)$. An affine flex of a framework $(G, \mathbf{p})$ in $\mathbb{E}^d$ is an an affine map A such that $(G, A(\mathbf{p}))$ is equivalent to $(G, \mathbf{p})$. An affine flex is non-trivial if A is not a Euclidean motion.*

The importance of conics at infinity comes from their close connection to affine flexes of frameworks.

Proposition 1.4 ([6, 7])**.** *The edge directions of $(G, \mathbf{p})$ lie on a conic at infinity if and only if there is a non-trivial affine flex of $(G, \mathbf{p})$.*

Remark 1.5. *Connelly [8] has shown that if $(G, \mathbf{p})$ has a non-trivial affine flex A, then there is a continuous path A_t of non-trivial affine flexes. Thus, Proposition 1.4 implies that if the edges of $(G, \mathbf{p})$ lie on a conic at infinity, then $(G, \mathbf{p})$ is not even "locally rigid" (see [14] for detailed definitions).*

Figure 2 shows a framework in $\mathbb{E}^2$ with its edge directions on a conic at infinity. A Euclidean equivalent framework on the right is obtained obtained from $(G, \mathbf{p})$ through a single affine (but non Euclidean) transformation.

For any specific framework $(G, \mathbf{p})$, we can efficiently test whether the edge directions are on a conic at infinity by solving a linear system. However, the usual application of Proposition 1.4 is to classes of frameworks. This motivates the question of when $(G, \mathbf{p})$ does not, a priori, have its edge directions on a conic at infinity.

A fundamental result along these lines is due to Connelly [7]:

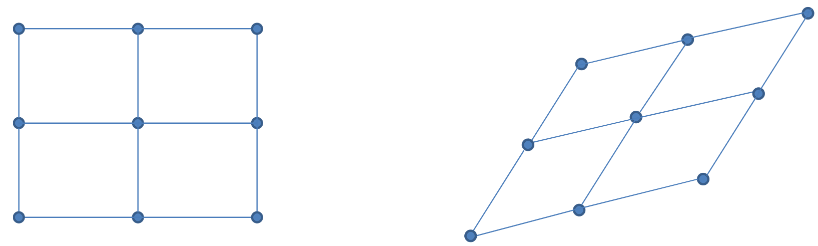


Figure 2: (Left) A framework $(G, \mathbf{p})$ with its edge directions at a conic at infinity. (Right) A Euclidean equivalent framework obtained from $(G, \mathbf{p})$ through a single affine (but non Euclidean) transformation.

Theorem 1.6 ([7])**.** *Let $(G, \mathbf{p})$ be a framework in $\mathbb{E}^d$. Suppose that each vertex of G has degree at least d. Furthermore suppose that $\mathbf{p}$ is a generic configuration. Then the edge directions of $(G, \mathbf{p})$ do not lie on a conic at infinity of $\mathbb{E}^d$.*

In Theorem 1.6, genericity means that the coordinates of $\mathbf{p}$ do not satisfy any non-trivial algebraic equations with coefficients in $\mathbb{Q}$. This means that, while the theorem holds for almost every $\mathbf{p}$ in configuration space, there can be exceptions. Moreover, if one restricts oneself to some special class of frameworks, say, with some imposed non-trivial symmetry, then the theorem may not hold anywhere in that class.

Before continuing, we need two definitions:

Definition 1.7. *Let $\mathcal{S}^n$ be the set of symmetric $n \times n$ matrices. Let $C(G)$, the* graph supported matrices*, be the linear space of matrices (of any rank) in $\mathcal{S}^n$, that vanish on the $\{ij\}$ entries corresponding to non-edges of G.*

Definition 1.8. *A* stress matrix *Ω is a matrix (of any rank) in $C(G)$ with the added property that the all-ones vector is in its kernel. We say that a d-dimensional framework $(G, \mathbf{p})$ is* in the kernel *of Ω if each of its d coordinate n-vectors is in the kernel. In this case, we say that Ω is an* equilibrium stress matrix *for $(G, \mathbf{p})$.*

(The property of Ω being an equilibrium stress matrix for $(G, \mathbf{p})$ does not depend on the choice affine coordinates for $\mathbb{E}^d$.)

In this paper, we are going to consider frameworks $(G, \mathbf{p})$ that are "neighborhood affine rigid" or have an equilibrium stress matrix of rank $n - d - 1$. This additional assumption (which is not present in Theorem 1.6) will allow us to prove essentially tight results about which frameworks have their edge directions on a conic at infinity.

An earlier result using a similar setup is by Alfakih [4].

Theorem 1.9 ([4])**.** *Suppose that a framework $(G, \mathbf{p})$ in $\mathbb{E}^d$ has an equilibrium stress matrix Ω of rank $n - d - 1$. Moreover suppose that each (inclusive) neighborhood of each vertex has a full d-dimensional affine span. Then its edge directions do not lie on a conic at infinity of $\mathbb{E}^d$.*

The exact statement from [4] also requires that the stress matrix is PSD. However, the proof, which is reliant on a long series of linear-algebraic manipulations, only uses the rank of the stress.

Our main result is strictly stronger than Theorem 1.9. The technique is also more conceptual. We relate a notion (described in Section 2) called neighborhood affine rigidity to that of edge directions not lying on a conic at infinity. With this perspective, a direct and simple constructive argument can be used to establish our theorem. The statement of our results needs a definition.

Definition 1.10. *We say that a framework* $(G, \mathbf{p})$ *in* $\mathbb{E}^d$ *is* ruled on a single quadric *if all the vertices* $\mathbf{p}_i$*, and all of points on all of the edges of the framework, lie on some non-trivial, but possibly degenerate, (possibly) inhomogeneous quadric* $\mathcal{Q}$ *in* $\mathbb{E}^d$*. We can assume that, like* $\mathbf{p}$*,* $\mathcal{Q}$ *has a full dimensional affine span. For brevity, we will simply refer to this property as* ruled*. (This property does not depend on the choice affine coordinates for* $\mathbb{E}^d$*.)*

We may describe $\mathcal{Q}$ *by a defining polynomial* $Q(\mathbf{x}) = \mathbf{x}^t\mathbf{Q}\mathbf{x} + \mathbf{l}^t\mathbf{x} + c$ *where* $\mathbf{Q}$ *is a symmetric* $d \times d$ *matrix,* $\mathbf{l}$ *is some vector in* $\mathbb{R}^d$ *and* c *is some constant. Alternatively we may describe* $Q(\mathbf{x})$ *in homogeneous coordinates as* $\hat{\mathbf{x}}^t\hat{\mathbf{Q}}\hat{\mathbf{x}} = 0$ *where* $\hat{\mathbf{Q}}$ *is a symmetric* $(d+1) \times (d+1)$ *matrix.*

A ruled framework is quite special. Indeed, assuming that G is a connected graph, a ruled framework in $\mathbb{E}^2$ must be entirely contained within two intersecting lines! (See Figure 5 below.)

The main theorem of this paper is:

Theorem 1.11. *Suppose that a framework* $(G, \mathbf{p})$ *in* $\mathbb{E}^d$ *is neighborhood affine rigid. Then its edge directions lie on a conic at infinity of* $\mathbb{E}^d$ *iff* $(G, \mathbf{p})$ *is ruled.*

Corollary 1.12. *Suppose that a framework* $(G, \mathbf{p})$ *in* $\mathbb{E}^d$ *has an equilibrium stress matrix* Ω *of rank* $n - d - 1$*. Then its edge directions lie on a conic at infinity of* $\mathbb{E}^d$ *iff* $(G, \mathbf{p})$ *is ruled.*

Proof. Proposition 5.9 of [14] explains that a equilibrium stress matrix of rank $n-d-1$ is a sufficient (but not necessary) certificate of neighborhood affine rigidity. □

Proposition 3.4, below, tells us that any framework with even a subset of d vertices in general affine position, each with a neighborhood of full affine span, cannot be ruled. This makes Corollary 1.12 stronger than Theorem 1.9.

Using our theorem, we will also show as a corollary, that the property of a framework being "super stable" is preserved by invertible projective transforms of $\mathbb{E}^d$ as well as the "coning" and "slicing" operations.

Finally, we will describe a relationship between the notion of a ruled framework to the notion of a matrix having "the Strong Arnold Property" and a related connection that has recently been made in another paper by Alfakih [3].

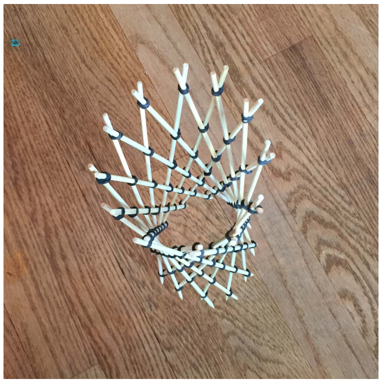
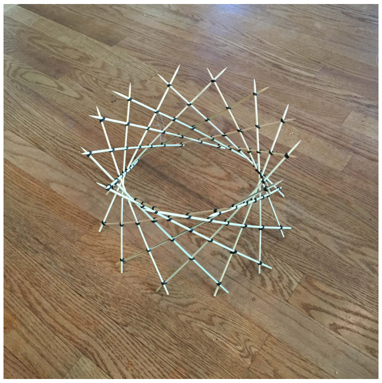



Figure 3: Two Euclidean equivalent "frameworks" related by an affine flex. The frameworks are neigborhood affine rigid and are ruled.

2 Neighborhood Affine Rigidity

First we review a few definitions about affine rigidity from [14].

Definition 2.1. *Let $(G, \mathbf{p})$ be a framework in $\mathbb{E}^d$. We say that $(G, \mathbf{p})$ is neigborhood affine preequivalent to a second framework $(G, \mathbf{q})$ if for each vertex i, the point $\mathbf{p}_i$ and all of the points $\mathbf{p}_j$, where vertex j is a neighbor of vertex i, can be mapped to their associated positions in $\mathbf{q}$ by a (possibly singular) affine transform depending only on i.*

We say that $\mathbf{p}$ is affine precongruent to $\mathbf{q}$ if all the vertices in $\mathbf{p}$ can be mapped to their positions in $\mathbf{q}$ by a (possibly singular) affine transform.

(The inclusion of singular transforms is done for technical reasons [14]. This prevents affine preequivalence (and also precongruence) from being a symmetric relation, and is the source for the "pre" terminology.)

We say that $(G, \mathbf{p})$ is neighborhood affine rigid *if for any other framework $(G, \mathbf{q})$, to which $(G, \mathbf{p})$ is neighborhood affine prequivalent, we always have that $\mathbf{p}$ is affine precongruent to $\mathbf{q}$.*

Definition 2.2. *Let $\mathbf{Q}$ be a symmetric $d \times d$ matrix. We define an associated* perturbation map *m acting on a point $\mathbf{x}$ in $\mathbb{E}^d$ to be*

$$m(\mathbf{x}) := \mathbf{x} + [\mathbf{x}^t \mathbf{Q} \mathbf{x}]\mathbf{v}$$

where $\mathbf{v}$ is some chosen non-zero vector in $\mathbb{R}^d$.

We denote by $m(\mathbf{p})$ the configuration defined by mapping all of the points of $\mathbf{p}$ by m.

Proposition 2.3. *Suppose $(G, \mathbf{p})$ has its edge directions on a conic at infinity defined by a non-zero matrix $\mathbf{Q}$. Let m be an associated perturbation map. Then $(G, \mathbf{p})$ is neighborhood affine preequivalent to $(G, m(\mathbf{p}))$.*

Proof. Let $\mathbf{q} := m(\mathbf{p})$. We just need to show that for each vertex i, there is an affine transform that maps each vertex $\mathbf{p}_j$ of the (inclusive) neighborhood of $\mathbf{p}_i$, to its position in the configuration $\mathbf{q}$.

From our assumption that the edge directions are at a conic at infinity,

$$0 \quad = \quad (\mathbf{p}_j - \mathbf{p}_i)^t \mathbf{Q} (\mathbf{p}_j - \mathbf{p}_i)$$

we get

$$\mathbf{p}_j^t \mathbf{Q} \mathbf{p}_j \quad = \quad -\mathbf{p}_i^t \mathbf{Q} \mathbf{p}_i + 2\mathbf{p}_i^t \mathbf{Q} \mathbf{p}_j$$

when vertex j is a neighbor of vertex i. The same is trivially true when $j = i$.

Thus on the (inclusive) neighborhood of $\mathbf{p}_i$, the action of m can be modeled with the affine transform:

$$m(\mathbf{x}) = \mathbf{x} + [-\mathbf{p}_i^t \mathbf{Q} \mathbf{p}_i + 2\mathbf{p}_i^t \mathbf{Q} \mathbf{x}]\mathbf{v}$$

□

Lemma 2.4. *Let $\mathbf{Q}$ be a non-zero symmetric $d \times d$ matrix and let m be an associated perturbation map. Suppose that $\mathbf{p}$ is affine precongruent to $m(\mathbf{p})$. Then all of the $\mathbf{p}_i$ must lie on a (possibly) inhomogeneous quadric with its quadratic term defined by $\mathbf{Q}$.*

Proof. Assume, w.l.o.g., that $\|v\| = 1$. Expanding the definition of m and rearranging, affine precongruence implies there is an affine map $A(\mathbf{x}) := \mathbf{A}\mathbf{x} + \mathbf{t}$ (where $\mathbf{A}$ is a $d \times d$ matrix, and $\mathbf{t}$ is a vector in $\mathbb{R}^d$) such that for all $\mathbf{x} = \mathbf{p}_i$

$$m(\mathbf{x}) = \mathbf{x} + [\mathbf{x}^t\mathbf{Q}\mathbf{x}]\mathbf{v} = \mathbf{A}\mathbf{x} + \mathbf{t}$$

and

$$[\mathbf{x}^t\mathbf{Q}\mathbf{x}]\mathbf{v} = (\mathbf{A} - \mathbf{I})\mathbf{x} + \mathbf{t} =: \mathbf{A}'\mathbf{x} + \mathbf{t}$$

Multiplying by $\mathbf{v}^t$ on the left, we obtain

$$\mathbf{x}^t\mathbf{Q}\mathbf{x} = \left[\mathbf{v}^t\mathbf{A}'\right]\mathbf{x} + \mathbf{v}^t\mathbf{t}$$

which gives a non-trivial quadric for the $\mathbf{p}_i$, since the left-hand side, at least, is non-zero. □

Lemma 2.5. *Let $\mathcal{Q}$ be a (possibly) inhomogeneous quadric with quadratic terms defined by a non-zero symmetric matrix $\mathbf{Q}$. Suppose that two points $\mathbf{x}_1$ and $\mathbf{x}_2$ are both on $\mathcal{Q}$, and that for the edge vector $\mathbf{e} := \mathbf{x}_2 - \mathbf{x}_1$ we have $\mathbf{e}^t\mathbf{Q}\mathbf{e} = 0$. Then all the points on the line spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$ are on $\mathcal{Q}$.*

Proof. Thinking projectively (see Remark 1.2), we have the points $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{e}$ (at infinity) on a line ℓ in $\mathbb{P}^d$. If Q is the polynomial defining $\mathcal{Q}$, then $Q(\mathbf{e}) = \mathbf{e}^t\mathbf{Q}\mathbf{e}$, since $\mathbf{e}$ is at infinity. Thus $\mathcal{Q}$ has more than 2 intersection points with ℓ, so Q vanishes identically on it. □

Remark 2.6. *We could prove Lemma 2.5 without leaving the affine setting. Suppose that $\mathcal{Q}$ is defined by the polynomial $Q(\mathbf{x}) = \mathbf{x}^t\mathbf{Q}\mathbf{x} + \mathbf{l}^t\mathbf{x} + c$. The hypothesis about the edge vector implies the identity $\mathbf{x}_1^t\mathbf{Q}\mathbf{x}_1 + \mathbf{x}_2^t\mathbf{Q}\mathbf{x}_2 = 2\mathbf{x}_1^t\mathbf{Q}\mathbf{x}_2$. We then compute that $Q(\frac{1}{2}[\mathbf{x}_1 + \mathbf{x}_2]) = \frac{1}{2}[Q(\mathbf{x}_1) + Q(\mathbf{x}_2)] = 0$, which lets us proceed as above.*

We will see a different approach to Lemma 2.5 in the discussion about the Strong Arnold Property in Section 5, below.

Proposition 2.7. *Suppose $(G, \mathbf{p})$ has its edge directions on a conic at infinity defined by $\mathbf{Q}$ and let m be an associated perturbation map. Suppose that $\mathbf{p}$ is affine precongruent to $m(\mathbf{p})$. Then $(G, \mathbf{p})$ must be a ruled framework.*

Proof. From Lemma 2.4, we have each point $\mathbf{p}_i$ on a (possibly) inhomogeneous quadric, with its quadratic terms defined by a $d \times d$ matrix $\mathbf{Q}$. By assumption, for each edge vector, $\mathbf{e}_{ij} := \mathbf{p}_j - \mathbf{p}_i$, we have $\mathbf{e}_{ij}^t\mathbf{Q}\mathbf{e}_{ij} = 0$. Thus from Lemma 2.5 we see that $(G, \mathbf{p})$ must be ruled. □

We are now ready to prove our main result.

Proof of Theorem 1.11. Let $(G, \mathbf{p})$ be a framework that is neighborhood affine rigid. If $(G, \mathbf{p})$ is ruled on a quadric $\mathcal{Q}$, the points of $\mathcal{Q}$ at infinity are a conic at infinity. As in the proof of Lemma 2.5, the edge direction vectors are on this conic.

For the "only if" direction, assume that it has its edge directions on a conic at infinity. Let m be an associated perturbation map. Then from Proposition 2.3 $(G, \mathbf{p})$ is neighborhood affine preequivalent to $(G, m(\mathbf{p}))$. Since $(G, \mathbf{p})$ is neighborhood affine rigid, this implies that $\mathbf{p}$ is affine precongruent to $m(\mathbf{p})$. So from Proposition 2.7, $(G, \mathbf{p})$ must be ruled. □

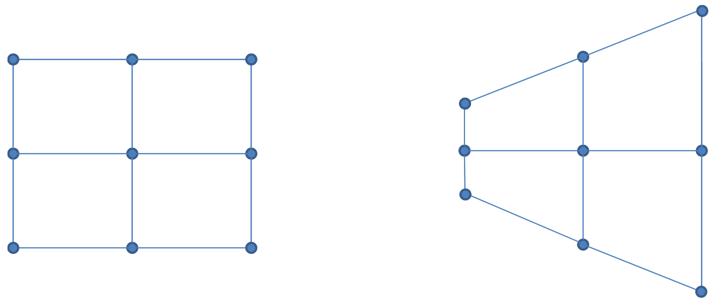


Figure 4: (Left) A framework $(G, \mathbf{p})$ with its edge directions at a conic at infinity. (Right) A framework $(G, \mathbf{q})$ with the property that each vertex neighborhood in $(G, \mathbf{q})$ can be obtained from its corresponding neighborhood in $(G, \mathbf{p})$ through an affine transform. There is no global affine transformation that maps $\mathbf{p}$ to $\mathbf{q}$.

Figure 4 shows a simple example of the construction used in the proof of Theorem 1.11. In the original framework $(G, \mathbf{p})$, shown on the left, all of the edge directions are either horizontal or vertical. Thus, they lie on the conic at infinity defined by the equation $xy = 0$. However, this framework is not ruled. We will see that it is not neighborhood affine rigid.

To define the perturbation map, we set the origin point to be the center vertex of the framework, which, for simplicity, we will assume has Cartesian coordinates $[0, 0]^t$ and we set the vector $\mathbf{v}$ to be $[0, 1]^t$ (in the vertical direction). The resulting map can be described, in coordinates, by

$$[x, y]^t \mapsto [x, y + xy]^t$$

On the right we show $(G, \mathbf{q})$, the image of $(G, \mathbf{p})$ under this map. It is easy to see that each vertex neighborhood of $(G, \mathbf{q})$ can be obtained from its configuration in $(G, \mathbf{p})$ under some affine transform, but that the full configuration $\mathbf{q}$ cannot be obtained from $\mathbf{p}$ using a single, global affine transform.

3 What do ruled frameworks look like?

We now unpack what it means to have a ruled framework.

Definition 3.1. *Let S be a point set in $\mathbb{E}^d$. We say that $\mathbf{x}$ is a* cone point *of S if for any other point $\mathbf{y}$ on S, we have the entire line spanned by $\mathbf{x}$ and $\mathbf{y}$ is in S.*

The following is standard (see e.g. [15, Example 3.3 and Lecture 22]). Every non-trivial quadric $\mathcal{Q}$ can be put in canonical form under a projective transform by diagonalizing and normalizing $\hat{\mathbf{Q}}$, the $(d+1) \times (d+1)$ symmetric matrix that describes $\mathcal{Q}$ in homogenized form. The resulting canonical matrix will have some number of $+1$, -1 and 0 diagonal entries. Call the rank of this matrix r. If the matrix is definite, or semi-definite, then $\mathcal{Q}$ cannot have full affine span. So let us now assume that the matrix is indefinite. When $r = d + 1$, the quadric is smooth. Otherwise, (we have $1 < r < d + 1$), the quadric is the cone over a smooth quadric of dimension $r - 2$ in $\mathbb{E}^{r-1}$, with a cone point set comprising an affine space of dimension $d - r$. In particular, all non-smooth points must be cone points. (See Figure 6 below.)

This canonical picture immediately gives us the following two Lemmas.

Lemma 3.2. *Let $\mathbf{x}$ be a point of a quadric $\mathcal{Q}$ in $\mathbb{E}^d$ that has d linearly independent ruling directions (lines within the conic through that point). Then $\mathbf{x}$ must be a cone point of $\mathcal{Q}$.*

Proof. The tangent hyperplane at a smooth point must include all ruling directions through that point. So having d ruling directions precludes the existence of such a tangent. □

Lemma 3.3. *Suppose $\mathcal{Q}$ is a non-trivial (possibly) inhomogeneous quadric with a full affine span in $\mathbb{E}^d$. Then $\mathcal{Q}$ cannot have d cone points in general affine position.*

Proof. The full affine span means that the rank of the quadric must be greater than 1 and indefinite. Thus, from our canonical picture of quadrics, the dimension of the affine space of cone points must be no larger than $d-2$. □

Proposition 3.4. *Suppose $(G, \mathbf{p})$, a framework in $\mathbb{E}^d$, has a subset of d vertices in general affine position, each with neighborhood in $(G, \mathbf{p})$ with full affine span. Then $(G, \mathbf{p})$ cannot be ruled.*

Proof. By assumption, $(G, \mathbf{p})$ has a full affine span. Suppose that $(G, \mathbf{p})$ were ruled by a non-trivial quadric $\mathcal{Q}$. Then from Lemma 3.2, each of the d vertices would be a cone point. But this would contradict Lemma 3.3. □

Remark 3.5. *We can now see that Corollary 1.12 is stronger than Theorem 1.9. In light of Proposition 3.4 the assumed affine span condition of Theorem 1.9 is strictly stronger than being non-ruled.*

4 Super Stability

Recall that a framework in $\mathbb{E}^d$ is universally rigid if there is no second framework $(G, \mathbf{q})$ in any dimension that is equivalent but not congruent to $(G, \mathbf{p})$. Universal rigidity (unlike infinitesimal rigidity) is known not to be invariant under projective transformations or under the "slicing" operation described below [9]. A slightly stronger property than universal rigidity is called super stability [5, 13]. In this section, we show that super stability is well-behaved with respect to these operations. We start with some definitions.

Definition 4.1. *A framework $(G, \mathbf{p})$ in $\mathbb{E}^d$ with a full dimensional affine span is called* super stable *if it has a positive semidefinite (PSD) equilibrium stress matrix Ω of rank $n-d-1$, and its edge directions do not lie on a conic at infinity.*

Theorem 4.2 ([5])**.** *Super stability implies universal rigidity.*

Definition 4.3. *A* cone graph *is a graph with $n+1$ vertices where vertex 0 is connected to all the others. We will consider* cone frameworks *of this graph in $\mathbb{E}^{d+1}$ denoted as $\mathbf{p}_0 * (G, \mathbf{p})$. Here G is the subgraph on n vertices induced by removing vertex 0, $\mathbf{p}$ is a configuration of the n vertices of G in $\mathbb{E}^{d+1}$, and $\mathbf{p}_0$ is another point in $\mathbb{E}^{d+1}$ that is not coincident with any of the points of $\mathbf{p}$.*

We define several operations on cone frameworks.

Definition 4.4. *Given a cone framework $\mathbf{p}_0 * (G, \mathbf{p})$ in $\mathbb{E}^{d+1}$, the process of* sliding *denotes moving points of $\mathbf{p}$ along their lines connecting them to $\mathbf{p}_0$, while avoiding $\mathbf{p}_0$ itself.*

Definition 4.5. *We call a cone framework $\mathbf{p}_0 * (G, \mathbf{p})$ in $\mathbb{E}^{d+1}$,* flat *if each of the n vertices of $\mathbf{p}$ lies in a d-dimensional Euclidean subspace that does not include $\mathbf{p}_0$.*

Definition 4.6. *Given a framework $(G, \mathbf{p})$ in $\mathbb{E}^d$, we can* cone *it by placing $\mathbb{E}^d$ in a hyperplane in $\mathbb{E}^{d+1}$, and then adding a cone vertex at some location $\mathbf{p}_0$ (outside of the affine span of $\mathbf{p}$) to create a flat cone framework in $\mathbb{E}^{d+1}$.*

Given a cone framework $\mathbf{p}_0 * (G, \mathbf{p})$ in $\mathbb{E}^{d+1}$, we can slice *it, by sliding all points of $\mathbf{p}$ to create a flat cone framework. And then we can consider the resulting subframework of G as living in $\mathbb{E}^d$.*

This section's main result is:

Theorem 4.7. *Super stability is invariant under coning and slicing.*

Corollary 4.8. *Super stability is invariant with respect to invertible projective transformations in $\mathbb{E}^d$ that do not send any vertices to infinity.*

Proof. Any projective transformation on $(G, \mathbf{p})$ can be modeled by coning, followed by a linear transformation on $\mathbb{E}^{d+1}$, followed by slicing. □

4.1 Proof of Theorem 4.7

We start with two technical lemmas.

Lemma 4.9. *Suppose that a cone framework $\mathbf{p}_0 * (G, \mathbf{p})$ has an equilibrium stress matrix Ω. Then any framework $\mathbf{p}_0 * (G, \mathbf{q})$ obtained by sliding has an equilibrium stress matrix Ω' of the same rank and signature.*

The proof is based on ideas of [17, Theorem 8]. See also [16, Lemma 4.11].

Proof. By translating, we can assume that $\mathbf{p}_0$ is at the origin, and that sliding is then represented by scaling: $\mathbf{q}_i := s_i \mathbf{p}_i$.

Let Ω be an equilibrium stress matrix for $\mathbf{p}_0 * (G, \mathbf{p})$ with signature (a, b, c) (a negative eigenvalues, b zero eigenvalues and c positive eigenvalues). Let Ψ be the matrix obtained from Ω by removing the row and column corresponding to the cone vertex. Let it have signature (f, e, g). Because the kernel of Ω contains a vector that is non-zero on the coordinate corresponding to the cone vertex (such as the all-ones vector), we know that the rank of Ψ is the same as the rank of Ω, thus it has one less zero eigenvalue. Then from the eigenvalue interleaving theorem, we must have $f = a$ and $g = c$.

Since $\mathbf{p}_0$ is at the origin, we must still have the $d + 1$ coordinate vectors of $\mathbf{p}$ in the kernel of Ψ, though the all-ones vector is no longer in the kernel.

When scaling $\mathbf{p}$ to obtain $\mathbf{q}$, we can define the matrix $\Psi' := S \Psi S$, where S is a full rank diagonal matrix with entries $1/s_i$. The matrix Ψ' will have the same signature as Ψ and will have the coordinates of $\mathbf{q}$ in its kernel.

Finally we will augment Ψ' with a row and column corresponding to the cone vertex so that the all-ones vector is in the kernel of the resulting stress Ω'. To do this, we can first add a column which is the negative sum of the n columns of Ψ'. Since $\mathbf{p}_0$ is at the origin, the coordinates of $\mathbf{p}_0 * (G, \mathbf{q})$ are in the kernel of this matrix. Then we can add a row which is the negative sum of the n rows of this intermediate matrix. The coordinates of $\mathbf{p}_0 * (G, \mathbf{q})$ must also be annihilated by this last row (as it is just a linear combination of the other rows). Thus we have an equilibrium stress matrix for $\mathbf{p}_0 * (G, \mathbf{q})$. The rank of Ω' is the same as Ψ' thus it has one more zero eigenvalue. Again, from the eigenvalue interleaving theorem, it must have signature (a, b, c). □

Lemma 4.10. *A cone framework $\mathbf{p}_0 * (G, \mathbf{p})$ has its edge directions on a conic at infinity if and only if it is ruled.*

Proof. Since $\mathbf{p}_0$ is connected to all other vertices, this automatically makes the framework neighborhood affine rigid. Then we can apply Theorem 1.11. □

We can now prove an intermediate result that is interesting in its own right.

Proposition 4.11. *Super stability of a cone framework* $\mathbf{p}_0 * (G, \mathbf{p})$ *is invariant with respect to sliding.*

Proof. If $\mathbf{p}_0 * (G, \mathbf{p})$ has its edge directions on a conic at infinity, then from Lemma 4.10, it is ruled on a quadric $\mathcal{Q}$. For each edge $\{i, j\} \in E$, the three edges of the triangle $\{\mathbf{p}_0, \mathbf{p}_i, \mathbf{p}_j\}$ are contained in $\mathcal{Q}$, so its entire supporting plane must be too. So any $\mathbf{p}_0 * (G, \mathbf{q})$ obtained by sliding is also ruled. Thus, for cone frameworks, having edge directions on a conic at infinity is invariant with respect to sliding. Lemma 4.9 says that having a PSD equilibrium stress matrix of rank $n - d - 1$ is as well. □

We can now complete the proof of Theorem 4.7. The main observation is that for both $(G, \mathbf{p})$ in $\mathbb{E}^d$ and $\mathbf{p}_0 * (G, \mathbf{p})$ in $\mathbb{E}^{d+1}$, the necessary rank for super stability is $n - d - 1$. Starting with an equilibrium stress for $(G, \mathbf{p})$ in $\mathbb{E}^d$, we can simply add a row and column of zeros and obtain an equilibrium stress matrix for its coned framework in $\mathbb{E}^{d+1}$ of the same rank and positive/negative signature.

Conversely, for a flat cone framework $\mathbf{p}_0 * (G, \mathbf{p})$ in $\mathbb{E}^{d+1}$, any equilibrium stress Ω, must have $\Omega_{0i} = 0$ for all i. (The equilibrium condition can be thought of as a balance of forces along the edges of the framework, $\sum_{j \neq i} \Omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0}$. Any non-zero force along the cone edge at $\mathbf{p}_i$ cannot be matched by the forces arising from edges within G as these forces all lie in a hyperplane not containing $\mathbf{p}_0$). By simply discarding the row and column for $\mathbf{p}_0$, we obtain an equilibrium stress matrix for $(G, \mathbf{p})$ in $\mathbb{E}^d$ of the same rank and positive/negative signature.

Proceeding in the coning (easy) direction, if $(G, \mathbf{p})$ has a PSD equilibrium stress matrix of maximum rank, then from the above observation, so too must the result of coning. Meanwhile, if the edge directions of $(G, \mathbf{p})$ are not on a conic at infinity, then neither are the edges of the coned result.

Proceeding in the slicing (harder) direction, in light of Proposition 4.11, we can start with a flat coned framework $\mathbf{p}_0 * (G, \mathbf{p})$ that is super stable. From the observation above, if $\mathbf{p}_0 * (G, \mathbf{p})$ has a PSD equilibrium stress matrix of maximum rank, then so too must the sliced result $(G, \mathbf{p})$ in $\mathbb{E}^d$.

Meanwhile, if the edge directions of $\mathbf{p}_0 * (G, \mathbf{p})$ are not at a conic at infinity, then $\mathbf{p}_0 * (G, \mathbf{p})$ is certainly not ruled. Importantly, this implies that $(G, \mathbf{p})$ in $\mathbb{E}^d$ is not ruled either. From Corollary 1.12 then, $(G, \mathbf{p})$ does not have its edge directions on a conic at infinity. Thus it is super stable. □

4.2 Remarks

Lemma 4.9 is implicit in [11]. Also, we can prove Corollary 4.8 without Theorem 4.7 using Corollary 1.12, the fact that being ruled is preserved under projective transforms, and a result from [11] that the rank and signature of stress matrices is preserved by invertible projective maps.

Another connection is to the notion of dimensional rigidity, which was introduced by Alfakih [1]. A framework $(G, \mathbf{p})$ in $\mathbb{E}^d$ is *dimensionally rigid* if there are no equivalent frameworks with a higher dimensional span. Alfakih [2] has shown that $(G, \mathbf{p})$ is dimensionally rigid but not universally rigid if and only if its edge directions are on a conic at infinity. Connelly and Gortler [9, 10] showed that dimensional rigidity is invariant with respect to projective transformations and coning/sliding/slicing.

Universal rigidity is preserved under coning and sliding, but it is neither projectively invariant nor is it preserved by slicing, since having edge directions on a conic at infinity isn't preserved by projective transforms or by coning [9]. The counter-examples are necessarily not neighborhood affine rigid, and not ruled. In contrast, being ruled is invariant with respect to invertible projective transforms and to coning.

5 Strong Arnold Property

In the literature on the Colin de Verdière graph parameter, there is an central non-degeneracy property of a matrix called the Strong Arnold Property [19].

Definition 5.1. *Let r be some rank. Let D_r be the determinantal variety of matrices in $\mathcal{S}^n$ with rank no greater than r.*

A graph supported matrix, $\Psi \in C(G)$, with rank r is said to satisfy the Strong Arnold Property (SAP) *if D_r and $C(G)$ intersect transversely at Ψ.*

Recently, Laurent and Varvisiotis [16] began an exploration on the relationship between universal rigidity, PSD matrix completion and the Strong Arnold Property. They quote an older result of Godsil, which we translate into our language and specialize to stress matrices.

Theorem 5.2 ([12, Theorem 3.2])**.** *Let Ω be a stress matrix with rank $n-d-1$. Let $(G, \mathbf{p})$ be a framework in its kernel with a d-dimensional affine span. Then Ω does not have the SAP if and only if $(G, \mathbf{p})$ is ruled.*

Remark 5.3. *Godsil's proof of Theorem 5.2 provides another approach to proving Lemma 2.5. He shows that the conditions for a non-transverse intersection at Ω are equivalent to the existence of a non-zero symmetric $(d+1) \times (d+1)$ matrix $\hat{\mathbf{Q}}$ such that for all vertices i, we have $\hat{\mathbf{p}}_i^t \hat{\mathbf{Q}} \hat{\mathbf{p}}_i = 0$ and that for all edges $\{ij\}$, we have $\hat{\mathbf{p}}_i^t \hat{\mathbf{Q}} \hat{\mathbf{p}}_j = 0$ (where $\hat{\mathbf{p}}_i$ are as in Remark 1.2). Then $\hat{\mathbf{Q}}$ is the matrix defining the ruling quadric $\mathcal{Q}$.*

Other closely related results can be found in [16, 18].

Meanwhile, Alfakih proves the following:

Theorem 5.4 ([3, Corollary 2])**.** *Let Ω be a stress matrix with rank $n-d-1$. Let $(G, \mathbf{p})$ be a framework in its kernel with a d-dimensional affine span. Then Ω does not have the SAP if and only if the edge directions of $(G, \mathbf{p})$ are on a conic at infinity.*

Remark 5.5. *Similarly to the proof of Theorem 5.2, Alfakih's proof of Theorem 5.4 also (implicitly) involves the existence of a non-zero symmetric $(d+1) \times (d+1)$ matrix $\hat{\mathbf{Q}}$ such that for all i, we have $\hat{\mathbf{p}}_i^t \hat{\mathbf{Q}} \hat{\mathbf{p}}_i = 0$ and that for all edges $\{ij\}$ we have $\hat{\mathbf{p}}_i^t \hat{\mathbf{Q}} \hat{\mathbf{p}}_j = 0$. But in this case, a fair amount of extra work is needed to get to this condition starting from the assumed Ω and assumed conic at infinity.*

We can now see that Corollary 1.12, Theorem 5.2 and Theorem 5.4 form a cycle of relationships. Any two of them imply the third.

6 Examples

In $\mathbb{E}^2$ a ruled framework must lie in the intersection of two lines. See Figure 5.

What do neighborhood affine rigid, ruled frameworks in $\mathbb{E}^3$ look like? See Figure 6.

One possibility in $\mathbb{E}^3$ is a framework on a doubly ruled quadric such as a hyperbolic paraboloid or a hyperboloid of one sheet (smooth, rank 4). In this case, each vertex can have at most a two dimensional neighborhood affine span.

Figure 5: (Left) A ruled framework on two lines that is neighborhood affine rigid. (Right) After adding two long bracing edges, the ruled framework has a PSD equilibrium stress matrix of rank $n - d - 1$.

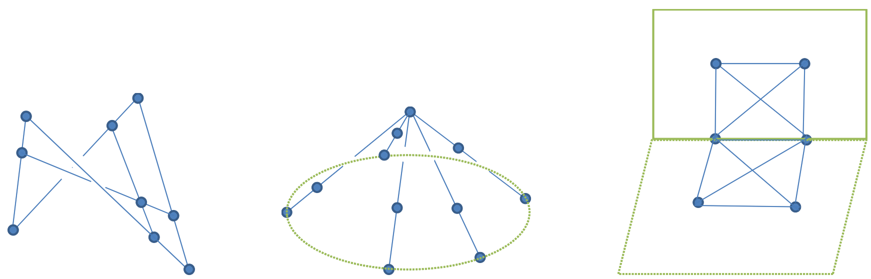


Figure 6: (Left) A ruled framework on a hyperbolic paraboloid that is neighborhood affine rigid. By adding six long bracing edges along the rulings, as in Figure 5, it will have a PSD equilibrium stress matrix of rank $n - d - 1$. (Middle) A ruled framework on an elliptical cone that is neighborhood affine rigid. It has one vertex at a cone point. Even with bracing edges along the rulings, it will not have an equilibrium stress matrix of rank $n - d - 1$. The green ellipse is added for visualization context only. (Right) A ruled framework on two planes that is neighborhood affine rigid and has a PSD equilibrium stress matrix of rank $n - d - 1$. It has two vertices at cone points, along the intersection line of the two planes. The green planes are added for visualization context only.

Proposition 6.1. *With an appropriate vertex and edge set, we can construct a framework on any doubly ruled quadric $\mathcal{Q}$ in $\mathbb{E}^3$ that has a PSD equilibrium stress matrix of rank $n - 4$ (and thus is also neighborhood affine rigid).*

Proof. The construction is as follows. Start with a collection of lines $\{\ell_1, \dots, \ell_s, m_1, \dots, m_t\}$ on $\mathcal{Q}$ so that: (1) the ℓ_i are in one ruling and then m_j in the other; (2) the (necessarily bipartite) intersection graph of the ℓ_i and m_j is connected and has minimum degree 3; (3) ℓ_1 and ℓ_2 intersect all the m_j. Now we construct a framework by putting vertices at the intersection points of lines and all the edges between vertices on the same line. Call this framework $(G, \mathbf{p})$.

Any framework of a complete graph with 3 or more vertices, all on a line, supports a PSD equilibrium stress matrix Ω with corank 2. Taking a positive linear combination of these shows that $(G, \mathbf{p})$ carries a PSD stress matrix Ω, that forces any framework $(G, \mathbf{q})$ in its kernel to have the vertices partitioned into the same collinear subsets.

To finish up, we observe that ℓ_1 and ℓ_2 span at most a 3-dimensional affine space. Since the m_i intersect ℓ_1 and ℓ_2, they are in the same space. Each of the ℓ_i, $i \geq 3$ intersect at least 2 of the m_i, establishing that any such kernel framework, $(G, \mathbf{q})$, has at most 3-dimensional span. Equivalently, Ω has rank $n - 4$. □

Another possibility in $\mathbb{E}^3$ is a framework on a quadric with a single cone point, such as an elliptic cone (rank 3). In this case, there can be at most one vertex with a full dimensional neighborhood affine span. Any other vertex can only have a one dimensional neighborhood affine span. With an appropriate edge set, we can construct such a framework that is neighborhood affine rigid. But we can not construct such a framework to have a equilibrium stress matrix of rank $n - d - 1$ unless the entire framework lies on three intersecting lines!

The last possibility is a framework contained entirely within two intersecting planes (rank 2). We can construct such a framework that is neighborhood affine rigid or even to have a PSD equilibrium stress matrix of rank $n - d - 1$. A framework contained within two planes can have at most two points in general affine position, that have neighborhoods with full three dimensional affine spans. (Such two points must be on the line of intersection between the two planes.)

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References

[1] A. Y. Alfakih. On dimensional rigidity of bar-and-joint frameworks. *Discrete Appl. Math.*, 155(10):1244–1253, 2007. ISSN 0166-218X. doi: 10.1016/j.dam.2006.11.011. URL `http://dx.doi.org/10.1016/j.dam.2006.11.011`.

[2] A. Y. Alfakih. Local, dimensional and universal rigidities: a unified Gram matrix approach. In *Rigidity and symmetry*, volume 70 of *Fields Inst. Commun.*, pages 41–60. Springer, New York, 2014. doi: 10.1007/978-1-4939-0781-6_3. URL `http://dx.doi.org/10.1007/978-1-4939-0781-6_3`.

[3] A. Y. Alfakih. Universal rigidity of bar frameworks via the geometry of spectrahedra. Preprint, arXiv:1504.00578, 2015. URL `https://arxiv.org/abs/1504.00578`.

[4] A. Y. Alfakih and V.-H. Nguyen. On affine motions and universal rigidity of tensegrity frameworks. *Linear Algebra Appl.*, 439(10):3134–3147, 2013. ISSN 0024-3795. doi: 10.1016/j.laa.2013.08.016. URL `http://dx.doi.org/10.1016/j.laa.2013.08.016`.

[5] R. Connelly. Rigidity and energy. *Invent. Math.*, 66(1):11–33, 1982. ISSN 0020-9910. doi: 10.1007/BF01404753. URL `http://dx.doi.org/10.1007/BF01404753`.

[6] R. Connelly. Rigidity. In *Handbook of convex geometry, Vol. A, B*, pages 223–271. North-Holland, Amsterdam, 1993.

[7] R. Connelly. Generic global rigidity. *Discrete Comput. Geom.*, 33(4):549–563, 2005. ISSN 0179-5376. doi: 10.1007/s00454-004-1124-4. URL `http://dx.doi.org/10.1007/s00454-004-1124-4`.

[8] R. Connelly. Tensegrities and global rigidity. In *Shaping space*, pages 267–278. Springer, New York, 2013. doi: 10.1007/978-0-387-92714-5_21. URL `http://dx.doi.org/10.1007/978-0-387-92714-5_21`.

[9] R. Connelly and S. J. Gortler. Iterative Universal Rigidity. *Discrete Comput. Geom.*, 53(4): 847–877, 2015. ISSN 0179-5376. doi: 10.1007/s00454-015-9670-5. URL http://dx.doi.org/10.1007/s00454-015-9670-5.

[10] R. Connelly and S. J. Gortler. Universal rigidity of complete bipartite graphs. Preprint, arXiv:1502.02278, 2015. URL https://arxiv.org/abs/1502.02278.

[11] R. Connelly and W. J. Whiteley. Global rigidity: the effect of coning. *Discrete Comput. Geom.*, 43(4):717–735, 2010. ISSN 0179-5376. doi: 10.1007/s00454-009-9220-0. URL http://dx.doi.org/10.1007/s00454-009-9220-0.

[12] C. Godsil. The Colin de Verdière number. Online notes, 2001. URL http://quoll.uwaterloo.ca/mine/Notes/cdv.pdf.

[13] S. J. Gortler and D. P. Thurston. Characterizing the universal rigidity of generic frameworks. *Discrete Comput. Geom.*, 51(4):1017–1036, 2014. ISSN 0179-5376. doi: 10.1007/s00454-014-9590-9. URL http://dx.doi.org/10.1007/s00454-014-9590-9.

[14] S. J. Gortler, C. Gotsman, L. Liu, and D. Thurston. On affine rigidity. *JoCG*, 4(1):160–181, 2013. URL http://jocg.org/index.php/jocg/article/view/49.

[15] J. Harris. *Algebraic geometry: a first course*, volume 133. Springer Science & Business Media, 2013.

[16] M. Laurent and A. Varvitsiotis. Positive semidefinite matrix completion, universal rigidity and the strong Arnold property. *Linear Algebra Appl.*, 452:292–317, 2014. ISSN 0024-3795. doi: 10.1016/j.laa.2014.03.015. URL http://dx.doi.org/10.1016/j.laa.2014.03.015.

[17] L. Lovász. Steinitz representations of polyhedra and the Colin de Verdière number. *J. Combin. Theory Ser. B*, 82(2):223–236, 2001. ISSN 0095-8956. doi: 10.1006/jctb.2000.2027. URL http://dx.doi.org/10.1006/jctb.2000.2027.

[18] H. van der Holst. Graphs with magnetic Schrödinger operators of low corank. *J. Combin. Theory Ser. B*, 84(2):311–339, 2002. ISSN 0095-8956. doi: 10.1006/jctb.2001.2087. URL http://dx.doi.org/10.1006/jctb.2001.2087.

[19] H. van der Holst, L. Lovász, and A. Schrijver. The Colin de Verdière graph parameter. In *Graph theory and combinatorial biology (Balatonlelle, 1996)*, volume 7 of *Bolyai Soc. Math. Stud.*, pages 29–85. János Bolyai Math. Soc., Budapest, 1999.