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Multifractal spectra and multifractal zeta-functions

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Abstract. We introduce multifractal zetafunctions providing precise information of a very general class of multifractal spectra, including, for example, the multifractal spectra of self-conformal measures and the multifractal spectra of ergodic Birkhoff averages of continuous functions. More precisely, we prove that these and more general multifractal spectra equal the abscissae of convergence of the associated zeta-functions.

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1. Introduction

Measures with widely varying intensity are called multifractals and have during the past 20 years been the focus of enormous attention in the mathematical literature. Loosely speaking there are two main ingredients in multifractal analysis: the multifractal spectrum and the Renyi dimensions. One of the main goals in multifractal analysis is to understand these two ingredients and their relationship with each other. It is generally believed by experts that the multifractal spectrum and the Renyi dimensions of a measure encode important information about the measure, and it is therefore of considerable importance to find explicit formulas for these quantities. In [29,37–39] the authors used the zeta-function technique introduced and pioneered by M. Lapidus et al in the intriguing books [27,28] in order to find explicit formulas for the Renyi dimensions of a self-similar measure. At this point we note that it is generally believed that analysing the multifractal spectrum of a measure is considerably more difficult and challenging than analysing its Renyi dimensions, and the main purpose of this paper is to address the substantially more difficult problem of finding explicit formulas for the multifractal spectrum of a self-similar measure similar to the explicit formulas for its Renyi dimensions found in [29,37–39]. In particular, and as a first step in this direction, we introduce

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multifractal zeta-functions providing precise information of very general classes of multifractal spectra, including, for example, the multifractal spectra of self-conformal measures and the multifractal spectra of ergodic Birkhoff averages of continuous functions. More precisely, we prove that these, and more general multifractal spectra, equal the abscissae of convergence of the associated zeta-functions.

1.1. The first ingredient in multifractal analysis: multifractal spectra

For a Borel measure μ on $\mathbb{R}^d$ with support equal to K and a positive number α, let us consider the set $\Delta_\mu(\alpha)$ of those points x in $\mathbb{R}^d$ for which the measure $\mu(B(x,r))$ of the ball $B(x,r)$ with center x and radius r behaves like r^α for small r, i.e. the set

$$\Delta_\mu(\alpha) = \left\{ x \in K \,\middle|\, \lim_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}.$$

If the intensity of the measure μ varies very widely, it may happen that the sets $\Delta_\mu(\alpha)$ display a fractal-like character for a range of values of α. In this case it is natural to study the Hausdorff dimension of the sets $\Delta_\mu(\alpha)$ as α varies. We therefore define the multifractal spectrum of μ by

$$f_\mu(\alpha) = \dim_{\mathsf{H}} \Delta_\mu(\alpha)\,, \tag{2.1}$$

where $\dim_{\mathsf{H}}$ denotes the Hausdorff dimension. Here and below we use the following convention, namely, we define the Hausdorff dimension of the empty set to be $-\infty$, i.e. we put

$$\dim_{\mathsf{H}} \varnothing = -\infty.$$

One of the main problems in multifractal analysis is to study this and related functions. The function $f_\mu(\alpha)$ was first explicitly defined by the physicists Halsey et al. in 1986 in their seminal paper [16].

The multifractal spectrum f_μ is defined using the Hausdorff dimension. There is an alternative approach using "box-counting" arguments leading to the coarse multifractal spectrum. Namely, for a Borel probability measure μ on $\mathbb{R}^d$ with support equal to K and a real number α, the coarse multifractal spectrum is defined as follows. For positive real numbers $r > 0$ and $\delta > 0$, we write

$$N_{\mu,\delta}(\alpha; r) = \sup \left\{ |I| \,\middle|\, \begin{array}{l} (B(x_i,\delta))_{i\in I} \text{ is a finite family of balls such that:} \\ x_i \in K \text{ for all } i. \\ \\ B(x_i,\delta) \cap B(x_j,\delta) = \varnothing \quad \text{for all } i \neq j, \end{array} \right.$$

$$\alpha - r \leq \frac{\log \mu(B(x_i,\delta))}{\log \delta} \leq \alpha + r \quad \text{for all } i \Bigg\} , \tag{1.3}$$

and define the r-approximate coarse multifractal spectrum $f_\mu^{\mathsf{c}}(\alpha;r)$ of μ by

$$f_\mu^{\mathsf{c}}(\alpha;r) = \liminf_{\delta\searrow 0} \frac{\log N_{\mu,\delta}(\alpha;r)}{-\log\delta}. \tag{1.4}$$

Finally, the coarse multifractal spectrum $f_\mu^{\mathsf{c}}(\alpha)$ of μ is defined by

$$f_\mu^{\mathsf{c}}(\alpha) = \lim_{r\searrow 0} f_\mu^{\mathsf{c}}(\alpha;r) \tag{1.5}$$

(it is clear that this limit exists since $f_\mu^{\mathsf{c}}(\alpha;r)$ is a monotone function of r). We note that it is easily seen that

$$f_\mu(\alpha) \leq f_\mu^{\mathsf{c}}(\alpha),$$

and that this inequality may be strict, see, for example, [10].

1.2. The second ingredient in multifractal analysis: Renyi dimensions

Renyi dimensions quantify the varying intensity of a measure by analyzing its moments at different scales. Formally, Renyi dimensions are defined as follows. Let μ be a Borel measure on $\mathbb{R}^d$. For $E \subseteq \mathbb{R}^d$, $q \in \mathbb{R}$ and $\delta > 0$, we define the q-moment $M_{\mu,\delta}(q;E)$ of μ on E at scale δ by

$$M_{\mu,\delta}(q;E) = \sup\left\{ \sum_{i\in I} \mu(B(x_i,\delta))^q \,\middle|\, \begin{array}{l} (B(x_i,\delta))_{i\in I} \text{ is a finite family of balls such that:} \\ x_i \in K \quad \text{for all} \quad i. \\ B(x_i,\delta)\cap B(x_j,\delta) = \varnothing \quad \text{for all } i \neq j \end{array} \right\},$$

We now define the lower and upper Renyi spectra $\underline{\tau}_\mu(\cdot;E), \overline{\tau}_\mu(\cdot;E) : \mathbb{R} \to [-\infty,\infty]$ of μ by

$$\underline{\tau}_\mu(q;E) = \liminf_{\delta\searrow 0} \frac{\log M_{\mu,\delta}(q;E)}{-\log\delta},$$

$$\overline{\tau}_\mu(q;E) = \limsup_{\delta\searrow 0} \frac{\log M_{\mu,\delta}(q;E)}{-\log\delta}.$$

If E equals the support $\operatorname{supp}\mu$ of μ, then we will use the following shorter notation

$$M_{\mu,\delta}(q) = M_{\mu,\delta}(q;\operatorname{supp}\mu), \quad \underline{\tau}_\mu(q) = \underline{\tau}_\mu(q;\operatorname{supp}\mu), \quad \overline{\tau}_\mu(q) = \overline{\tau}_\mu(q;\operatorname{supp}\mu).$$

We note that the q-moment $M_{\mu,\delta}(q;E)$ is closely related to the box dimension $\dim_{\mathsf{B}} E$ of E. Indeed, if we let $M_\delta(E)$ denote the greatest number of pairwise disjoint balls of radii δ with centers in E, then it follows from the definition of the box dimension that $\dim_{\mathsf{B}} E = \lim_{\delta\to 0} \frac{\log M_\delta(E)}{-\log\delta}$ (provided the limit exists) and we clearly have

$$M_\delta(E) = M_{\mu,\delta}(0; E). \tag{1.6}$$

It is also possible to define an integral version of the q-moments $M_{\mu,\delta}(q; E)$. Namely, for $E \subseteq \mathbb{R}^d$, $q \in \mathbb{R}$ and $\delta > 0$, we define the integral q-moment $V_{\mu,\delta}(q)$ of μ on E at scale δ by

$$V_{\mu,\delta}(q; E) = \int\limits_{B(E,\delta)} \mu(B(x,\delta))^q \, d\mathcal{L}^d(x)$$

where $B(E,\delta) = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x,E) \le \delta\}$ and $\mathcal{L}^d$ denotes the Lebesgue measure in $\mathbb{R}^d$. We now define the lower and upper integral Renyi spectra $\underline{T}_\mu(\cdot; E), \overline{T}_\mu(\cdot; E) : \mathbb{R} \to [-\infty, \infty]$ of μ by

$$\underline{T}_\mu(q; E) = \liminf_{\delta\searrow 0} \frac{\log V_{\mu,\delta}(q; E)}{-\log\delta},$$

$$\overline{T}_\mu(q; E) = \limsup_{\delta\searrow 0} \frac{\log V_{\mu,\delta}(q; E)}{-\log\delta}.$$

As above, if E equals the support $\operatorname{supp}\mu$ of μ, then we will use the following shorter notation

$$V_{\mu,\delta}(q) = V_{\mu,\delta}(q; \operatorname{supp}\mu), \quad \underline{T}_\mu(q) = \underline{T}_\mu(q; \operatorname{supp}\mu), \quad \overline{T}_\mu(q) = \overline{T}_\mu(q; \operatorname{supp}\mu).$$

As above, we note that the integral q-moment $V_{\mu,\delta}(q; E)$ is also closely related to the Minkowski volume of E and the box dimension $\dim_{\mathsf{B}} E$ of E. Namely, if we let $V_\delta(E)$ denote the δ approximate Minkowski volume of E, i.e. $V_\delta(E) = \mathcal{L}^d(\, B(E,\delta)\,)$, then it is well-known that $\dim_{\mathsf{B}} E = \lim_{\delta\to 0} \frac{\log(\frac{1}{r^d} V_\delta(E))}{-\log\delta}$ (provided the limit exists) and we clearly have

$$V_\delta(E) = V_{\mu,\delta}(0; E). \tag{1.7}$$

1.3. The multifractal formalism

Based on a remarkable insight together with a clever heuristic argument, it was suggested by theoretical physicists Halsey et al. [16] that the multifractal spectra f_μ and f_μ^{c} can be computed using the Renyi dimensions. This result is known as the "Multifractal Formalism" in the physics literature. More precisely, the "Multifractal Formalism" says that the multifractal spectra equal the Legendre transform of the Renyi dimensions. Recall that if $\varphi : \mathbb{R} \to \mathbb{R}$ is a real valued function, then the Legendre transform $\varphi^* : \mathbb{R} \to [-\infty, \infty]$ of φ is defined by

$$\varphi^*(x) = \inf_y (xy + \varphi(y)). \tag{1.8}$$

We can now state the "Multifractal Formalism".

The multifractal formalism—a physics Folklore theorem The multifractal spectrum f_μ of μ and the coarse multifractal spectrum f_μ^{c} of μ equal the Legendre transforms $\underline{\tau}_\mu^*$, $\overline{\tau}_\mu^*$, $(\underline{T}_\mu)^*$ and $(\overline{T}_\mu)^*$ of the Renyi dimensions, i.e.

$$f_\mu(\alpha) = f_\mu^{\mathsf{c}}(\alpha) = \underline{\tau}_\mu^*(\alpha) = \overline{\tau}_\mu^*(\alpha) = \underline{T}_\mu^*(\alpha) = \overline{T}_\mu^*(\alpha)$$

for all α.

During the past 20 years there has been an enormous interest in verifying the Multifractal Formalism and computing the multifractal spectra of measures in the mathematical literature. In the mid 1990's Cawley and Mauldin [6] and Arbeiter and Patzschke [1] verified the Multifractal Formalism for self-similar measures satisfying the OSC, and within the last 15 years the multifractal spectra of various classes of measures in the Euclidean space $\mathbb{R}^d$ exhibiting some degree of self-similarity have been computed rigorously, cf. the textbooks [11, 43] and the references therein. Summarizing the previous paragraph somewhat more succinctly, we can say that previous work has almost entirely concentrated on the following problem:

Previous work: Previous work has concentrated on finding the limiting behaviour of the following ratios, namely,

$$\frac{\log M_{\mu,\delta}(q)}{-\log\delta}$$

and

$$\frac{\log N_{\mu,\delta}(\alpha;r)}{-\log\delta}.$$

Indeed, computing the Renyi dimensions $\underline{\tau}_\mu(q)$ and $\overline{\tau}_\mu(q)$ involves analysing the limiting behaviour of $\frac{\log M_{\mu,r}(q)}{-\log r}$, and computing the coarse multifractal spectrum $f_\mu^{\mathsf{c}}(\alpha;r)$ involves analysing the limiting behaviour of $\frac{\log N_{\mu,\delta}(\alpha;r)}{-\log\delta}$. Due to the importance of the quantities $M_{\mu,\delta}(q)$ and $N_{\mu,\delta}(\alpha;r)$ it is clearly desirable not only to find expressions for the limiting behaviour of $\frac{\log M_{\mu,\delta}(q)}{-\log\delta}$ and $\frac{\log N_{\mu,\delta}(\alpha;r)}{-\log\delta}$, but to find explicit expressions for the quantities $M_{\mu,\delta}(q)$ and $N_{\mu,\delta}(\alpha;r)$ themselves. The purpose of this work can be seen as a first step in this direction. Again, summarizing this somewhat more succinctly, in the present work we concentrate on the following problem:

Present work: This work explores methods of finding explicit expressions for

$$M_{\mu,\delta}(q)$$

and

$$N_{\mu,\delta}(\alpha;r).$$

It is clear that finding explicit expressions for $M_{\mu,\delta}(q)$ and $N_{\mu,\delta}(\alpha;r)$ is a more challenging undertaking than determining the limiting behaviour of the ratios $\frac{\log M_{\mu,\delta}(q)}{-\log\delta}$ and $\frac{\log N_{\mu,\delta}(\alpha;r)}{-\log\delta}$; indeed, if explicit expressions for $M_{\mu,\delta}(q)$ and $N_{\mu,\delta}(\alpha;r)$ are known, then the limiting behaviour of the ratios $\frac{\log M_{\mu,\delta}(q)}{-\log\delta}$ and $\frac{\log N_{\mu,\delta}(\alpha;r)}{-\log\delta}$ can be computed directly from these expressions.

We will now describe our strategy for analysing the quantities $M_{\mu,\delta}(q)$ and $N_{\mu,\delta}(\alpha;r)$. Very loosely speaking, the quantities $M_{\mu,\delta}(q)$ and $N_{\mu,\delta}(\alpha;r)$ "count" the number of balls $B(x,\delta)$ satisfying certain conditions. There are two distinct and widely used techniques for analysing the asymptotic behaviour of such (and similar) "counting functions", namely, (1) using ideas from renewal theory or (2) using the Mellin transform and the residue theorem to express the "counting functions" as sums involving the residues of suitably defined zeta-functions. Indeed, renewal theory techniques were introduced and pioneered by Lalley [19–21] in the 1980's, and later investigated further by Gatzouras [15], Winter [48] and most recently Kesseböhmer and Kombrink [18], in order to analyse the asymptotic behaviour of the "counting function" $M_\delta(E) = M_{\mu,\delta}(0,E) = M_{\mu,\delta}(0)$ for self-similar sets E (see (1.6)) and similar "counting functions" from fractal geometry. However, while renewal theory techniques are powerful tools for analysing the asymptotic behaviour of "counting functions", they do not yield "explicit" formulas. This is clearly unsatisfactory and it would be desirable if "explicit" expressions could be found. However, in spite of the difficulties, the problem of finding "explicit" formulas of "counting functions" in fractal geometry has recently attracted considerable interest. In particular, Lapidus and collaborators [22–24,27,28] have with spectacular success during the past 20 years pioneered the use of applying the Mellin transform to suitably defined zeta-functions in order to obtain explicit formulas for the Minkowski volume $V_\delta(E) = V_{\mu,\delta}(0,E) = V_{\mu,\delta}(0)$ of self-similar fractal subsets E of the line (see (1.7)).

It would clearly be desirable if similar formulas could be found for the multifractal quantities $M_{\mu,\delta}(q)$ and $N_{\mu,\delta}(\alpha;r)$ of self-similar (and more general) multifractal measures μ. In multifractal analysis it is generally believed that analysing the q-moments $M_{\mu,\delta}(q)$ and the associated Renyi dimensions $\underline{\tau}^*_\mu(\alpha)$ and $\overline{\tau}^*_\mu(\alpha)$ is less difficult than analysing the "counting function" $N_{\mu,\delta}(\alpha;r)$ and the associated multifractal spectra f_μ and f^{c}_μ. Indeed, in [29,37] (see also the surveys [38,39]) the authors introduced a one-parameter family of multifractal zeta-functions and established explicit formulas for the integral q-moments $V_{\mu,\delta}(q)$ expressing $V_{\mu,\delta}(q)$ as a sum involving the residues of these zeta-functions, and in [34] the asymptotic behaviour of the q-moments $M_{\mu,\delta}(q)$ were analysed using techniques from renewal theory. In addition, we note that

Lapidus and collaborators have introduced various intriguing multifractal zeta-functions [25, 26]. However, the multifractal zeta-functions in [25, 26] serve very different purposes and are significantly different from the multifractal zeta-functions introduced in [29, 35, 37]. The purpose of this paper is to address the significantly more difficult and challenging problem of performing a similar analysis of the multifractal spectrum "counting function" $N_{\mu,\delta}(\alpha;r)$. In particular, the final aim is to introduce a class of multifractal zeta-functions allowing us to derive explicit formulas for the "counting function" $N_{\mu,\delta}(\alpha;r)$ expressing $N_{\mu,\delta}(\alpha;r)$ as a sum involving the residues of these zeta-functions. As a first step in this direction, in this work we introduce multifractal zeta-functions providing precise information of very general classes of multifractal spectra, including, for example, the spectra f_μ and f_μ^{c} of self-similar multifractal measures μ. More precisely, we prove that the multifractal spectra equal the abscissae of convergence of the associated zeta-functions. It is our hope that a more careful analysis of these zeta-functions will provide explicit formulas for the "counting function" $N_{\mu,\delta}(\alpha;r)$ allowing us to express $N_{\mu,\delta}(\alpha;r)$ as a sum involving the residues of these zeta-functions; this will be explored in [32]. In order to illustrate the ideas involved we now consider a simple example.

1.4. An example illustrating the ideas: self-similar measures

To illustrate the above ideas in a simple setting, we consider the following example involving self-similar measures. Recall, that self-similar measures are defined as follows. Let $(S_1,\dots,S_N)$ be a list of contracting similarities $S_i:\mathbb{R}^d\to\mathbb{R}^d$ and let r_i denote the similarity ratio of S_i. Also, let $(p_1,\dots,p_N)$ be a probability vector. Then there is a unique Borel probability measure μ on $\mathbb{R}^d$ such that

$$\mu=\sum_i p_i\mu\circ S_i^{-1}\,, \tag{1.9}$$

see [10, 17]. The measure μ is called the self-similar measure associated with the list $(S_1,\dots,S_N,p_1,\dots,p_N)$. If the so-called Open Set Condition (OSC) is satisfied, then the multifractal spectra f_μ and f_μ^{c} are given by the following formula. Namely, if the OSC is satisfied and if we define $\beta:\mathbb{R}\to\mathbb{R}$ by

$$\sum_i p_i^q r_i^{\beta(q)}=1\,, \tag{1.10}$$

then it follows from, [6, 42] that

$$f_\mu(\alpha)=f_\mu^{\mathsf{c}}(\alpha)=\beta^*(\alpha)$$

for all $\alpha\in\mathbb{R}$ where β^* denotes the Legendre transform of β (recall, that the definition of the Legendre transform is given in (1.8)).

For $\alpha \in \mathbb{R}$, we are now attempting to introduce a "natural" self-similar multifractal zeta-function $\zeta_{\alpha}^{\mathsf{sim}}$ whose abscissa of convergence equals $f_{\mu}(\alpha)$. To do this we first introduce the following notation. Write $\Sigma^{*} = \{\mathbf{i} = i_1 \cdots i_n \,|\, n \in \mathbb{N}\,,\, i_j \in \{1, \ldots, N\}\}$ i.e. Σ^{*} is the set of all finite strings $\mathbf{i} = i_1 \cdots i_n$ with $n \in \mathbb{N}$ and $i_j \in \{1, \ldots, N\}$. For a finite string $\mathbf{i} = i_1 \cdots i_n \in \Sigma^{*}$ of length n, we write $|\mathbf{i}| = n$, and we write $r_{\mathbf{i}} = r_{i_1} \cdots r_{i_n}$ and $p_{\mathbf{i}} = p_{i_1} \cdots p_{i_n}$. With this notation, we can now motivate the introduction of a "natural" multifractal zeta-function as follows. Namely, since $f_{\mu}(\alpha)$ measures the size of the set of points x for which $\lim_{\delta \searrow 0} \frac{\log \mu(B(x,\delta))}{\log \delta} = \alpha$ and since $\frac{\log \mu(B(x,\delta))}{\log \delta}$ has the same form as $\frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}}$, it is natural to define the self-similar multifractal zeta-function $\zeta_{\alpha}^{\mathsf{sim}}$ by

$$\zeta_{\alpha}^{\mathsf{sim}}(s) = \sum_{\substack{\mathbf{i} \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} = \alpha}} r_{\mathbf{i}}^{s} \tag{1.11}$$

for those complex numbers s for which the series converges absolutely. An easy and straightforward calculation (which we present below) shows that the abscissa of convergence $\sigma_{\mathsf{ab}}(\,\zeta_{\alpha}^{\mathsf{sim}}\,)$ of $\zeta_{\mu}^{\mathsf{sim}}$ is less than $f_{\mu}(\alpha)$, i.e.

$$\sigma_{\mathsf{ab}}(\,\zeta_{\alpha}^{\mathsf{sim}}\,) \leq f_{\mu}(\alpha) = f_{\mu}^{\mathsf{c}}(\alpha). \tag{1.12}$$

Indeed, if $\alpha \notin [\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i}]$, then it is easily seen that for all $\mathbf{i} \in \Sigma^{*}$, we have $\frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \neq \alpha$, whence $\sigma_{\mathsf{ab}}(\,\zeta_{\alpha}^{\mathsf{sim}}\,) = -\infty$, and inequality (1.12) is therefore trivially satisfied. On the other hand, if $\alpha \in [\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i}]$, then it follows from [6, 10, 42] that there is a (unique) $q \in \mathbb{R}$ with $f_{\mu}(\alpha) = f_{\mu}^{\mathsf{c}}(\alpha) = \alpha q + \beta(q)$. Hence, for each $\varepsilon > 0$, we have (using the fact that $\sum_i p_i^q r_i^{\beta(q)+\varepsilon} < 1$)

$$\begin{aligned}
\zeta_{\alpha}^{\mathsf{sim}}\big(\, f_{\mu}(\alpha) + \varepsilon \,\big) &= \sum_{\substack{\mathbf{i} \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} = \alpha}} r_{\mathbf{i}}^{f_{\mu}(\alpha)+\varepsilon} \\
&= \sum_{\substack{\mathbf{i} \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} = \alpha}} r_{\mathbf{i}}^{\alpha q + \beta(q) + \varepsilon} \\
&= \sum_{\substack{\mathbf{i} \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} = \alpha}} p_{\mathbf{i}}^{q} r_{\mathbf{i}}^{\beta(q)+\varepsilon} \\
&\leq \sum_{\mathbf{i}} p_{\mathbf{i}}^{q} r_{\mathbf{i}}^{\beta(q)+\varepsilon}
\end{aligned}$$

$$= \sum_n \sum_{|\mathbf{i}|=n} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)+\varepsilon}$$
$$= \sum_n \left(\sum_i p_i^q r_i^{\beta(q)+\varepsilon} \right)^n$$
$$< \infty.$$

This shows that $\sigma_{\mathsf{ab}}(\,\zeta_\alpha^{\mathsf{sim}}\,) \le f_\mu(\alpha) + \varepsilon$. Letting $\varepsilon \searrow 0$, now gives $\sigma_{\mathsf{ab}}(\,\zeta_\alpha^{\mathsf{sim}}\,) \le f_\mu(\alpha)$. This proves (1.12).

However, it is also clear that we, in general, do not have equality in (1.12). Indeed, the set $\{\frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \,|\, \mathbf{i} \in \Sigma^*\}$ is clearly countable (because Σ^* is countable) and if $\alpha \in \mathbb{R} \setminus \{\frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \,|\, \mathbf{i} \in \Sigma^*\}$, then $\sigma_{\mathsf{ab}}(\,\zeta_\alpha\,) = -\infty$ (because the series (1.11) that defines $\zeta_\alpha^{\mathsf{sim}}(s)$ is obtained by summing over the empty set). Since it also follows from [6,10,42] that $f_\mu(\alpha) = f_\mu^{\mathsf{c}}(\alpha) > 0$ for all $\alpha \in (\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i})$, we therefore conclude that:

$$\sigma_{\mathsf{ab}}(\,\zeta_\alpha^{\mathsf{sim}}\,) = -\infty < 0 < f_\mu(\alpha) = f_\mu^{\mathsf{c}}(\alpha)$$
$$\text{for all except at most countably many } \alpha \in (\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i}). \tag{1.13}$$

It follows from the above discussion that while the definition of $\zeta_\alpha^{\mathsf{sim}}(s)$ is "natural", it does not encode sufficient information for us to recover the multifractal spectra $f_\mu(\alpha)$ and $f_\mu^{\mathsf{c}}(\alpha)$. The reason for the strict inequality in (1.13) is, of course, clear: even though there are no strings $\mathbf{i} \in \Sigma^*$ for which the ratio $\frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}}$ equals α if $\alpha \in (\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i}) \setminus \{\frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \,|\, \mathbf{i} \in \Sigma^*\}$, there are nevertheless many sequences $(\mathbf{i}_n)_n$ of strings $\mathbf{i}_n \in \Sigma^*$ for which the ratios $\frac{\log p_{\mathbf{i}_n}}{\log r_{\mathbf{i}_n}}$ converge to α. In order to capture this, it is necessary to ensure that those strings $\mathbf{i}$ for which the ratio $\frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}}$ is "close" to α are also included in the series defining the multifractal zeta-function. For this reason, we modify the definition of $\zeta_\alpha^{\mathsf{sim}}$ and introduce a self-similar multifractal zeta-function obtained by replacing the original small "target" set $\{\alpha\}$ by a larger "target" set I (for example, we may choose the enlarged "target" set I to be a non-degenerate interval). In order to make this idea precise we proceed as follows. For a closed interval I, we define the self-similar multifractal zeta-function ζ_I^{sim} by

$$\zeta_I^{\mathsf{sim}}(s) = \sum_{\substack{\mathbf{i} \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \in I}} r_{\mathbf{i}}^s \tag{1.14}$$

for those complex numbers s for which the series converges absolutely. Observe that if $I=\{\alpha\}$, then

$$\zeta_I^{\mathsf{sim}}(s)=\zeta_\alpha^{\mathsf{sim}}(s).$$

We can now proceed in two equally natural ways. Either, we can consider a family of enlarged "target" sets shrinking to the original main "target" $\{\alpha\}$; this approach will be referred to as the shrinking target approach. Or, alternatively, we can consider a fixed enlarged "target" set and regard this as our original main "target"; this approach will be referred to as the fixed target approach. We now discuss these approaches in more detail.

(1) The shrinking target approach. For a given (small) "target" $\{\alpha\}$, we consider the following family $\big([\alpha-r,\alpha+r]\big)_{r>0}$ of enlarged "target" sets $[\alpha-r,\alpha+r]$ shrinking to the original main "target" $\{\alpha\}$ as $r\searrow 0$, and attempt to relate the limiting behaviour of the abscissa convergence of $\zeta_{[\alpha-r,\alpha+r]}^{\mathsf{sim}}$ to the multifractal spectrum $f_\mu(\alpha)$ at α. In order to make this idea formal we proceed as follows. For each $\alpha\in\mathbb{R}$ and for each $r>0$, we define the zeta-function $\zeta_\alpha^{\mathsf{sim}}(\cdot;r)$ by

$$\begin{aligned}\zeta_\alpha^{\mathsf{sim}}(s;r)&=\zeta_{[\alpha-r,\alpha+r]}^{\mathsf{sim}}(s)\\&=\sum_{\substack{\mathbf{i}\\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}}\in[\alpha-r,\alpha+r]}} r_{\mathbf{i}}^s\\&=\sum_{\substack{\mathbf{i}\\ \left|\frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}}-\alpha\right|\le r}} r_{\mathbf{i}}^s.\end{aligned}\tag{1.15}$$

The next result, which is an application of one of our main results (namely Theorem 3.6), shows that the multifractal zeta-functions $\zeta_\alpha^{\mathsf{sim}}(\cdot;r)$ encode sufficient information for us to recover the multifractal spectra $f_\mu(\alpha)$ and $f_\mu^{\mathsf{c}}(\alpha)$ by letting $r\searrow 0$.

Theorem 1.1. (Shrinking targets) *Assume that the list $(S_1,\dots S_N)$ satisfies the OSC and let μ be the self-similar measure defined by (1.9). For $\alpha\in\mathbb{R}$ and $r>0$, let $\zeta_\alpha^{\mathsf{sim}}(\cdot;r)$ be defined by (1.15). Then we have*

$$\lim_{r\searrow 0}\sigma_{\mathsf{ab}}\big(\zeta_\alpha^{\mathsf{sim}}(\cdot;r)\big)=f_\mu(\alpha)=f_\mu^{\mathsf{c}}(\alpha)$$

where $\sigma_{\mathsf{ab}}\big(\zeta_\alpha^{\mathsf{sim}}(\cdot;r)\big)$ denotes the abscissa of convergence of the zeta-function $\zeta_\alpha^{\mathsf{sim}}(\cdot;r)$.

(2) The fixed target approach. Alternatively we can keep the enlarged "target" set I fixed and attempt to relate the abscissa of convergence of the multifractal

zeta-function ζ_I^{sim} associated with the enlarged "target" set I to the values of the multifractal spectrum $f_\mu(\alpha)$ for $\alpha \in I$. Of course, inequality (1.13) shows that if the "target" set I is "too small", then this is not possible. However, if the enlarged "target" set I satisfies a mild non-degeneracy condition, namely condition (1.16), guaranteeing that I is sufficiently "big", then the next result, which is also an application of one of our main results (namely Theorem 3.6), shows that this is possible. More precisely the result shows that if the enlarged "target" set I satisfies condition (1.16), then the multifractal zeta-function ζ_I^{sim} associated with the enlarged "target" set I encode sufficient information for us to recover the suprema $\sup_{\alpha\in I} f_\mu(\alpha)$ and $\sup_{\alpha\in I} f_\mu^{\mathsf{c}}(\alpha)$ of the multifractal spectra $f_\mu(\alpha)$ and $f_\mu^{\mathsf{c}}(\alpha)$ for $\alpha \in I$.

Theorem 1.2. (Fixed targets) *Assume that the list* $(S_1, \ldots S_N)$ *satisfies the OSC and let* μ *be the self-similar measure defined by* (1.9). *For a closed interval* I, *let* ζ_I^{sim} *be defined by* (1.14). *If*

$$\mathring{I} \cap \left(\min_i \tfrac{\log p_i}{\log r_i}, \max_i \tfrac{\log p_i}{\log r_i} \right) \neq \varnothing \tag{1.16}$$

(where $\mathring{I}$ *denotes the interior of* I*), then we have*

$$\sigma_{\mathsf{ab}}\left(\zeta_I^{\mathsf{sim}} \right) = \sup_{\alpha\in I} f_\mu(\alpha) = \sup_{\alpha\in I} f_\mu^{\mathsf{c}}(\alpha)$$

where $\sigma_{\mathsf{ab}}\left(\zeta_I^{\mathsf{sim}} \right)$ *denotes the abscissa of convergence of the zeta-function* ζ_I^{sim}.

We emphasise that Theorems 1.1 and 1.2 are presented in order to motivate this work and are special cases of the substantially more general theory of multifractal zeta-functions developed in this paper.

The next section, i.e. Sect. 2, describes the general framework developed in this paper and list our main results. In Sect. 3 we will discuss a number of examples, including, mixed and non-mixed multifractal spectra of self-similar and self-conformal measures, and multifractal spectra of Birkhoff ergodic averages.

2. Statements of main results

2.1. Main definitions: the zeta-functions $\zeta_C^{U,\Lambda}(\cdot)$ and $\zeta_C^{U,\Lambda}(\cdot; r)$

In this section we describe the framework developed in this paper and list our main results. We first recall and introduce some useful notation. Fix a positive integer N. Let $\Sigma = \{1, \ldots, N\}$ and for a positive integer n, write

$$\Sigma^n = \{1, \ldots, N\}^n ,$$
$$\Sigma^* = \bigcup_m \Sigma^m ,$$

i.e. Σ^n is the family of all strings $\mathbf{i} = i_1 \cdots i_n$ of length n with $i_j \in \{1, \ldots, N\}$ and Σ^* is the family of all finite strings $\mathbf{i} = i_1 \cdots i_m$ with $m \in \mathbb{N}$ and $i_j \in \{1, \ldots, N\}$. Also write

$$\Sigma^{\mathbb{N}} = \{1, \ldots, N\}^{\mathbb{N}},$$

i.e. $\Sigma^{\mathbb{N}}$ is the family of all infinite strings $\mathbf{i} = i_1 i_2 \ldots$ with $i_j \in \{1, \ldots, N\}$. For an infinite string $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$ and a positive integer n, we will write $\mathbf{i}|n = i_1 \cdots i_n$. In addition, for a positive integer n and a finite string $\mathbf{i} = i_1 \cdots i_n \in \Sigma^n$ with length equal to n, we will write $|\mathbf{i}| = n$, and we let $[\mathbf{i}]$ denote the cylinder generated by $\mathbf{i}$, i.e.

$$[\mathbf{i}] = \left\{ \mathbf{j} \in \Sigma^{\mathbb{N}} \,\middle|\, \mathbf{j}|n = \mathbf{i} \right\}.$$

Also, let $S : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ denote the shift map. Finally, we denote the family of Borel probability measures on $\Sigma^{\mathbb{N}}$ by $\mathcal{P}(\Sigma^{\mathbb{N}})$ and we equip $\mathcal{P}(\Sigma^{\mathbb{N}})$ with the weak topology.

The multifractal zeta-function framework developed in this paper depend on a space X and two maps U and Λ satisfying various conditions. We will now introduce the space X and the maps U and Λ.

(1) First, we fix a metric space X.
(2) Next, we fix a continuous map $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$.
(3) Finally, we fix a function $\Lambda : \Sigma^{\mathbb{N}} \to \mathbb{R}$ satisfying the following three conditions:
  (C1) The function Λ is continuous;
  (C2) There are constants $c_{\min}$ and $c_{\max}$ with $-\infty < c_{\min} \le c_{\max} < 0$ such that $c_{\min} \le \Lambda \le c_{\max}$;
  (C3) There is a constant c with $c \ge 1$ such that for all positive integers n and all $\mathbf{i}, \mathbf{j} \in \Sigma^{\mathbb{N}}$ with $\mathbf{i}|n = \mathbf{j}|n$, we have

$$\frac{1}{c} \le \frac{\exp \sum_{k=0}^{n-1} \Lambda S^k \mathbf{i}}{\exp \sum_{k=0}^{n-1} \Lambda S^k \mathbf{j}} \le c.$$

  Condition (C2) is clearly motivated by the hyperbolicity condition from dynamical systems, and Condition (C3) is equally clearly motivated by the bounded distortion property from dynamical systems.

Associated with the space X and the maps U and Λ, we now define the following multifractal zeta-functions.

Definition. (*The zeta-functions* $\zeta_C^{U,\Lambda}$ **and** $\zeta_C^{U,\Lambda}(\cdot; r)$ *associated with the space* X *and the maps* U *and* Λ) For a finite string $\mathbf{i} \in \Sigma^n$, let

$$s_{\mathbf{i}} = \sup_{\mathbf{k} \in [\mathbf{i}]} \exp \sum_{k=0}^{n-1} \Lambda S^k \mathbf{k},$$

and for a positive integer n and an infinite string $\mathbf{i} \in \Sigma^{\mathbb{N}}$, let $L_n : \Sigma^{\mathbb{N}} \to \mathcal{P}(\Sigma^{\mathbb{N}})$ be defined by

$$L_n\mathbf{i} = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{S^k\mathbf{i}}.$$

For $C \subseteq X$, we define the zeta-function $\zeta_C^{U,\Lambda}$ associated with the space X and the maps U and Λ by

$$\zeta_C^{U,\Lambda}(s) = \sum_{\substack{\mathbf{i} \\ UL_{|\mathbf{i}|}[\mathbf{i}] \subseteq C}} s_{\mathbf{i}}^s$$

for those complex numbers s for which the series converges absolutely, and for $r > 0$ and $C \subseteq X$, we define the zeta-function $\zeta_C^{U,\Lambda}(\cdot; r)$ associated with the space X and the maps U and Λ by

$$\begin{aligned}\zeta_C^{U,\Lambda}(s; r) &= \zeta_{B(C,r)}^{U,\Lambda}(s) \\ &= \sum_{\substack{\mathbf{i} \\ UL_{|\mathbf{i}|}[\mathbf{i}] \subseteq B(C,r)}} s_{\mathbf{i}}^s\end{aligned}$$

for those complex numbers s for which the series converges absolutely and where $B(C,r) = \{x \in X \mid \operatorname{dist}(x,C) \leq r\}$ denotes the closed neighborhood r of C.

Next, we formally define the abscissa of convergence (of a zeta-function).

Definition. (*Abscissa of convergence*) Let $(\, a_{\mathbf{i}} \,)_{\mathbf{i} \in \Sigma^*}$ be a family of positive numbers and define the (zeta-)function ζ by

$$\zeta(s) = \sum_{\mathbf{i}} a_{\mathbf{i}}^s$$

for those complex numbers s for which the series converges. The abscissa of convergence of ζ is defined by

$$\sigma_{\mathsf{ab}}(\zeta) = \inf \left\{ t \in \mathbb{R} \,\middle|\, \text{the series } \sum_{\mathbf{i}} a_{\mathbf{i}}^t \text{ converges absolutely} \right\}.$$

Our main results, i.e. Theorems 2.1 and 2.2 below, relate the abscissa of converge of the zeta-functions $\zeta_C^{U,\Lambda}(\cdot; r)$ and $\zeta_C^{U,\Lambda}$ to various multifractal quantities, including, the coarse multifractal spectrum associated with the space X and the maps U and Λ. In order to state Theorems 2.1 and 2.2 we will now define the coarse multifractal spectra.

Definition. (*The coarse multifractal spectra associated with the space X and the maps U and* Λ) For $\mathbf{i} = i_1 \cdots i_n \in \Sigma^*$, we let $\widehat{\mathbf{i}} = i_1 \cdots i_{n-1} \in \Sigma^*$ denote the "parent" of $\mathbf{i}$. Next, for $\mathbf{i} \in \Sigma^*$ and $\delta > 0$, we write

$$s_{\mathbf{i}} \approx \delta$$

if and only if $s_{\mathbf{i}} \leq \delta < s_{\widehat{\mathbf{i}}}$. For $r > 0$ and $C \subseteq X$, let

$$\Pi_{\delta}^{U,\Lambda}(C,r) = \left\{ \mathbf{i} \,\middle|\, s_{\mathbf{i}} \approx \delta \,,\, UL_{|\mathbf{i}|}[\mathbf{i}] \subseteq B(C,r) \right\}$$

and

$$N_{\delta}^{U,\Lambda}(C,r) = \left| \, \Pi_{\delta}^{U,\Lambda}(C,r) \, \right|.$$

We define the lower and upper r-approximate coarse multifractal spectrum associated with the space X and the maps U and Λ by

$$\underline{f}^{U,\Lambda}(C,r) = \liminf_{\delta \searrow 0} \frac{\log N_{\delta}^{U,\Lambda}(C,r)}{-\log \delta} \,,$$

$$\overline{f}^{U,\Lambda}(C,r) = \limsup_{\delta \searrow 0} \frac{\log N_{\delta}^{U,\Lambda}(C,r)}{-\log \delta} \,,$$

and we define the lower and upper coarse multifractal spectrum associated with the space X and the maps U and Λ by

$$\underline{f}^{U,\Lambda}(C) = \lim_{r \searrow 0} \underline{f}^{U,\Lambda}(C,r) \,,$$

$$\overline{f}^{U,\Lambda}(C) = \lim_{r \searrow 0} \overline{f}^{U,\Lambda}(C,r).$$

Below we state our main results. As suggested by the discussion in Sect. 1.4, we will attempt to relate the abscissae of convergence of the multifractal zeta-functions $\zeta_C^{U,\Lambda}$ and $\zeta_C^{U,\Lambda}(\cdot;r)$ to various multifractal spectra using two different but equally natural approaches: the shrinking target approach or the fixed target approach. The shrinking target approach is discussed in Sect. 2.2 and the fixed target approach is discussed in Sect. 2.3.

2.2. First main result: the shrinking target approach: finding $\lim_{r \searrow 0} \sigma_{\mathsf{ab}} \left(\zeta_C^{U,\Lambda}(\cdot;r) \right)$

For a given "target" C, we consider the following family $\big(B(C,r) \big)_{r>0}$ of enlarged "target" sets $B(C,r)$ shrinking to the original main "target" C as $r \searrow 0$, and attempt to relate the limiting behaviour of the abscissa of convergence of the zeta-function $\zeta_C^{U,\Lambda}(\cdot;r) = \zeta_{B(C,r)}^{U,\Lambda}$ to the coarse multifractal spectrum $\underline{f}^{U,\Lambda}(C)$ and other multifractal quatities. Our first main result, i.e. Theorem 2.1 below, shows that this approach is possible. More precisely, Theorem 2.1 shows that the abscissa of convergence of the zeta-function $\zeta_C^{U,\Lambda}(\cdot;r)$ converges as $r \searrow 0$, and that this limit equals the coarse multifractal spectrum of C. We also show that the limit can be obtained by a variational principle involving the supremum of the entropy of all shift invariant Borel probability measures $\mu \in \mathcal{P}(\Sigma^{\mathbb{N}})$ with $U\mu \in C$. In Sect. 3 we show that in many important cases the limit $\lim_{r \searrow 0} \sigma_{\mathsf{ab}} \big(\zeta_C^{U,\Lambda}(\cdot;r) \big)$ equals the traditional multifractal spectra.

Theorem 2.1. (Shrinking targets) *Let X be a metric space and let $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a closed subset of X.*

(1) *The lower coarse multifractal spectrum associated with the space X and the maps U and Λ: we have*

$$\lim_{r \searrow 0} \sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}(\cdot; r)\big) = \underline{f}^{U,\Lambda}(C).$$

(2) *The variational principle: we have*

$$\lim_{r \searrow 0} \sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}(\cdot; r)\big) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}}) \\ U\mu \in C}} -\frac{h(\mu)}{\int \Lambda \, d\mu};$$

here $\mathcal{P}_S(\Sigma^{\mathbb{N}})$ denotes the family of shift invariant Borel probability measures on $\Sigma^{\mathbb{N}}$ and $h(\mu)$ denotes the entropy of $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$.

In order to prove Theorem 2.1 it suffices to prove the following three inequalities:

$$\limsup_{r \searrow 0} \sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}(\cdot; r)\big) \le \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}}) \\ U\mu \in C}} -\frac{h(\mu)}{\int \Lambda \, d\mu}, \tag{2.1}$$

$$\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}}) \\ U\mu \in C}} -\frac{h(\mu)}{\int \Lambda \, d\mu} \le \underline{f}^{U,\Lambda}(C), \tag{2.2}$$

$$\underline{f}^{U,\Lambda}(C) \le \liminf_{r \searrow 0} \sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}(\cdot; r)\big). \tag{2.3}$$

Inequality (2.1) is proven in Sect. 5 using techniques from the theory of large deviations. Inequality (2.2) is proven in Sect. 6 using techniques from ergodic theory. Finally, inequality (2.3) follows directly from the definitions and is proved in Sect. 7.

2.3. Second main result: the fixed target approach: finding $\boldsymbol{\sigma}_{\mathsf{ab}}\big(\boldsymbol{\zeta_C^{U,\Lambda}}\big)$

Alternatively, instead of choosing a family of "target" sets that shrinks to the given "target" C, we can keep the given "target" set C fixed and attempt to relate the abscissa of convergence of the multifractal zeta-function $\zeta_C^{U,\Lambda}$ associated with the "target" set C to the values of the coarse multifractal spectrum $\underline{f}^{U,\Lambda}(C)$. Of course, the example in Sect. 1.4 shows that if the "target" set C is "too small", then this is not possible. However, if the coarse multifractal spectrum $\underline{f}^{U,\Lambda}$ satisfies a continuity condition at C guaranteeing that the interior of C is "sufficiently big", then our second main result, i.e. Theorem 2.2 below, shows that this approach is possible. More precisely, Theorem 2.2 shows that if the coarse multifractal spectrum $\underline{f}^{U,\Lambda}$ is inner continuous at C (the definition of inner continuity will be given below), then the abscissa of convergence of the zeta-function $\zeta_C^{U,\Lambda}$ equals the coarse multifractal spectrum of C. In analogy with Theorem 2.1, we also show that the abscissa of convergence of $\zeta_C^{U,\Lambda}$ can be obtained by a variational principle involving the supremum of the entropy of all shift invariant Borel probability measures $\mu \in \mathcal{P}(\Sigma^{\mathbb{N}})$ with $U\mu \in C$. However, before stating Theorem 2.2, we first define the continuity condition that the coarse multifractal spectrum $\underline{f}^{U,\Lambda}$ is required to satisfy.

Definition. (*Inner continuity*) Let $P(X)$ denote the family of subsets of X and for $C \subseteq X$ and $r > 0$, write

$$I(C,r) = \left\{ x \in C \,\middle|\, \operatorname{dist}(x, \partial C) \geq r \right\}.$$

We say that a function $\Phi : P(X) \to \mathbb{R}$ is inner continuous at $C \subseteq X$ if

$$\Phi\big(\, I(C,r) \,\big) \to \Phi(C) \quad \text{as } r \searrow 0.$$

We can now state Theorem 2.2.

Theorem 2.2. (Fixed targets) *Fix a positive integer M. Let $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to \mathbb{R}^M$ be continuous with respect to the weak topology. Let $C \subseteq \mathbb{R}^M$ be a closed subset of $\mathbb{R}^M$ and assume that $\underline{f}^{U,\Lambda}$ is inner continuous at C.*

(1) *The lower coarse multifractal spectrum associated with $\mathbb{R}^M$ and the maps U and Λ: we have*

$$\sigma_{\mathsf{ab}}\big(\, \zeta_C^{U,\Lambda} \,\big) = \underline{f}^{U,\Lambda}(C).$$

(2) *The variational principle: we have*

$$\sigma_{\mathsf{ab}}\big(\, \zeta_C^{U,\Lambda} \,\big) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}}) \\ U\mu \in C}} -\frac{h(\mu)}{\int \Lambda \, d\mu}\,;$$

here $\mathcal{P}_S(\Sigma^{\mathbb{N}})$ denotes the family of shift invariant Borel probability measures on $\Sigma^{\mathbb{N}}$ and $h(\mu)$ denotes the entropy of $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$.

Theorem 2.2 follows easily from Theorem 2.1 and is proved in Sect. 8.

2.4. Euler product

We will now prove that the multifractal zeta-function $\zeta_C^{U,\Lambda}$ has a natural Euler product. We begin with a definition.

Definition. (*Composite and prime*) A finite string $\mathbf{i} \in \Sigma^*$ is called composite (or peiodic) if there are $\mathbf{u} \in \Sigma^*$ and a positive integer $n > 1$ such that $\mathbf{i} = \mathbf{u} \cdots \mathbf{u}$ where $\mathbf{u}$ is repeated n times. A finite string $\mathbf{i} \in \Sigma^*$ is called prime if it is not composite.

Theorem 2.3 shows that $\zeta_C^{U,\Lambda}$ has an Euler product. In Theorem 2.3 we use the following notation, namely, if f is a holomorphic function that does not attain the value 0, then we let Lf denote the logarithmic derivative of f, i.e. $Lf = \frac{f'}{f}$. We can now state Theorem 2.3.

Theorem 2.3. (Euler product) *Let X be a metric space and let $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Assume that*

$$s_{\mathbf{ij}} = s_{\mathbf{i}} s_{\mathbf{j}}$$

for all $\mathbf{i}, \mathbf{j} \in \Sigma^*$. Let $C \subseteq X$ be a closed subset of X.

(1) *For complex numbers s with $\mathrm{Re}(s) > \sigma_{\mathsf{ab}}(\zeta_C^{U,\Lambda})$, the product*

$$Q_C^{U,\Lambda}(s) = \prod_{\substack{\mathbf{i} \\ \mathbf{i} \text{ is prime} \\ UL_{|\mathbf{i}|}[\mathbf{i}] \subseteq C}} \left(\frac{1}{1 - s_{\mathbf{i}}^s} \right)^{\frac{1}{\log s_{\mathbf{i}}}}$$

converges and $Q_C^{U,\Lambda}(s) \neq 0$. The product $Q_C^{U,\Lambda}(s)$ is called the Euler product of $\zeta_C^{U,\Lambda}$.

(2) *For all complex numbers s with $\mathrm{Re}(s) > \sigma_{\mathsf{ab}}(\zeta_C^{U,\Lambda})$, we have*

$$\zeta_C^{U,\Lambda}(s) = L\, Q_C^{U,\Lambda}(s).$$

Theorem 2.3 is proved in Sect. 9.

3. Applications: multifractal spectra of measures and multifractal spectra of ergodic Birkhoff averages

We will now consider several applications of Theorems 2.1 and 2.2 to multifractal spectra of measures and ergodic averages. In particular, we consider the following examples:

- Section 3.1: Multifractal spectra of self-conformal measures.
- Section 3.2: Mixed multifractal spectra of self-conformal measures.
- Section 3.3: Multifractal spectra of self-similar measures.
- Section 3.4: Multifractal spectra of ergodic Birkhoff averages.

3.1. Multifractal spectra of self-conformal measures

Since our examples are formulated in the setting of self-conformal (or self-similar) measures we begin be recalling the definition of self-conformal (and self-similar) measures. A conformal iterated function system is a list

$$\big(V \,,\, X \,,\, (S_i)_{i=1,\ldots,N} \big)$$

where

(1) V is an open, connected subset of $\mathbb{R}^d$.
(2) X is a compact set with $X \subseteq V$ and $X^{\circ\,-} = X$.
(3) $S_i : V \to V$ is a contractive $C^{1+\gamma}$ diffeomorphism with $0 < \gamma < 1$ such that $S_i X \subseteq X$ for all i.
(4) The Conformality Condition: For each $x \in V$, we have that $(DS_i)(x)$ is a contractive similarity map, i.e. there exists $r_i(x) \in (0,1)$ such that $|(DS_i)(x)u - (DS_i)(x)v| = r_i(x)|u - v|$ for all $u, v \in \mathbb{R}^d$; here $(DS_i)(x)$ denotes the derivative of S_i at x.

It follows from [17] that there exists a unique non-empty compact set K with $K \subseteq X$ such that

$$K = \bigcup_i S_i K. \tag{3.1}$$

The set K is called the self-conformal set associated with the list $\big(V\,,\,X, (S_i)_{i=1,\ldots,N}\,\big)$; in particular, if each map S_i is a contracting similarity, then the set K is called the self-similar set associated with the list $\big(V\,,\,X\,,\,(S_i)_{i=1,\ldots,N}\,\big)$. In addition, if $(p_i)_{i=1,\ldots,N}$ is a probability vector then it follows from [17] that there is a unique probability measure μ with $\operatorname{supp}\mu = K$ such that

$$\mu = \sum_i p_i\, \mu \circ S_i^{-1}. \tag{3.2}$$

The measure μ is called the self-conformal measure associated with the list $\big(V\,,\,X\,,\,(S_i)_{i=1,\ldots,N}\,,\,(p_i)_{i=1,\ldots,N}\,\big)$; if each map S_i is a contracting similarity, then the measure μ is called the self-similar measure associated with the list $\big(V\,,\,X\,,\,(S_i)_{i=1,\ldots,N}\,,\,(p_i)_{i=1,\ldots,N}\,\big)$. We will frequently assume that the list $\big(V\,,\,X\,,\,(S_i)_{i=1,\ldots,N}\,\big)$ satisfies the Open Set Condition defined below. Namely, the list $\big(V\,,\,X\,,\,(S_i)_{i=1,\ldots,N}\,\big)$ satisfies the Open Set Condition (OSC) if there exists an open, non-empty and bounded set O with $O \subseteq X$ and $S_i O \subseteq O$ for all i such that $S_i O \cap S_j O = \varnothing$ for all i, j with $i \neq j$.

Next, we define the natural projection map $\pi : \Sigma^{\mathbb{N}} \to K$. However, we first make the following definitions. Namely, for $\mathbf{i} = i_1 \cdots i_n \in \Sigma^*$, write

$$\begin{aligned} S_{\mathbf{i}} &= S_{i_1} \cdots S_{i_n}\,, \\ K_{\mathbf{i}} &= S_{\mathbf{i}} K. \end{aligned}$$

The natural projection map $\pi : \Sigma^{\mathbb{N}} \to K$ is now defined by

$$\Big\{\, \pi(\mathbf{i}) \,\Big\} = \bigcap_n S_{\mathbf{i}|n} K$$

for $\mathbf{i} \in \Sigma^{\mathbb{N}}$.

Finally, we collect the definitions and results from multifractal analysis of self-conformal measures that we need in order to state our main results. We first recall, that the Hausdorff multifractal spectrum f_μ of μ is defined by

$$f_\mu(\alpha) = \dim_{\mathsf{H}} \left\{ x \in K \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu B(x,r)}{\log r} = \alpha \right\},$$

for $\alpha \in \mathbb{R}$ where $\dim_{\mathsf{H}}$ denotes the Hausdorff dimension. In the late 1990's Patzschke [42], building on works by Cawley & Mauldin [6] and Arbeiter & Patzschke [1], succeeded in computing the multifractal spectra $f_\mu(\alpha)$ assuming the OSC. In order to state Patzschke's result we make the following definitions. Define $\Phi, \Lambda : \Sigma^{\mathbb{N}} \to \mathbb{R}$ by $\Phi(\mathbf{i}) = \log p_{i_1}$ and $\Lambda(\mathbf{i}) = \log |DS_{i_1}(\pi S\mathbf{i})|$ for $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$, and for $q \in \mathbb{R}$, let $\beta(q)$ be the unique real number such that

$$0 = P\big(\, \beta(q)\Lambda + q\Phi \,\big)\,;$$

here, and below, we use the following standard notation, namely if $\varphi : \Sigma^{\mathbb{N}} \to \mathbb{R}$ is a Hölder continuous function, then $P(\varphi)$ denotes the pressure of φ. Also, recall that the Legendre transform is defined in (1.8). We can now state Patzschke's result.

Theorem A. [P] *Let μ be defined by (3.2) and $\alpha \in \mathbb{R}$. If the OSC is satisfied, then we have*

$$f_\mu(\alpha) = \beta^*(\alpha).$$

Of course, in general, the limit $\lim_{r \searrow 0} \frac{\log \mu B(x,r)}{\log r}$ may not exist. Indeed, recently Barreira and Schmeling [4] (see also Olsen and Winter [40, 41], Xiao, Wu and Gao [49] and Moran [31]) have shown that the set of divergence points, i.e. the set

$$\Delta_\mu = \left\{ x \in K \,\middle|\, \text{the expression } \frac{\log \mu B(x,r)}{\log r} \text{ diverges as } r \searrow 0 \right\}$$

of points x for which the limit $\lim_{r \searrow 0} \frac{\log \mu B(x,r)}{\log r}$ does not exist, is typically highly "visible" and "observable", namely it has full Hausdorff dimension. More precisely, it follows from [4] that if the OSC is satisfied and t denotes the Hausdorff dimension of K, then

$$\left\{ x \in K \,\middle|\, \text{the expression } \frac{\log \mu B(x,r)}{\log r} \text{ diverges as } r \searrow 0 \right\} = \varnothing$$

provided μ is proportional to the t-dimensional Hausdorff measure restricted to K, and

$$\dim_{\mathsf{H}} \left\{ x \in K \,\middle|\, \text{the expression } \frac{\log \mu B(x,r)}{\log r} \text{ diverges as } r \searrow 0 \right\} = \dim_{\mathsf{H}} K$$

provided μ is not proportional to the t-dimensional Hausdorff measure restricted to K. This suggests that the set Δ_μ has a surprisingly rich and complex fractal structure, and in order to explore this more carefully Olsen and Winter [40, 41] introduced various generalised multifractal spectra functions designed to "see" different sets of divergence points. In order to define these spectra we introduce the following notation. If M is a metric space and $\varphi : (0,\infty) \to M$ is a function, then we write $\mathrm{acc}_{r\searrow 0} f(r)$ for the set of accumulation points of f as $r \searrow 0$, i.e.

$$\operatorname*{acc}_{r\searrow 0} \varphi(r) = \Big\{ x \in M \,\Big|\, x \text{ is an accumulation point of } f \text{ as } r \searrow 0 \Big\}.$$

In [40] Olsen and Winter introduced and investigated the generalised Hausdorff multifractal spectrum F_μ of μ defined by

$$F_\mu(C) = \dim_{\mathsf{H}} \left\{ x \in K \,\middle|\, \operatorname*{acc}_{r\searrow 0} \frac{\log \mu B(x,r)}{\log r} \subseteq C \right\}$$

for $C \subseteq \mathbb{R}$. Note that the generalised spectrum is a genuine extension of the traditional multifractal spectrum $f_\mu(\alpha)$, namely if $C = \{\alpha\}$ is a singleton consisting of the point α, then clearly $F_\mu(C) = f_\mu(\alpha)$. There is a natural divergence point analogue of Theorem A. Indeed, the following divergence point analogue of Theorem A was first obtained by Moran [31] and Olsen and Winter [40], and later in a less restrictive setting by Li, Wu and Xiong [30] (see also [5,46] for earlier but related results in a slightly different setting).

Theorem B. [30,31,40] *Let μ be defined by (3.2) and let C be a closed subset of $\mathbb{R}$. If the OSC is satisfied, then we have*

$$F_\mu(C) = \sup_{\alpha\in C} \beta^*(\alpha).$$

As a first application of Theorems 2.1 and 2.2 we obtain a zeta-function whose abscissa of convergence equals the generalised multifractal spectrum $F_\mu(C)$ of a self-conformal measure μ. The is the content of the next theorem.

Theorem 3.1. (Multifractal zeta-functions for multifractal spectra of self-conformal measures) *Let $(p_1,\ldots,p_N)$ be a probability vector, and let μ denote the self-conformal measure associated with the list $\big(V\,,\,X\,,\,(S_i)_{i=1,\ldots,N}, (p_i)_{i=1,\ldots,N}\,\big)$, i.e. μ is the unique probability measure such that $\mu = \sum_i p_{l,i}\mu\circ S_i^{-1}$.*

For $\mathbf{i} \in \Sigma^*$, let

$$s_{\mathbf{i}} = \sup_{\mathbf{u}\in\Sigma^{\mathbb{N}}} |DS_{\mathbf{i}}(\pi\mathbf{u})|.$$

For a closed set $C \subseteq \mathbb{R}$, we define the self-conformal multifractal zeta-function by

$$\zeta_C^{\mathsf{con}}(s) = \sum_{\substack{\mathbf{i}\\ \frac{\log p_{\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}}\in C}} s_{\mathbf{i}}^{s}\,,$$

For a closed set $C \subseteq \mathbb{R}$ and $r > 0$, we define the self-conformal multifractal zeta-function by

$$\begin{aligned}\zeta_C^{\mathsf{con}}(s;r) &= \zeta_{B(C,r)}^{\mathsf{con}}(s)\\ &= \sum_{\substack{\mathbf{i}\\ \operatorname{dist}\left(\frac{\log p_{\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}},\, C\right)\le r}} s_{\mathbf{i}}^{s}\,,\end{aligned}$$

and if $\alpha \in \mathbb{R}$ and $C = \{\alpha\}$ is the singleton consisting of α, then we write $\zeta_C^{\mathsf{con}}(s;r) = \zeta_\alpha^{\mathsf{con}}(s;r)$, i.e. we write

$$\zeta_\alpha^{\mathsf{con}}(s;r) = \sum_{\substack{\mathbf{i}\\ \left|\frac{\log p_{\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}} - \alpha\right|\le r}} s_{\mathbf{i}}^{s}\,.$$

Define $\Phi, \Lambda : \Sigma^{\mathbb{N}} \to \mathbb{R}$ by $\Phi(\mathbf{i}) = \log p_{i_1}$ and $\Lambda(\mathbf{i}) = \log|DS_{i_1}(\pi S\mathbf{i})|$ for $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$. Define $\beta : \mathbb{R}^M \to \mathbb{R}$ by

$$0 = P\big(\,\beta(q)\Lambda + q\Phi\,\big)$$

for $q \in \mathbb{R}$. Let C be a closed subset of $\mathbb{R}$. Then the following hold:

(1.1) *We have*

$$\lim_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\,\zeta_C^{\mathsf{con}}(\cdot;r)\,\big) = \sup_{\alpha\in C} \beta^*(\alpha).$$

In particular, if $\alpha \in \mathbb{R}$, then we have

$$\lim_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\,\zeta_\alpha^{\mathsf{con}}(\cdot;r)\,\big) = \beta^*(\alpha).$$

(1.2) *If the OSC is satisfied, then we have*

$$\begin{aligned}\lim_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\,\zeta_C^{\mathsf{con}}(\cdot;r)\,\big) &= \sup_{\alpha\in C} \dim_{\mathsf{H}}\left\{x\in K \,\middle|\, \lim_{r\searrow 0}\frac{\log\mu(B(x,r))}{\log r} = \alpha\right\}\\ &= \dim_{\mathsf{H}}\left\{x\in K \,\middle|\, \operatorname*{acc}_{r\searrow 0}\frac{\log\mu(B(x,r))}{\log r} \subseteq C\right\}.\end{aligned}$$

In particular, if the OSC is satisfied and $\alpha \in \mathbb{R}$, then we have

$$\lim_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\,\zeta_\alpha^{\mathsf{con}}(\cdot;r)\,\big) = \dim_{\mathsf{H}}\left\{x\in K \,\middle|\, \lim_{r\searrow 0}\frac{\log\mu(B(x,r))}{\log r} = \alpha\right\}.$$

(2.1) *If C is an interval and $\overset{\circ}{C} \cap \big(-\beta'(\mathbb{R})\big) \neq \varnothing$, then we have*

$$\sigma_{\mathsf{ab}}\big(\,\zeta_C^{\mathsf{con}}\,\big) = \sup_{\alpha\in C} \beta^*(\alpha).$$

(2.2) *If C is an interval and $\overset{\circ}{C} \cap \big(-\beta'(\mathbb{R})\big) \neq \varnothing$ and the OSC is satisfied, then we have*

$$\sigma_{\mathsf{ab}}\big(\zeta_C^{\mathsf{con}}\big) = \sup_{\alpha \in C} \dim_{\mathsf{H}} \left\{ x \in K \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}$$
$$= \dim_{\mathsf{H}} \left\{ x \in K \,\middle|\, \operatorname*{acc}_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} \subseteq C \right\}.$$

Proof. This follows immediately from the more general Theorem 3.2 in Sect. 3.2 by putting $M = 1$. □

3.2. Mixed multifractal spectra of self-conformal measures

Recently mixed (or simultaneous) multifractal spectra have generated an enormous interest in the mathematical literature, see [3,31,35,36]. Indeed, previous results (Theorems A and B) only considered the scaling behaviour of a single measure. Mixed multifractal analysis investigates the *simultaneous* scaling behaviour of finitely many measures. Mixed multifractal analysis thus combines local characteristics which depend simultaneously on various different aspects of the underlying dynamical system, and provides the basis for a significantly better understanding of the underlying dynamics. We will now make these ideas precise. For $m = 1, \ldots, M$, let $(p_{m,1}, \ldots, p_{m,N})$ be a probability vector, and let μ_m denote the self-conformal measure associated with the list $\big(V\,,\,X\,,\,(S_i)_{i=1,\ldots,N}\,,\,(p_{m,i})_{i=1,\ldots,N}\,\big)$, i.e. μ_m is the unique probability measure such that

$$\mu_m = \sum_i p_{m,i}\mu_m \circ S_i^{-1}. \tag{3.3}$$

The mixed multifractal spectrum $f_{\boldsymbol{\mu}}$ of the list $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_M)$ is defined by

$$f_{\boldsymbol{\mu}}(\boldsymbol{\alpha}) = \dim_{\mathsf{H}} \left\{ x \in K \,\middle|\, \lim_{r \searrow 0} \left(\frac{\log \mu_1(B(x,r))}{\log r}, \ldots, \frac{\log \mu_M(B(x,r))}{\log r} \right) = \boldsymbol{\alpha} \right\}$$

for $\boldsymbol{\alpha} \in \mathbb{R}^M$. Of course, it is also possible to define generalised mixed multifractal spectra designed to "see" different sets of divergence points. Namely, we define the generalised mixed Hausdorff multifractal spectrum $F_{\boldsymbol{\mu}}$ of the list $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_M)$ by

$$F_{\boldsymbol{\mu}}(C) = \dim_{\mathsf{H}} \left\{ x \in K \,\middle|\, \operatorname*{acc}_{r \searrow 0} \left(\frac{\log \mu_1(B(x,r))}{\log r}, \ldots, \frac{\log \mu_M(B(x,r))}{\log r} \right) \subseteq C \right\}$$

for $C \subseteq \mathbb{R}^M$. Again we note that the generalised mixed multifractal spectrum is a genuine extension of the traditional mixed multifractal spectrum $F_\mu(\boldsymbol{\alpha})$, namely, if $C = \{\boldsymbol{\alpha}\}$ is a singleton consisting of the point $\boldsymbol{\alpha}$, then clearly $F_\mu(C) = f_\mu(\boldsymbol{\alpha})$. Assuming the OSC, the generalised mixed multifractal spectrum $F_\mu(C)$ can be computed [31,35]. In order to state the result from [31,35], we introduce the following definitions. Define $\Lambda, \Phi_m : \Sigma^{\mathbb{N}} \to \mathbb{R}$ for $m = 1, \dots, M$ by $\Lambda(\mathbf{i}) = \log |DS_{i_1}(\pi S\mathbf{i})|$ and $\Phi_m(\mathbf{i}) = \log p_{m,i_1}$ for $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$, and write $\boldsymbol{\Phi} = (\Phi_1, \dots, \Phi_M)$. Define $\beta : \mathbb{R}^M \to \mathbb{R}$ by

$$0 = P\big(\, \beta(\mathbf{q})\Lambda + \langle \mathbf{q} | \boldsymbol{\Phi} \rangle \,\big)$$

for $\mathbf{q} \in \mathbb{R}^M$; recall that if $\varphi : \Sigma^{\mathbb{N}} \to \mathbb{R}$ is a Hölder continuous map, then $P(\varphi)$ denotes the pressure of φ. Also, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^M$, we let $\langle \mathbf{x} | \mathbf{y} \rangle$ denote the usual inner product of $\mathbf{x}$ and $\mathbf{y}$, and if $\varphi : \mathbb{R}^M \to \mathbb{R}$ is a function, we define the Legendre transform $\varphi^* : \mathbb{R}^M \to [-\infty, \infty]$ of φ by

$$\varphi^*(\mathbf{x}) = \inf_{\mathbf{y}} (\langle \mathbf{x} | \mathbf{y} \rangle + \varphi(\mathbf{y})).$$

The generalised mixed multifractal spectra $f_{\boldsymbol{\mu}}$ and $F_{\boldsymbol{\mu}}$ are now given by the following theorem.

Theorem C. [31,35] *Let $\mu_1, \dots, \mu_M$ be defined by (3.3) and let $C \subseteq \mathbb{R}^M$ be a closed set. Put $\boldsymbol{\mu} = (\mu_1, \dots, \mu_M)$. If the OSC is satisfied, then we have*

$$F_{\boldsymbol{\mu}}(C) = \sup_{\boldsymbol{\alpha} \in C} \beta^*(\boldsymbol{\alpha}).$$

In particular, if the OSC is satisfied and $\boldsymbol{\alpha} \in \mathbb{R}^M$, then we have

$$f_{\boldsymbol{\mu}}(\boldsymbol{\alpha}) = \beta^*(\boldsymbol{\alpha}).$$

As a second application of Theorems 2.1 and 2.2 we obtain a zeta-function whose abscissa of convergence equals the generalised mixed multifractal spectrum $F_{\boldsymbol{\mu}}(C)$ of a list $\boldsymbol{\mu}$ of self-conformal measures. This is the content of the next theorem.

Theorem 3.2. (Multifractal zeta-functinons for mixed multifractal spectra of self-conformal measures) *For $m = 1, \dots, M$, let $(p_{m,1}, \dots, p_{m,N})$ be a probability vector, and let μ_m denote the self-conformal measure associated with the list $\big(\, V\,,\, X\,,\, (S_i)_{i=1,\dots,N}\,,\, (p_{m,i})_{i=1,\dots,N}\,\big)$, i.e. μ_m is the unique probability measure such that $\mu_m = \sum_i p_{m,i} \mu_m \circ S_i^{-1}$.*

For $\mathbf{i} \in \Sigma^*$, let

$$s_{\mathbf{i}} = \sup_{\mathbf{u} \in \Sigma^{\mathbb{N}}} |DS_{\mathbf{i}}(\pi \mathbf{u})|.$$

For a closed set $C \subseteq \mathbb{R}^M$, we define the self-conformal multifractal zeta-function by

$$\zeta_C^{\mathsf{con}}(s) = \sum_{\substack{\mathbf{i} \\ \left(\frac{\log p_{1,\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}}, \dots, \frac{\log p_{M,\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}}\right) \in C}} s_{\mathbf{i}}^s .$$

For a closed set $C\subseteq\mathbb{R}^M$ *and* $r>0$*, we define the self-conformal multifractal zeta-function by*

$$\zeta^{\mathsf{con}}_C(s;r)=\zeta^{\mathsf{con}}_{B(C,r)}(s;r)=\sum_{\substack{\mathbf{i}\\ \operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}},\dots,\frac{\log p_{M,\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}}\right),C\right)\le r}} s_{\mathbf{i}}^s.$$

Define $\Lambda,\Phi_m:\Sigma^{\mathbb{N}}\to\mathbb{R}$ *for* $m=1,\dots,M$ *by* $\Lambda(\mathbf{i})=\log|DS_{i_1}(\pi S\mathbf{i})|$ *and* $\Phi_m(\mathbf{i})=\log p_{m,i_1}$ *for* $\mathbf{i}=i_1i_2\ldots\in\Sigma^{\mathbb{N}}$*, and write* $\mathbf{\Phi}=(\Phi_1,\dots,\Phi_M)$*. Define* $\beta:\mathbb{R}^M\to\mathbb{R}$ *by*

$$0=P\big(\beta(\mathbf{q})\Lambda+\langle\mathbf{q}|\mathbf{\Phi}\rangle\big)$$

for $\mathbf{q}\in\mathbb{R}^M$*. Let* C *be a closed subset of* $\mathbb{R}^M$*. Then the following hold:*

(1.1) *We have*

$$\lim_{r\searrow 0}\sigma_{\mathsf{ab}}\big(\zeta^{\mathsf{con}}_C(\cdot;r)\big)=\sup_{\boldsymbol{\alpha}\in C}\beta^*(\boldsymbol{\alpha}).$$

(1.2) *If the OSC is satisfied, then we have*

$$\begin{aligned}\lim_{r\searrow 0}&\sigma_{\mathsf{ab}}\big(\zeta^{\mathsf{con}}_C(\cdot;r)\big)\\ &=\sup_{\boldsymbol{\alpha}\in C}\dim_{\mathsf{H}}\left\{x\in K\,\middle|\,\lim_{r\searrow 0}\left(\frac{\log\mu_1(B(x,r))}{\log r},\dots,\frac{\log\mu_M(B(x,r))}{\log r}\right)=\boldsymbol{\alpha}\right\}\\ &=\dim_{\mathsf{H}}\left\{x\in K\,\middle|\,\operatorname*{acc}_{r\searrow 0}\left(\frac{\log\mu_1(B(x,r))}{\log r},\dots,\frac{\log\mu_M(B(x,r))}{\log r}\right)\subseteq C\right\}.\end{aligned}$$

(2.1) *If* C *is convex and* $\mathring{C}\cap\big(-\nabla\beta(\mathbb{R}^M)\big)\neq\varnothing$*, then we have*

$$\sigma_{\mathsf{ab}}\big(\zeta^{\mathsf{con}}_C\big)=\sup_{\boldsymbol{\alpha}\in C}\beta^*(\boldsymbol{\alpha}).$$

(2.2) *If* C *is convex and* $\mathring{C}\cap\big(-\nabla\beta(\mathbb{R}^M)\big)\neq\varnothing$ *and the OSC is satisfied, then we have*

$$\begin{aligned}\sigma_{\mathsf{ab}}&\big(\zeta^{\mathsf{con}}_C\big)\\ &=\sup_{\boldsymbol{\alpha}\in C}\dim_{\mathsf{H}}\left\{x\in K\,\middle|\,\lim_{r\searrow 0}\left(\frac{\log\mu_1(B(x,r))}{\log r},\dots,\frac{\log\mu_M(B(x,r))}{\log r}\right)=\boldsymbol{\alpha}\right\}\\ &=\dim_{\mathsf{H}}\left\{x\in K\,\middle|\,\operatorname*{acc}_{r\searrow 0}\left(\frac{\log\mu_1(B(x,r))}{\log r},\dots,\frac{\log\mu_M(B(x,r))}{\log r}\right)\subseteq C\right\}.\end{aligned}$$

We will now prove Theorem 3.2. Recall that the function $\Lambda:\Sigma^{\mathbb{N}}\to\mathbb{R}$ is defined by

$$\Lambda(\mathbf{i})=\log|DS_{i_1}(\pi S\mathbf{i})| \tag{3.4}$$

for $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$. It is well-known that Λ satisfies Conditions (C1)–(C3) in Sect. 2.1. Also, a straightforward calculation shows that $\sup_{\mathbf{k}\in[\mathbf{i}]} \exp \sum_{k=0}^{|\mathbf{i}|-1} \Lambda S^k \mathbf{k} = \sup_{\mathbf{u}\in\Sigma^{\mathbb{N}}} |DS_{\mathbf{i}}(\pi\mathbf{u})| = s_{\mathbf{i}}$ for $\mathbf{i} \in \Sigma^*$. Next, recall that $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_M)$ where $\Phi_m : \Sigma^{\mathbb{N}} \to \mathbb{R}$ is defined by $\Phi_m(\mathbf{i}) = \log p_{m,i_1}$ for $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$. For $\mu \in \mathcal{P}(\Sigma^{\mathbb{N}})$, we will write $\int \mathbf{\Phi}\, d\mu = (\int \Phi_1\, d\mu, \ldots, \int \Phi_M\, d\mu)$. Finally, define $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to \mathbb{R}^M$ by

$$U\mu = \frac{\int \mathbf{\Phi}\, d\mu}{\int \Lambda\, d\mu}, \tag{3.5}$$

and note that if $\mathbf{i} \in \Sigma^*$, then

$$UL_{|\mathbf{i}|}[\mathbf{i}] = \left\{ \left(\frac{\log p_{1,\mathbf{i}}}{\log |DS_{\mathbf{i}}(\pi\mathbf{u})|}, \ldots, \frac{\log p_{M,\mathbf{i}}}{\log |DS_{\mathbf{i}}(\pi\mathbf{u})|} \right) \middle| \mathbf{u} \in \Sigma^{\mathbb{N}} \right\}.$$

Hence, for $C \subseteq \mathbb{R}^M$ we have

$$\begin{aligned} \zeta_C^{U,\Lambda}(s;r) &= \sum_{\substack{\mathbf{i} \\ UL_{|\mathbf{i}|}[\mathbf{i}] \subseteq B(C,r)}} s_{\mathbf{i}}^s \\ &= \sum_{\substack{\mathbf{i} \\ \left\{ \left(\frac{\log p_{1,\mathbf{i}}}{\log |DS_{\mathbf{i}}(\pi\mathbf{u})|}, \ldots, \frac{\log p_{M,\mathbf{i}}}{\log |DS_{\mathbf{i}}(\pi\mathbf{u})|} \right) \middle| \mathbf{u}\in\Sigma^{\mathbb{N}} \right\} \subseteq B(C,r)}} s_{\mathbf{i}}^s \\ &= \sum_{\substack{\mathbf{i} \\ \forall \mathbf{u}\in\Sigma^{\mathbb{N}} \,:\, \operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log |DS_{\mathbf{i}}(\pi\mathbf{u})|}, \ldots, \frac{\log p_{M,\mathbf{i}}}{\log |DS_{\mathbf{i}}(\pi\mathbf{u})|} \right), C \right) \le r}} s_{\mathbf{i}}^s. \end{aligned} \tag{3.6}$$

In order to prove Theorem 3.2, we first prove the following three auxiliary results, namely, Propositions 3.3–3.5.

Proposition 3.3. *Let U and Λ be defined by (3.5) and (3.4), respectively. For $\boldsymbol{\alpha} \in \mathbb{R}^M$, we have*

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}}) \\ U\mu=\boldsymbol{\alpha}}} -\frac{h(\mu)}{\int \Lambda\, d\mu} = \beta^*(\boldsymbol{\alpha}).$$

Proof. This result is folklore for $M = 1$. The proof of Proposition 3.3 for an arbitrary positive integer can (with some modifications) be modelled on the argument for $M = 1$. However, for the sake of brevity we have decided to omit the proof. ∎

Proposition 3.4. *Let U and Λ be defined by (3.5) and (3.4), respectively. Let C be a closed subset of $\mathbb{R}^M$. If C is convex and $\overset{\circ}{C} \cap \big(-\nabla\beta(\mathbb{R}^M)\big) \neq \varnothing$, then $\underline{f}^{U,\Lambda}$ is inner continuous at C.*

Proof. Note that it follows from Theorem 2.1 and Proposition 3.3 that if W is a closed subset of $\mathbb{R}^M$, then

$$\begin{aligned}\underline{f}^{U,\Lambda}(W) &= \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in W}} -\frac{h(\mu)}{\int \Lambda\, d\mu}\\ &= \sup_{\boldsymbol{\alpha}\in W}\ \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu=\boldsymbol{\alpha}}} -\frac{h(\mu)}{\int \Lambda\, d\mu}\\ &= \sup_{\boldsymbol{\alpha}\in W} \beta^*(\boldsymbol{\alpha}). \end{aligned} \tag{3.8}$$

Also, since the function β^* satisfies $\{\boldsymbol{\alpha}\in\mathbb{R}^M \mid \beta^*(\boldsymbol{\alpha}) > -\infty\} = \nabla\beta(\mathbb{R}^M)$ (see [44, Corollary 26.4.1]) and the set C is convex with $\overset{\circ}{C} \cap \big(-\nabla\beta(\mathbb{R}^M)\big) \neq \varnothing$, we conclude immediately from (3.8) that $\underline{f}^{U,\Lambda}$ is inner continuous at C. $\square$

Proposition 3.5. *Let U and Λ be defined by (3.5) and (3.4), respectively.*

(1) *There is a sequence $(\Delta_n)_n$ with $\Delta_n > 0$ and $\Delta_n \to 0$ such that for all closed subsets C of $\mathbb{R}^M$ and for all $n\in\mathbb{N}$, $\mathbf{i}\in\Sigma^n$ and $\mathbf{u}\in\Sigma^{\mathbb{N}}$, we have*

$$\begin{aligned}&\operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|},\ldots,\frac{\log p_{M,\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\right), C\right)\\ &\qquad\le \operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}},\ldots,\frac{\log p_{M,\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}}\right), C\right) + \Delta_n,\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}},\ldots,\frac{\log p_{M,\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}}\right), C\right)\\ &\qquad\le \operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|},\ldots,\frac{\log p_{M,\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\right), C\right) + \Delta_n.\end{aligned} \tag{3.10}$$

(2) *For all closed subsets W of $\mathbb{R}^M$ and all $r>0$, we have*

$$\sigma_{\mathsf{ab}}\big(\zeta_W^{U,\Lambda}(\cdot;r)\big) \le \sigma_{\mathsf{ab}}\big(\zeta_{B(W,2r)}^{\mathsf{con}}\big), \tag{3.11}$$

$$\sigma_{\mathsf{ab}}\big(\zeta_{B(W,r)}^{\mathsf{con}}\big) \le \sigma_{\mathsf{ab}}\big(\zeta_W^{U,\Lambda}(\cdot;2r)\big). \tag{3.12}$$

(3) *Let C be a closed subset of $\mathbb{R}^M$. We have*

$$\lim_{r\searrow 0}\sigma_{\mathsf{ab}}\big(\zeta_C^{\mathsf{con}}(\cdot;r)\big)=\lim_{r\searrow 0}\sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}(\cdot;r)\big).$$

(4) *Let C be a closed subset of $\mathbb{R}^M$. If C is convex and $\mathring{C}\cap\big(-\nabla\beta(\mathbb{R}^M)\big)\neq\varnothing$, then we have*

$$\sigma_{\mathsf{ab}}\big(\zeta_C^{\mathsf{con}}\big)=\sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}\big).$$

Proof. (1) It is well-known that there is a constant $c_0>0$ such that for all $\mathbf{i}\in\Sigma^*$ and all $\mathbf{u}\in\Sigma^{\mathbb{N}}$, we have $\frac{1}{c_0}\leq\frac{\operatorname{diam}K_{\mathbf{i}}}{|DS_{\mathbf{i}}(\pi\mathbf{u})|}\leq c_0$, see, for example, [11] or [42]. It is not difficult to see that the desired result follows from this and the fact that the function $\Lambda:\Sigma^{\mathbb{N}}\to\mathbb{R}$ defined by $\Lambda(\mathbf{i})=\log|DS_{i_1}(\pi S\mathbf{i})|$ for $\mathbf{i}=i_1i_2\ldots\in\Sigma^{\mathbb{N}}$ satisfies Conditions (C1)–(C3) in Sect. 2.1.

(2) Fix $r>0$. Let $(\Delta_n)_n$ be the sequence from (1). Since $\Delta_n\to 0$, we can find a positive integer N_r such that if $n\geq N_r$, then $\Delta_n\leq r$. Consequently, using (3.10) in Part (1), for $s\in\mathbb{R}$, we have

$$\begin{aligned}
\zeta_W^{U,\Lambda}(s;r)&=\sum_{\substack{\mathbf{i}\\ \forall\mathbf{u}\in\Sigma^{\mathbb{N}}\,:\,\operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|},\ldots,\frac{\log p_{M,\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\right),W\right)\leq r}} s_{\mathbf{i}}^{s}\\
&\leq\sum_{\substack{\mathbf{i}\\|\mathbf{i}|<N_r}}s_{\mathbf{i}}^{s}+\sum_{\substack{\mathbf{i}\\|\mathbf{i}|\geq N_r\\ \forall\mathbf{u}\in\Sigma^{\mathbb{N}}\,:\,\operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|},\ldots,\frac{\log p_{M,\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\right),W\right)\leq r}} s_{\mathbf{i}}^{s}\\
&\leq\sum_{\substack{\mathbf{i}\\|\mathbf{i}|<N_r}}s_{\mathbf{i}}^{s}+\sum_{\substack{\mathbf{i}\\|\mathbf{i}|\geq N_r\\ \operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}},\ldots,\frac{\log p_{M,\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}}\right),W\right)\leq r+\Delta_{|\mathbf{i}|}}} s_{\mathbf{i}}^{s}\\
&\leq\sum_{\substack{\mathbf{i}\\|\mathbf{i}|<N_r}}s_{\mathbf{i}}^{s}+\sum_{\substack{\mathbf{i}\\|\mathbf{i}|\geq N_r\\ \operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}},\ldots,\frac{\log p_{M,\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}}\right),W\right)\leq 2r}} s_{\mathbf{i}}^{s}\\
&\leq\sum_{\substack{\mathbf{i}\\|\mathbf{i}|<N_r}}s_{\mathbf{i}}^{s}+\sum_{\substack{\mathbf{i}\\ \operatorname{dist}\left(\left(\frac{\log p_{1,\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}},\ldots,\frac{\log p_{M,\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}}\right),W\right)\leq 2r}} s_{\mathbf{i}}^{s}
\end{aligned}$$

$$= \sum_{\substack{\mathbf{i} \\ |\mathbf{i}|<N_r}} s_{\mathbf{i}}^s \ + \ \zeta^{\mathsf{con}}_{B(W,2r)}(s). \tag{3.13}$$

A similar argument using (3.1) in Part 1 shows that

$$\zeta^{\mathsf{con}}_{B(W,r)}(s) \leq \sum_{\substack{\mathbf{i} \\ |\mathbf{i}|<N_r}} s_{\mathbf{i}}^s \ + \ \zeta^{U,\Lambda}_{W}(s;2r). \tag{3.14}$$

The desired results follow immediately from inequalities (3.13) and (3.14).
(3) This result follows from Part (2) by letting $r \searrow 0$.
(4) "$\leq$" It follows from (3.12) and Theorem 2.1 that

$$\begin{aligned}
\sigma_{\mathsf{ab}}\big(\,\zeta^{\mathsf{con}}_{C}\,\big) &\leq \liminf_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\,\zeta^{\mathsf{con}}_{B(C,r)}\,\big) && [\text{since } C \subseteq B(C,r)]\\
&\leq \liminf_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\,\zeta^{U,\Lambda}_{C}(\cdot;2r)\,\big) && [\text{by (3.12)}]\\
&= \underline{f}^{U,\Lambda}(C) && [\text{by Theorem 2.1}]. \qquad (3.15)
\end{aligned}$$

Next, since C is convex and $\overset{\circ}{C} \cap \big(-\nabla\beta(\mathbb{R}^M)\big) \neq \varnothing$, we conclude from Proposition 3.4 that $\underline{f}^{U,\Lambda}$ is inner continuous at C, and it therefore follows from Theorem 2.2 that $\underline{f}^{U,\Lambda}(C) = \sigma_{\mathsf{ab}}\big(\,\zeta^{U,\Lambda}_{C}\,\big)$. The desired result follows from this and (3.15).
"$\geq$" Let $\varepsilon > 0$. For all $r > 0$ with $2r < \varepsilon$, it follows from (3.11) applied to $W = I(C,\varepsilon)$ (recall that $I(C,\varepsilon) = \{x \in C \,|\, \mathrm{dist}(x,\partial C) \geq \varepsilon\}$, see Sect. 2.3) that

$$\sigma_{\mathsf{ab}}\big(\,\zeta^{U,\Lambda}_{I(C,\varepsilon)}(\cdot;r)\,\big) \leq \sigma_{\mathsf{ab}}\big(\,\zeta^{\mathsf{con}}_{B(I(C,\varepsilon),2r)}\,\big). \tag{3.16}$$

However, for $2r < \varepsilon$ it is not difficult to see that $B(I(C,\varepsilon),2r) \subseteq C$ (see, for example, the proof of Lemma 8.2), whence $\sigma_{\mathsf{ab}}\big(\,\zeta^{\mathsf{con}}_{B(I(C,\varepsilon),2r)}\,\big) \leq \sigma_{\mathsf{ab}}\big(\,\zeta^{\mathsf{con}}_{C}\,\big)$, and we therefore conclude from (3.16) that if $2r < \varepsilon$, then

$$\sigma_{\mathsf{ab}}\big(\,\zeta^{U,\Lambda}_{I(C,\varepsilon)}(\cdot;r)\,\big) \leq \sigma_{\mathsf{ab}}\big(\,\zeta^{\mathsf{con}}_{C}\,\big). \tag{3.17}$$

Letting $r \searrow 0$ in (3.17) we now deduce that

$$\limsup_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\,\zeta^{U,\Lambda}_{I(C,\varepsilon)}(\cdot;r)\,\big) \leq \sigma_{\mathsf{ab}}\big(\,\zeta^{\mathsf{con}}_{C}\,\big). \tag{3.18}$$

Next, since $I(C,\varepsilon)$ is closed, we deduce from Theorem 2.1 that $\limsup_{r\searrow 0} \sigma_{\mathsf{ab}} \big(\,\zeta^{U,\Lambda}_{I(C,\varepsilon)}(\cdot;r)\,\big) = \underline{f}^{U,\Lambda}(\,I(C,\varepsilon)\,)$, and (3.18) therefore implies that

$$\underline{f}^{U,\Lambda}(\,I(C,\varepsilon)\,) \leq \sigma_{\mathsf{ab}}\big(\,\zeta^{\mathsf{con}}_{C}\,\big). \tag{3.19}$$

Finally, it follows from Proposition 3.4 that $\underline{f}^{U,\Lambda}$ is inner continuous at C, whence $\lim_{\varepsilon\searrow 0} \underline{f}^{U,\Lambda}(\,I(C,\varepsilon)\,) = \underline{f}^{U,\Lambda}(C)$. The desired result follows from this and (3.19). $\square$

We can now prove Theorem 3.2.

Proof of Theorem 3.2. (1.1) and (2.1) The statements in Part (1.1) and Part (2.1) of Theorem 3.2 follow immediately from Theorem 2.1, Propositions 3.3 and 3.5.
(1.2) and (2.2) The statements in Part (1.2) and Part (2.2) of Theorem 3.2 follow immediately from Part (1.1) and Part (2.1) using Theorem 2.2 and Theorem C. □

3.3. Multifractal spectra of self-similar measures

Due to the important role self-similar measures play in fractal geometry, it is instructive to note the following special case of Theorem 3.1.

Theorem 3.6. (Multifractal zeta-functinons for multifractal spectra of self-similar measures) *Assume that the maps $S_1,\ldots,S_N$ are contracting similarities and let r_i denote the contraction ratio of S_i. For $\mathbf{i}=i_1\cdots i_n\in\Sigma^*$, let*

$$r_{\mathbf{i}}=r_{i_1}\cdots r_{i_n}.$$

Let $(p_1,\ldots,p_N)$ be a probability vector, and let μ denote the self-conformal measure associated with the list $\big(V\,,\,X\,,\,(S_i)_{i=1,\ldots,N}\,,\,(p_i)_{i=1,\ldots,N}\big)$, i.e. μ is the unique probability measure such that $\mu=\sum_i p_{l,i}\mu\circ S_i^{-1}$.

For a closed set $C\subseteq\mathbb{R}$, we define the self-similar multifractal zeta-function by

$$\zeta_C^{\mathsf{sim}}(s)=\sum_{\substack{\mathbf{i}\\ \frac{\log p_{\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}}\in C}} r_{\mathbf{i}}^s.$$

For a closed set $C\subseteq\mathbb{R}$ and $r>0$, we define the self-similar multifractal zeta-function by

$$\zeta_C^{\mathsf{sim}}(s;r)=\sum_{\substack{\mathbf{i}\\ \operatorname{dist}\left(\frac{\log p_{\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}}\,,\,C\right)\le r}} r_{\mathbf{i}}^s\,,$$

and if $\alpha\in\mathbb{R}$ and $C=\{\alpha\}$ is the singleton consisting of α, then we write $\zeta_C(s;r)=\zeta_\alpha(s;r)$, i.e. we write

$$\zeta_\alpha^{\mathsf{sim}}(s;r)=\sum_{\substack{\mathbf{i}\\ \left|\frac{\log p_{\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}}-\alpha\right|\le r}} r_{\mathbf{i}}^s.$$

Define $\beta:\mathbb{R}^M\to\mathbb{R}$ by

$$\sum_i p_i^q r_i^{\beta(q)}=1$$

for $q\in\mathbb{R}$. Let C be a closed subset of $\mathbb{R}$. Then the following hold:

(1.1) *We have*

$$\lim_{r\searrow 0}\sigma_{\mathsf{ab}}\big(\,\zeta_C^{\mathsf{sim}}(\cdot;r)\,\big)=\sup_{\alpha\in C}\beta^*(\alpha).$$

In particular, if $\alpha\in\mathbb{R}$, then we have

$$\lim_{r\searrow 0}\sigma_{\mathsf{ab}}\big(\,\zeta_\alpha^{\mathsf{sim}}(\cdot;r)\,\big)=\beta^*(\alpha).$$

(1.2) *If the OSC is satisfied, then we have*

$$\begin{aligned}\lim_{r\searrow 0}\sigma_{\mathsf{ab}}\big(\,\zeta_C^{\mathsf{sim}}(\cdot;r)\,\big)&=\sup_{\alpha\in C}\dim_{\mathsf{H}}\left\{x\in K\,\middle|\,\lim_{r\searrow 0}\frac{\log\mu(B(x,r))}{\log r}=\alpha\right\}\\&=\dim_{\mathsf{H}}\left\{x\in K\,\middle|\,\operatorname*{acc}_{r\searrow 0}\frac{\log\mu(B(x,r))}{\log r}\subseteq C\right\}.\end{aligned}$$

In particular, if the OSC is satisfied and $\alpha\in\mathbb{R}$, then we have

$$\lim_{r\searrow 0}\sigma_{\mathsf{ab}}\big(\,\zeta_\alpha^{\mathsf{sim}}(\cdot;r)\,\big)=\dim_{\mathsf{H}}\left\{x\in K\,\middle|\,\lim_{r\searrow 0}\frac{\log\mu(B(x,r))}{\log r}=\alpha\right\}.$$

(2.1) *If C is an interval and $\overset{\circ}{C}\cap\big(\min_i\frac{\log p_i}{\log r_i},\max_i\frac{\log p_i}{\log r_i}\big)\neq\varnothing$, then we have*

$$\sigma_{\mathsf{ab}}\big(\,\zeta_C^{\mathsf{sim}}\,\big)=\sup_{\alpha\in C}\beta^*(\alpha).$$

(2.2) *If C is an interval and $\overset{\circ}{C}\cap\big(\min_i\frac{\log p_i}{\log r_i},\max_i\frac{\log p_i}{\log r_i}\big)\neq\varnothing$ and the OSC is satisfied, then we have*

$$\begin{aligned}\sigma_{\mathsf{ab}}\big(\,\zeta_C^{\mathsf{sim}}\,\big)&=\sup_{\alpha\in C}\dim_{\mathsf{H}}\left\{x\in K\,\middle|\,\lim_{r\searrow 0}\frac{\log\mu(B(x,r))}{\log r}=\alpha\right\}\\&=\dim_{\mathsf{H}}\left\{x\in K\,\middle|\,\operatorname*{acc}_{r\searrow 0}\frac{\log\mu(B(x,r))}{\log r}\subseteq C\right\}.\end{aligned}$$

Proof. Theorem 3.6 follows immediately from Theorem 3.1. □

It is, of course, also possible to formulate a version of Theorem 3.2 for a finite list of self-similar measures. However, for sake of brevity we have decided not to do this.

3.4. Multifractal spectra of ergodic Birkhoff averages

We first fix $\gamma\in(0,1)$ and define the metric d_γ on $\Sigma^{\mathbb{N}}$ by $\mathsf{d}_\gamma(\mathbf{i},\mathbf{j})=\gamma^{\max\{n\,|\,\mathbf{i}|n=\mathbf{j}|n\}}$; throughout this section, we equip $\Sigma^{\mathbb{N}}$ with the metric d_γ and continuity and Lipschitz properties of functions $f:\Sigma^{\mathbb{N}}\to\mathbb{R}$ from $\Sigma^{\mathbb{N}}$ to $\mathbb{R}$ will always refer to the metric d_γ. Multifractal analysis of Birkhoff averages has received significant interest during the past 10 years, see, for example, [2,12–14,33,36,41]. The multifractal spectrum F_f^{erg} of ergodic Birkhoff averages of a continuous function $f:\Sigma^{\mathbb{N}}\to\mathbb{R}$ is defined by

$$F_f^{\mathsf{erg}}(\alpha)=\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\lim_n\frac{1}{n}\sum_{k=0}^{n-1}f(S^k\mathbf{i})=\alpha\right\}$$

for $\alpha\in\mathbb{R}$; recall that the projection map $\pi:\Sigma^{\mathbb{N}}\to\mathbb{R}^d$ is defined in Sect. 3.1 and that $S:\Sigma^{\mathbb{N}}\to\Sigma^{\mathbb{N}}$ denotes the shift map. One of the main problems in multifractal analysis of Birkhoff averages is the detailed study of the multifractal spectrum F_f^{erg}. For example, Theorem D below is proved in different settings and at various levels of generality in [12–14,33,36,41]. Before we can state our result we introduce the following notation. If $(x_n)_n$ is a sequence of real numbers, then we write $\operatorname{acc}_n x_n$ for the set of accumulation points of $(x_n)_n$, i.e.

$$\operatorname*{acc}_n x_n=\left\{x\in\mathbb{R}\,\middle|\,x\text{ is an accumulation point of }(x_n)_n\right\}.$$

Also, recall that $\mathcal{P}_S(\Sigma^{\mathbb{N}})$ denotes the family of shift invariant Borel probability measures on $\Sigma^{\mathbb{N}}$ and that $h(\mu)$ denotes the entropy of $\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})$. We can now state Theorem D.

Theorem D. [12–14,33,36,41] *Let $f:\Sigma^{\mathbb{N}}\to\mathbb{R}$ be a Lipschitz function. Define $\Lambda:\Sigma^{\mathbb{N}}\to\mathbb{R}$ by $\Lambda(\mathbf{i})=\log|DS_{i_1}(\pi S\mathbf{i})|$ for $\mathbf{i}=i_1i_2\ldots\in\Sigma^{\mathbb{N}}$. Let C be a closed subset of $\mathbb{R}$. If the OSC is satisfied, then*

$$\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\operatorname*{acc}_n\frac{1}{n}\sum_{k=0}^{n-1}f(S^k\mathbf{i})\subseteq C\right\}=\sup_{\alpha\in C}\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ \int f\,d\mu=\alpha}}-\frac{h(\mu)}{\int\Lambda\,d\mu}.$$

In particular, if the OSC is satisfied and $\alpha\in\mathbb{R}$, then we have

$$\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\lim_n\frac{1}{n}\sum_{k=0}^{n-1}f(S^k\mathbf{i})=\alpha\right\}=\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ \int f\,d\mu=\alpha}}-\frac{h(\mu)}{\int\Lambda\,d\mu}.$$

As a third application of Theorem 2.1 we obtain a zeta-function whose abscissa of convergence equals the multifractal spectrum F_f^{erg} of ergodic Birkhoff averages of a Lipschitz function f. This is the content of the next theorem.

Theorem 3.7. (Multifractal zeta-functinons for multifractal spectra of ergodic Birkhoff averages) *Let $f:\Sigma^{\mathbb{N}}\to\mathbb{R}$ be a Lipschitz function.*

For $\mathbf{i}\in\Sigma^*$, let

$$s_{\mathbf{i}}=\sup_{\mathbf{u}\in\Sigma^{\mathbb{N}}}|DS_{\mathbf{i}}(\pi\mathbf{u})|$$

and write $\bar{\mathbf{i}} = \mathbf{iii}\ldots \in \Sigma^{\mathbb{N}}$*. For a closed set* $C \subseteq \mathbb{R}^M$*, we define the self-similar multifractal zeta-function of* f *by*

$$\zeta_C^{\mathsf{erg}}(s;r) = \sum_{\substack{\mathbf{i} \\ \operatorname{dist}\left(\frac{1}{|\mathbf{i}|}\sum_{k=0}^{|\mathbf{i}|-1} f(S^k\bar{\mathbf{i}}),\, C\right) \le r}} s_{\mathbf{i}}^s\,,$$

and if $\alpha \in \mathbb{R}$ *and* $C = \{\alpha\}$ *is the singleton consisting of* α*, then we write* $\zeta_C(s;r) = \zeta_\alpha(s;r)$*, i.e. we write*

$$\zeta_\alpha^{\mathsf{erg}}(s;r) = \sum_{\substack{\mathbf{i} \\ \left|\frac{1}{|\mathbf{i}|}\sum_{k=0}^{|\mathbf{i}|-1} f(S^k\bar{\mathbf{i}}) - \alpha\right| \le r}} s_{\mathbf{i}}^s.$$

Define $\Lambda : \Sigma^{\mathbb{N}} \to \mathbb{R}$ *by* $\Lambda(\mathbf{i}) = \log|DS_{i_1}(\pi S\mathbf{i})|$ *for* $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$*. Then the following hold:*

(1) *We have*

$$\lim_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\zeta_C^{\mathsf{erg}}(\cdot;r)\big) = \sup_{\alpha\in C} \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}}) \\ \int f\,d\mu=\alpha}} -\frac{h(\mu)}{\int \Lambda\,d\mu}.$$

In particular, if $\alpha \in \mathbb{R}$*, then we have*

$$\lim_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\zeta_\alpha^{\mathsf{erg}}(\cdot;r)\big) = \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}}) \\ \int f\,d\mu=\alpha}} -\frac{h(\mu)}{\int \Lambda\,d\mu}.$$

(2) *If the OSC is satisfied, then we have*

$$\begin{aligned}\lim_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\zeta_C^{\mathsf{erg}}(\cdot;r)\big) &= \sup_{\alpha\in C} \dim_{\mathsf{H}} \pi\left\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \frac{1}{n}\sum_{k=0}^{n-1} f(S^k\mathbf{i}) = \alpha\right\} \\ &= \dim_{\mathsf{H}} \pi\left\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \operatorname*{acc}_n \frac{1}{n}\sum_{k=0}^{n-1} f(S^k\mathbf{i}) \subseteq C\right\}.\end{aligned}$$

In particular, if the OSC is satisfied and $\alpha \in \mathbb{R}$*, then we have*

$$\lim_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\zeta_\alpha^{\mathsf{erg}}(\cdot;r)\big) = \dim_{\mathsf{H}} \pi\left\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \frac{1}{n}\sum_{k=0}^{n-1} f(S^k\mathbf{i}) = \alpha\right\}.$$

We will now prove Theorem 3.7. Recall that the function $\Lambda : \Sigma^{\mathbb{N}} \to \mathbb{R}$ is defined by

$$\Lambda(\mathbf{i}) = \log|DS_{i_1}(\pi S\mathbf{i})| \tag{3.20}$$

for $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$. It is well-known that Λ satisfies Conditions (C1)–(C3) in Sect. 2.1. Also, a straightforward calculation shows that $\sup_{\mathbf{k}\in[\mathbf{i}]} \exp\sum_{k=0}^{|\mathbf{i}|-1}$

$\Lambda S^k \mathbf{k} = \sup_{\mathbf{u}\in\Sigma^{\mathbb{N}}} |DS_{\mathbf{i}}(\pi\mathbf{u})| = s_{\mathbf{i}}$ for $\mathbf{i} \in \Sigma^*$. Finally, define $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to \mathbb{R}^M$ by

$$U\mu = \int f\, d\mu \tag{3.21}$$

and note that if $\mathbf{i} \in \Sigma^*$, then

$$UL_{|\mathbf{i}|}[\mathbf{i}] = \left\{ \frac{1}{|\mathbf{i}|} \sum_{k=0}^{|\mathbf{i}|-1} f(S^k(\mathbf{iu})) \,\middle|\, \mathbf{u} \in \Sigma^{\mathbb{N}} \right\}.$$

Hence, for $C \subseteq \mathbb{R}$ we have

$$\begin{aligned}\zeta_C^{U,\Lambda}(s;r) &= \sum_{\substack{\mathbf{i} \\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^s \\ &= \sum_{\substack{\mathbf{i} \\ \left\{ \frac{1}{|\mathbf{i}|} \sum_{k=0}^{|\mathbf{i}|-1} f(S^k(\mathbf{iu})) \,\middle|\, \mathbf{u}\in\Sigma^{\mathbb{N}} \right\} \subseteq B(C,r)}} s_{\mathbf{i}}^s \\ &= \sum_{\substack{\mathbf{i} \\ \forall \mathbf{u}\in\Sigma^{\mathbb{N}} \,:\, \operatorname{dist}\left(\frac{1}{|\mathbf{i}|} \sum_{k=0}^{|\mathbf{i}|-1} f(S^k(\mathbf{iu}))\,,\, C\right) \le r}} s_{\mathbf{i}}^s. \end{aligned} \tag{3.22}$$

In order to prove Theorem 3.7, we first prove the following auxiliary result, namely, Proposition 3.8.

Proposition 3.8. *Let U and Λ be defined by (3.21) and (3.20), respectively.*

(1) *There is a sequence $(\Delta_n)_n$ with $\Delta_n > 0$ for all n and $\Delta_n \to 0$ such that for all closed subsets C of $\mathbb{R}$ and for all $n \in \mathbb{N}$, $\mathbf{i} \in \Sigma^n$ and $\mathbf{u} \in \Sigma^{\mathbb{N}}$, we have*

$$\operatorname{dist}\left(\frac{1}{|\mathbf{i}|} \sum_{k=0}^{|\mathbf{i}|-1} f(S^k(\mathbf{iu}))\,,\, C\right) \le \operatorname{dist}\left(\frac{1}{|\mathbf{i}|} \sum_{k=0}^{|\mathbf{i}|-1} f(S^k(\bar{\mathbf{i}}))\,,\, C\right) + \Delta_n\,,$$
$$\operatorname{dist}\left(\frac{1}{|\mathbf{i}|} \sum_{k=0}^{|\mathbf{i}|-1} f(S^k(\bar{\mathbf{i}}))\,,\, C\right) \le \operatorname{dist}\left(\frac{1}{|\mathbf{i}|} \sum_{k=0}^{|\mathbf{i}|-1} f(S^k(\mathbf{iu}))\,,\, C\right) + \Delta_n.$$

(2) *We have*

$$\lim_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\zeta_C^{\mathsf{erg}}(\cdot;r) \big) = \lim_{r\searrow 0} \sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}(\cdot;r) \big).$$

Proof. (1) Let Lip(f) denote the Lipschitz constant of f. It is clear that for all $n \in \mathbb{N}, \mathbf{i} \in \Sigma^n$ and $\mathbf{u} \in \Sigma^{\mathbb{N}}$, we have

$$\begin{aligned} \left| \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(\bar{\mathbf{i}})) - \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(\mathbf{iu})) \right| &\leq \frac{1}{n} \sum_{k=0}^{n-1} |f(S^k(\bar{\mathbf{i}})) - f(S^k(\mathbf{iu}))| \\ &\leq \text{Lip}(f) \frac{1}{n} \sum_{k=0}^{n-1} \mathsf{d}_\gamma\big(S^k(\bar{\mathbf{i}}), S^k(\mathbf{iu}) \big) \\ &\leq \text{Lip}(f) \frac{1}{n} \sum_{k=0}^{n-1} \gamma^k \\ &\leq \text{Lip}(f) \frac{1}{n(1-\gamma)}. \end{aligned} \tag{3.23}$$

It is not difficult to see that the desired result follows from (3.23).
(2) This statement follows from Part (1) by an argument very similar to the proof of Part (2) and Part (3) in Proposition 3.5, and the proof is therefore omitted. □

We can now prove Theorem 3.7.

Proof of Theorem 3.7. (1) This statement follows immediately from Theorem 2.1 and Proposition 3.8.
(2) This statement follows immediately from Part (1) using Theorem 2.2 and Theorem D. □

4. Preliminary results

The purpose of this short section is to prove Proposition 4.1 establishing various auxiliary results needed for the proof of Theorem 2.1. Let $c_{\min}$ and $c_{\max}$ be the constants from the Condition (C2) in Sect. 2.1 and write

$$\begin{aligned} s_{\min} &= e^{c_{\min}}, \\ s_{\max} &= e^{c_{\max}}. \end{aligned} \tag{4.1}$$

We can now state and prove Proposition 4.1. Recall, that for $\mathbf{i} \in \Sigma^n$, the number $s_{\mathbf{i}}$ is defined by $s_{\mathbf{i}} = \sup_{\mathbf{k} \in [\mathbf{i}]} \exp \sum_{k=0}^{n-1} \Lambda S^k \mathbf{k}$, see Sect. 2.1.

Proposition 4.1. *Let c be the constant from Condition (C3) in Sect. 2.1. Let $\mathbf{i}, \mathbf{j} \in \Sigma^*$.*

(1) $0 < s_{\min}^{|\mathbf{i}|} \leq s_{\mathbf{i}} \leq s_{\max}^{|\mathbf{i}|} < 1$.
(2) $s_{\mathbf{ij}} \leq s_{\mathbf{i}} s_{\mathbf{j}} \leq c s_{\mathbf{ij}}$.
(3) $s_{\mathbf{i}} < s_{\hat{\mathbf{i}}}$.
(4) *For $\mathbf{k} \in \Sigma^{\mathbb{N}}$ and a positive integer n, we have* $\exp \sum_{k=0}^{n-1} \Lambda S^k \mathbf{k} \leq s_{\mathbf{k}|n} \leq c \exp \sum_{k=0}^{n-1} \Lambda S^k \mathbf{k}$.

(5) *For* $\mathbf{k} \in \Sigma^{\mathbb{N}}$ *and a real number* α*, the following two statements are equivalent:*
   (i) $\frac{1}{n}\sum_{k=0}^{n-1} \Lambda S^k \mathbf{k} \to \alpha$.
   (ii) $\frac{1}{n}\log s_{\mathbf{k}|n} \to \alpha$.

Proof. Statements (1), (2) and (4) follow easily from the definitions. Statement (3) follows from (1) and (2), and statement (5) follows from (4). □

5. Proof of inequality (2.1)

The purpose of this section is to prove Theorem 5.5 providing a proof of inequality (2.1). The proof of (2.1) is based on results from large deviation theory. In particular, we need Varadhan's [45] large deviation theorem (Theorem 5.1.(i) below), and a non-trivial application of this (namely Theorem 5.1.(ii) below) providing first order asymptotics of certain "Boltzmann distributions".

Definition. Let X be a complete separable metric space and let $(P_n)_n$ be a sequence of probability measures on X. Let $(a_n)_n$ be a sequence of positive numbers with $a_n \to \infty$ and let $I : X \to [0, \infty]$ be a lower semicontinuous function with compact level sets. The sequence $(P_n)_n$ is said to have the large deviation property with constants $(a_n)_n$ and rate function I if the following two conditions hold:

(i) For each closed subset K of X, we have

$$\limsup_n \frac{1}{a_n} \log P_n(K) \leq -\inf_{x \in K} I(x).$$

(ii) For each open subset G of X, we have

$$\liminf_n \frac{1}{a_n} \log P_n(G) \geq -\inf_{x \in G} I(x).$$

Theorem 5.1. *Let* X *be a complete separable metric space and let* $(P_n)_n$ *be a sequence of probability measures on* X*. Assume that the sequence* $(P_n)_n$ *has the large deviation property with constants* $(a_n)_n$ *and rate function* I*. Let* $F : X \to \mathbb{R}$ *be a continuous function satisfying the following two conditions:*

(i) *For all* n*, we have*

$$\int \exp(a_n F)\, dP_n < \infty.$$

(ii) *We have*

$$\lim_{M\to\infty} \limsup_n \frac{1}{a_n} \log \int_{\{M \leq F\}} \exp(a_n F)\, dP_n = -\infty.$$

(Observe that the Conditions (i)–(ii) are satisfied if F *is bounded.) Then the following statements hold.*

(1) *We have*

$$\lim_n \frac{1}{a_n} \log \int \exp(a_n F)\, dP_n = - \inf_{x \in X} (I(x) - F(x)).$$

(2) *For each n define a probability measure Q_n on X by*

$$Q_n(E) = \frac{\int_E \exp(a_n F)\, dP_n}{\int \exp(a_n F)\, dP_n}.$$

Then the sequence $(Q_n)_n$ has the large deviation property with constants $(a_n)_n$ and rate function $(I - F) - \inf_{x \in X}(I(x) - F(x))$.

Proof. Statement (1) follows from [9, Theorem II.7.1] or [7, Theorem 4.3.1], and statement (2) follows from [9, Theorem II.7.2]. □

Using Theorem 5.1 we first establish the following auxiliary result.

Theorem 5.2. *Let X be a metric space and let $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a closed subset of X and $r > 0$.*

If $t \in \mathbb{R}$, then

$$\limsup_n \frac{1}{n} \log \sum_{\substack{|\mathbf{i}|=n \\ UL_n[\mathbf{i}] \subseteq B(C,r)}} s_{\mathbf{i}}^t \le \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}}) \\ U\mu \in B(C,r)}} \left(t \int \Lambda\, d\mu + h(\mu) \right).$$

Proof. We start by introducing some notation. If $\mathbf{i} \in \Sigma^*$, then we define $\bar{\mathbf{i}} \in \Sigma^{\mathbb{N}}$ by $\bar{\mathbf{i}} = \mathbf{i}\mathbf{i}\ldots$. We also define $M_n : \Sigma^{\mathbb{N}} \to \mathcal{P}_S(\Sigma^{\mathbb{N}})$ by

$$\begin{aligned} M_n \mathbf{i} &= L_n \left(\overline{\mathbf{i}|n} \right) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \delta_{S^k(\overline{\mathbf{i}|n})} \end{aligned}$$

for $\mathbf{i} \in \Sigma^{\mathbb{N}}$; recall, that the map $L_n : \Sigma^{\mathbb{N}} \to \mathcal{P}(\Sigma^{\mathbb{N}})$ is defined in Sect. 2. Furthermore, note that if $\mathbf{i} \in \Sigma^{\mathbb{N}}$, then $M_n \mathbf{i}$ is shift invariant, i.e. M_n maps $\Sigma^{\mathbb{N}}$ into $\mathcal{P}_S(\Sigma^{\mathbb{N}})$ as claimed. Next, let P denote the probability measure on $\Sigma^{\mathbb{N}}$ given by

$$P = \bigtimes_{\mathbb{N}} \sum_{i=1}^{N} \frac{1}{N} \delta_i.$$

Finally, we define $F : \mathcal{P}_S(\Sigma^{\mathbb{N}}) \to \mathbb{R}$ by

$$F(\mu) = t \int \Lambda\, d\mu.$$

Observe that since Λ is bounded, i.e. $\|\Lambda\|_\infty < \infty$, we conclude that $\|F\|_\infty = |t|\, \|\Lambda\|_\infty < \infty$. Also, for a positive integer n, define probability measures $P_n, Q_n \in \mathcal{P}(\mathcal{P}_S(\Sigma^{\mathbb{N}}))$ by

$$P_n = P \circ M_n^{-1}\,,$$
$$Q_n(E) = \frac{\int_E \exp(nF)\,dP_n}{\int \exp(nF)\,dP_n} \quad \text{for } E \subseteq \mathcal{P}_S(\Sigma^{\mathbb{N}}).$$

We now prove the following two claims. □

Claim 1. We have

$$\sum_{\substack{|\mathbf{k}|=n\\ UL_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t \le \sum_{\substack{|\mathbf{k}|=n\\ UM_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t.$$

Proof of Claim 1. Observe that if $|\mathbf{k}|=n$, then $M_n[\mathbf{k}] = \{M_n(\mathbf{kl})\,|\,\mathbf{l}\in\Sigma^{\mathbb{N}}\} = \{L_n(\,\overline{(\mathbf{kl})|n}\,)\,|\,\mathbf{l}\in\Sigma^{\mathbb{N}}\} = \{L_n\overline{\mathbf{k}}\,|\,\mathbf{l}\in\Sigma^{\mathbb{N}}\} = \{L_n\overline{\mathbf{k}}\} \subseteq L_n[\mathbf{k}]$. The desired result follows immediately from this inclusion. This proves Claim 1. □

Claim 2. We have

$$\sum_{\substack{|\mathbf{k}|=n\\ UM_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t \le N^n \int_{\left\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,\middle|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\right\}} s_{\mathbf{i}|n}^t\,dP(\mathbf{i}).$$

Proof of Claim 2. It follows that

$$\begin{aligned}
&\int_{\left\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,\middle|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\right\}} s_{\mathbf{i}|n}^t\,dP(\mathbf{i})\\
&\quad= \sum_{|\mathbf{k}|=n} \int_{[\mathbf{k}]\,\cap\,\left\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,\middle|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\right\}} s_{\mathbf{i}|n}^t\,dP(\mathbf{i})\\
&\quad= \sum_{|\mathbf{k}|=n} s_{\mathbf{k}}^t\; P\Big(\,[\mathbf{k}]\;\cap\;\Big\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,\Big|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\Big\}\Big)\\
&\quad\ge \sum_{\substack{|\mathbf{k}|=n\\ UM_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t\; P\Big(\,[\mathbf{k}]\;\cap\;\Big\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,\Big|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\Big\}\Big).
\end{aligned} \tag{5.1}$$

However, for $\mathbf{k}$ with $|\mathbf{k}|=n$ and $UM_n[\mathbf{k}]\subseteq B(C,r)$, it is clear that $[\mathbf{k}]\subseteq\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,|\,UM_n[\mathbf{j}|n]\subseteq B(C,r)\}$, whence $[\mathbf{k}]\cap\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,|\,UM_n[\mathbf{j}|n]\subseteq B(C,r)\} = [\mathbf{k}]$. This and (5.1) now imply that

$$\int_{\left\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,\middle|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\right\}} s_{\mathbf{i}|n}^t\,dP(\mathbf{i})$$

$$\geq \sum_{\substack{|\mathbf{k}|=n \\ UM_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t \, P\Big([\mathbf{k}] \ \cap \ \Big\{ \mathbf{j}\in\Sigma^{\mathbb{N}} \,\Big|\, UM_n[\mathbf{j}|n] \subseteq B(C,r) \Big\} \Big)$$
$$= \sum_{\substack{|\mathbf{k}|=n \\ UM_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t \, P\big([\mathbf{k}] \big)$$
$$= \sum_{\substack{|\mathbf{k}|=n \\ \operatorname{dist}(UL\mathbf{k},C)\leq r}} s_{\mathbf{k}}^t \, \frac{1}{N^n}.$$

Hence

$$\sum_{\substack{|\mathbf{k}|=n \\ UM_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t \leq N^n \int_{\{\mathbf{j}\in\Sigma^{\mathbb{N}} \,|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\}} s_{\mathbf{i}|n}^t \, dP(\mathbf{i}).$$

This completes the proof of Claim 2. □

Combining Claims 1 and 2 shows that

$$\sum_{\substack{|\mathbf{k}|=n \\ UL_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t \leq \sum_{\substack{|\mathbf{k}|=n \\ UM_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t \leq N^n \int_{\{\mathbf{j}\in\Sigma^{\mathbb{N}} \,|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\}} s_{\mathbf{i}|n}^t \, dP(\mathbf{i}). \tag{5.2}$$

Let c be the constant from Condition (C3) in Sect. 2.1, and notice that it follows from Proposition 4.1 that if $\mathbf{i}\in\Sigma^{\mathbb{N}}$ and n is a positive integer, then we have $s_{\mathbf{i}|n}^t \leq c^{|t|}\exp(\, t\sum_{k=0}^{n-1}\Lambda S^k(\,\overline{\mathbf{i}|n}\,)\,)$. We conclude from this and (5.2) that

$$\begin{aligned}
\sum_{\substack{|\mathbf{k}|=n \\ UL_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t &\leq N^n \int_{\{\mathbf{j}\in\Sigma^{\mathbb{N}} \,|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\}} s_{\mathbf{i}|n}^t \, dP(\mathbf{i}) \\
&\leq c^{|t|}\, N^n \int_{\{\mathbf{j}\in\Sigma^{\mathbb{N}} \,|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\}} \exp\left(t\sum_{k=0}^{n-1}\Lambda S^k\left(\overline{\mathbf{i}|n}\right)\right) dP(\mathbf{i}) \\
&= c^{|t|}\, N^n \int_{\{\mathbf{j}\in\Sigma^{\mathbb{N}} \,|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\}} \exp\left(tn\int\Lambda\, d(M_n\mathbf{i})\right) dP(\mathbf{i}) \\
&= c^{|t|}\, N^n \int_{\{\mathbf{j}\in\Sigma^{\mathbb{N}} \,|\, UM_n[\mathbf{j}|n]\subseteq B(C,r)\}} \exp\left(nF(M_n\mathbf{i})\right) dP(\mathbf{i}).
\end{aligned} \tag{5.3}$$

Noticing that $\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,|\,UM_n[\mathbf{j}|n]\subseteq B(C,r)\}\subseteq\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,|\,UM_n\mathbf{j}\subseteq B(C,r)\}=\{UM_n\in B(C,r)\}$, we now deduce from (5.3) that

$$\begin{aligned}\sum_{\substack{|\mathbf{k}|=n\\ UL_n[\mathbf{k}]\subseteq B(C,r)}} s_{\mathbf{k}}^t &\le c^{|t|}\,N^n \int_{\left\{\mathbf{j}\in\Sigma^{\mathbb{N}}\,\middle|\,UM_n[\mathbf{j}|n]\subseteq B(C,r)\right\}} \exp\left(nF(M_n\mathbf{i})\right)\,dP(\mathbf{i})\\ &\le c^{|t|}\,N^n \int_{\left\{UM_n\in B(C,r)\right\}} \exp\left(nF(M_n\mathbf{i})\right)\,dP(\mathbf{i})\\ &= c^{|t|}\,N^n \int_{\left\{U\in B(C,r)\right\}} \exp\left(nF\right)\,dP_n\\ &= c^{|t|}\,N^n\,Q_n\Big(\Big\{U\in B(C,r)\Big\}\Big)\int \exp\left(nF\right)\,dP_n. \end{aligned}\tag{5.4}$$

It follows immediately from (5.4) that

$$\begin{aligned}\limsup_n \frac{1}{n}\log \sum_{\substack{|\mathbf{i}|=n\\ UL_n[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t \le \log N \,+\, \limsup_n \frac{1}{n}\log Q_n\Big(\Big\{U\in B(C,r)\Big\}\Big)\\ +\,\limsup_n\frac{1}{n}\log\int \exp\left(nF\right)\,dP_n.\end{aligned}\tag{5.5}$$

Next, we observe that it follows from [9] that the sequence $(P_n=P\circ M_n^{-1})_n\subseteq\mathcal{P}\big(\mathcal{P}_S(\Sigma^{\mathbb{N}})\big)$ has the large deviation property with respect to the sequence $(n)_n$ and rate function $I:\mathcal{P}_S(\Sigma^{\mathbb{N}})\to\mathbb{R}$ given by $I(\mu)=\log N-h(\mu)$. We therefore conclude from Part (1) of Theorem 5.1 that

$$\limsup_n \frac{1}{n}\log\int \exp\left(nF\right)\,dP_n = -\inf_{\nu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})}(I(\nu)-F(\nu)).\tag{5.6}$$

Also, since the sequence $(P_n=P\circ M_n^{-1})_n\subseteq\mathcal{P}\big(\mathcal{P}_S(\Sigma^{\mathbb{N}})\big)$ has the large deviation property with respect to the sequence $(n)_n$ and rate function $I:\mathcal{P}_S(\Sigma^{\mathbb{N}})\to\mathbb{R}$ given by $I(\mu)=\log N-h(\mu)$, we conclude from Part (2) of Theorem 5.1 that the sequence $(Q_n)_n$ has the large deviation property with respect to the sequence $(n)_n$ and rate function $(I-F)-\inf_{\nu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})}(I(\nu)-F(\nu))$. As the set $\{U\in B(C,r)\}=U^{-1}(B(C,r))$ is closed, it therefore follows from the large deviation property that

$$\begin{aligned}&\limsup_n \frac{1}{n}\log Q_n\Big(\Big\{U\in B(C,r)\Big\}\Big)\\ &\le -\inf_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,r)}}\left((I(\mu)-F(\mu))-\inf_{\nu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})}(I(\nu)-F(\nu))\right).\end{aligned}\tag{5.7}$$

Combining (5.5). (5.6) and (5.7) now yields

$$\begin{aligned}\limsup_n \frac{1}{n}\log \sum_{\substack{|\mathbf{i}|=n\\ UL_n[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t &\le \log N + \limsup_n \frac{1}{n}\log Q_n\Big(\Big\{U\in B(C,r)\Big\}\Big)\\ &\quad + \limsup_n \frac{1}{n}\log\int \exp(nF)\,dP_n\\ &\le \log N - \inf_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,r)}}((I(\mu)-F(\mu))\\ &\quad -\inf_{\nu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})}(I(\nu)-F(\nu))\Bigg)\\ &\quad -\inf_{\nu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})}(I(\nu)-F(\nu))\\ &= \log N + \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,r)}}(F(\mu)-I(\mu))\\ &= \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,r)}}\left(t\int\Lambda\,d\mu+h(\mu)\right).\end{aligned}$$

This completes the proof. □

We will now use Theorem 5.2 to prove Theorem 5.5 providing a proof of inequality (2.1). However, we first prove two small lemmas.

Lemma 5.3. *Let X be a metric space and let $f,g : X\to\mathbb{R}$ be upper semi-continuous functions with $f,g\ge 0$. Then fg is upper semi-continuous.*

Proof. Since f and g are upper semi-continuous with $f,g\ge 0$, this result follows easily from the definition of upper semi-continuity, and the proof is therefore omitted. □

Lemma 5.4. *Let X be a metric space and let $\Phi : X\to\mathbb{R}$ be an upper semi-continuous function. Let $K_1,K_2,\ldots\subseteq X$ be non-empty compact subsets of X with $K_1\supseteq K_2\supseteq\cdots$. Then*

$$\inf_n \sup_{x\in K_n}\Phi(x) = \sup_{x\in\cap_n K_n}\Phi(x).$$

Proof. First note that it is clear that $\inf_n \sup_{x\in K_n}\Phi(x)\ge \sup_{x\in\cap_n K_n}\Phi(x)$. We will now prove the reverse inequality, namely, $\inf_n \sup_{x\in K_n}\Phi(x)\le \sup_{x\in\cap_n K_n}\Phi(x)$. Let $\varepsilon>0$. For each n, we can choose $x_n\in K_n$ such that $\Phi(x_n)\ge \sup_{x\in K_n}\Phi(x)-\varepsilon$. Next, since K_n is compact for all n and $K_1\supseteq K_2\supseteq\cdots$, we can find a subsequence $(x_{n_k})_k$ and a point $x_0\in\cap_n K_n$ such that $x_{n_k}\to x_0$. Also, since $K_{n_1}\supseteq K_{n_2}\supseteq\cdots$, we conclude that $\sup_{x\in K_{n_1}}\Phi(x)\ge \sup_{x\in K_{n_2}}\Phi(x)\ge\cdots$, whence $\inf_k \sup_{x\in K_{n_k}}\Phi(x)=\limsup_k \sup_{x\in K_{n_k}}\Phi(x)$.

This implies that $\inf_n \sup_{x\in K_n} \Phi(x) \le \inf_k \sup_{x\in K_{n_k}} \Phi(x) = \limsup_k \sup_{x\in K_{n_k}} \Phi(x) \le \limsup_k \Phi(x_{n_k}) + \varepsilon$. However, since $x_{n_k} \to x_0$, we deduce from the upper semi-continuity of the function Φ, that $\limsup_k \Phi(x_{n_k}) \le \Phi(x_0)$. Consequently $\inf_n \sup_{x\in K_n} \Phi(x) \le \limsup_k \Phi(x_{n_k}) + \varepsilon \le \Phi(x_0) + \varepsilon \le \sup_{x\in\cap_n K_n} \Phi(x) + \varepsilon$. Finally, letting $\varepsilon \searrow 0$ gives the desired result. □

We can now state and prove Theorem 5.5.

Theorem 5.5. *Let X be a metric space and let $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a closed subset of X and $r > 0$.*

(1) *We have*

$$\sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}(\cdot;r)\big) \le \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,r)}} -\frac{h(\mu)}{\int \Lambda\,d\mu}.$$

(2) *We have*

$$\limsup_{r\searrow 0}\ \sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}(\cdot;r)\big) \le \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in C}} -\frac{h(\mu)}{\int \Lambda\,d\mu}.$$

Proof. (1) For brevity write

$$u = \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,r)}} -\frac{h(\mu)}{\int \Lambda\,d\mu}.$$

We must now prove that if $t > u$, then

$$\sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t < \infty.$$

Let $t > u$ and write $\varepsilon = \frac{t-u}{3} > 0$. It follows from the definition of u that if $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$ with $U\mu \in B(C,r)$, then we have $-\frac{h(\mu)}{\int \Lambda\,d\mu} < u + \varepsilon = (u+2\varepsilon) - \varepsilon$, whence $-h(\mu) > (u+2\varepsilon)\int \Lambda\,d\mu - \varepsilon\int \Lambda\,d\mu$ where we have used the fact that $\int \Lambda\,d\mu < 0$ because $\Lambda < 0$. This implies that if $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$ with $U\mu \in B(C,r)$, then

$$\begin{aligned}(u+2\varepsilon)\int \Lambda\,d\mu + h(\mu) &\le \varepsilon\int \Lambda\,d\mu\\ &\le \varepsilon c_{\max}\\ &= -\varepsilon\,|c_{\max}|.\end{aligned}$$

We deduce from this inequality and Theorem 5.2 that

$$
\begin{aligned}
\limsup_n \frac{1}{n}\log \sum_{\substack{|\mathbf{i}|=n\\ UL_n[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t
&= \limsup_n \frac{1}{n}\log \sum_{\substack{|\mathbf{i}|=n\\ UL_n[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^{u+3\varepsilon}\\
&\le \limsup_n \frac{1}{n}\log \sum_{\substack{|\mathbf{i}|=n\\ UL_n[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^{u+2\varepsilon}\\
&\le \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,r)}} \left((u+2\varepsilon)\int \Lambda\, d\mu + h(\mu)\right) \qquad \text{[by Theorem 5.2]}\\
&\le -\varepsilon\,|c_{\max}|\\
&< -\tfrac{1}{2}\varepsilon\,|c_{\max}|. \qquad (5.8)
\end{aligned}
$$

Inequality (5.8) shows that there is an integer N_0 such that $\frac{1}{n}\log \sum_{|\mathbf{i}|=n\,,\,UL_n[\mathbf{i}]\subseteq B(C,r)} s_{\mathbf{i}}^t \le -\frac{1}{2}\varepsilon\,|c_{\max}|$ for all $n\ge N_0$, whence

$$
\sum_{\substack{|\mathbf{i}|=n\\ UL_n[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t \le e^{-\frac{1}{2}\varepsilon\,|c_{\max}|\,n} \qquad (5.9)
$$

for all $n\ge N_0$. Using (5.9) we now conclude that

$$
\begin{aligned}
\sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t
&= \sum_{n<N_0}\sum_{\substack{|\mathbf{i}|=n\\ UL_n[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t + \sum_{n\ge N_0}\sum_{\substack{|\mathbf{i}|=n\\ UL_n[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t\\
&\le \sum_{n<N_0}\sum_{\substack{|\mathbf{i}|=n\\ UL_n[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t + \sum_{n\ge N_0} e^{-\frac{1}{2}\varepsilon\,|c_{\max}|\,n}\\
&< \infty.
\end{aligned}
$$

This completes the proof of (1).
(2) It follows immediately from Part (1) that

$$
\limsup_{r\searrow 0}\ \sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}(\cdot;r)\big) \le \limsup_{r\searrow 0}\ \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,r)}} -\frac{h(\mu)}{\int\Lambda\,d\mu}. \qquad (5.10)
$$

Also, the function $r \to \sup_{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\,,\,U\mu\in B(C,r)} -\frac{h(\mu)}{\int\Lambda\,d\mu}$ is clearly increasing, and it therefore follows that

$$\limsup_{r\searrow 0} \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,r)}} -\frac{h(\mu)}{\int\Lambda\,d\mu} = \inf_k \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,\frac{1}{k})}} -\frac{h(\mu)}{\int\Lambda\,d\mu}. \tag{5.11}$$

Next, since the function $U:\mathcal{P}(\Sigma^{\mathbb{N}})\to X$ is continuous, we conclude that the set $U^{-1}B(C,\frac{1}{k})$ is closed, and it therefore follows that the set $K_k = \mathcal{P}_S(\Sigma^{\mathbb{N}})\cap U^{-1}B(C,\frac{1}{k})$ is compact. Also, since the entropy function $h:\mathcal{P}_S(\Sigma^{\mathbb{N}})\to\mathbb{R}$ is upper semi-continuous (see [47, Theorem 8.2]) with $h\geq 0$ and the function $f:\mathcal{P}_S(\Sigma^{\mathbb{N}})\to\mathbb{R}$ given by $f(\mu) = -\frac{1}{\int\Lambda\,d\mu}$ is continuous (because Λ is continuous) with $f\geq 0$, we conclude from Lemma 5.3 that the function $\Phi:\mathcal{P}_S(\Sigma^{\mathbb{N}})\to\mathbb{R}$ given by $\Phi(\mu) = f(\mu)h(\mu) = -\frac{h(\mu)}{\int\Lambda\,d\mu}$ is upper semi-continuous. Lemma 5.4 applied to Φ therefore implies that

$$\begin{aligned}\inf_k \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,\frac{1}{k})}} -\frac{h(\mu)}{\int\Lambda\,d\mu} &= \inf_k \sup_{\mu\in K_k} -\frac{h(\mu)}{\int\Lambda\,d\mu}\\ &= \sup_{\mu\in\cap_k K_k} -\frac{h(\mu)}{\int\Lambda\,d\mu}.\end{aligned} \tag{5.12}$$

However, clearly $\cap_k K_k = \cap_k(\mathcal{P}_S(\Sigma^{\mathbb{N}})\cap U^{-1}B(C,\frac{1}{k})) = \mathcal{P}_S(\Sigma^{\mathbb{N}})\cap U^{-1}C$, whence

$$\sup_{\mu\in\cap_k K_k} -\frac{h(\mu)}{\int\Lambda\,d\mu} = \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in C}} -\frac{h(\mu)}{\int\Lambda\,d\mu}. \tag{5.13}$$

Combining (5.12) and (5.13) gives

$$\inf_k \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in B(C,\frac{1}{k})}} -\frac{h(\mu)}{\int\Lambda\,d\mu} = \sum_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in C}} -\frac{h(\mu)}{\int\Lambda\,d\mu}. \tag{5.14}$$

Finally, the desired result follows by combining (5.10), (5.11) and (5.14). $\square$

6. Proof of inequality (2.2)

The purpose of this section is to prove Theorem 6.6 providing a proof of inequality (2.2).

We first state and prove a number of auxiliary results. For $\mathbf{i},\mathbf{j}\in\Sigma^{\mathbb{N}}$ with $\mathbf{i}\neq\mathbf{j}$, we will write $\mathbf{i}\wedge\mathbf{j}$ for the longest common prefix of $\mathbf{i}$ and $\mathbf{j}$ (i.e. $\mathbf{i}\wedge\mathbf{j} = \mathbf{u}$ where $\mathbf{u}$ is the unique element in Σ^* for which there are $\mathbf{k},\mathbf{l}\in\Sigma^{\mathbb{N}}$ with $\mathbf{k} =$

$k_1k_2\dots$ and $\mathbf{l} = l_1l_2\dots$ such that $k_1 \neq l_1$, $\mathbf{i} = \mathbf{uk}$ and $\mathbf{j} = \mathbf{ul}$). We will always equip $\Sigma^{\mathbb{N}}$ with the metric $\mathsf{d}_{\Sigma^{\mathbb{N}}}$ defined by

$$\mathsf{d}_{\Sigma^{\mathbb{N}}}(\mathbf{i},\mathbf{j}) = \begin{cases} 0 & \text{if } \mathbf{i} = \mathbf{j}; \\ s_{\mathbf{i}\wedge\mathbf{j}} & \text{if } \mathbf{i} \neq \mathbf{j}, \end{cases} \tag{6.1}$$

for $\mathbf{i},\mathbf{j} \in \Sigma^{\mathbb{N}}$. In the results below, we will always compute the Hausdorff dimension of a subset of $\Sigma^{\mathbb{N}}$ with respect to the metric $\mathsf{d}_{\Sigma^{\mathbb{N}}}$. Note that when $\Sigma^{\mathbb{N}}$ is equipped with the metric $\mathsf{d}_{\Sigma^{\mathbb{N}}}$, then

$$\operatorname{diam}[\mathbf{i}] = s_{\mathbf{i}} \tag{6.2}$$

for all $\mathbf{i} \in \Sigma^*$.

Lemma 6.1. *Let (X, d) be a metric space and let $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let C be a closed subset of X and $r > 0$.*

(1) *There is a positive integer M_r such that if $k \geq M_r$, $\mathbf{u} \in \Sigma^k$ and $\mathbf{k},\mathbf{l} \in \Sigma^{\mathbb{N}}$, then*

$$\mathsf{d}\big(UL_k(\mathbf{uk})\,,\,UL_k(\mathbf{ul})\big) \leq \frac{r}{2}.$$

(2) *There is a positive integer M_r such that if $m \geq M_r$, then*

$$\Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, UL_k\mathbf{i} \in B(C, \tfrac{r}{2}) \quad \text{for all } k \geq m\Big\} \subseteq \Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, UL_k[\mathbf{i}|k] \subseteq B(C, r) \quad \text{for all } k \geq m\Big\}.$$

Proof. (1) For a function $f : \Sigma^{\mathbb{N}} \to \mathbb{R}$, let $\operatorname{Lip}(f)$ denote the Lipschitz constant of f, i.e. $\operatorname{Lip}(f) = \sup_{\mathbf{i},\mathbf{j}\in\Sigma^{\mathbb{N}},\mathbf{i}\neq\mathbf{j}} \frac{|f(\mathbf{i})-f(\mathbf{j})|}{\mathsf{d}_{\Sigma^{\mathbb{N}}}(\mathbf{i},\mathbf{j})}$ and define the metric L in $\mathcal{P}(\Sigma^{\mathbb{N}})$ by

$$\mathsf{L}(\mu,\nu) = \sup_{\substack{f:\Sigma^{\mathbb{N}}\to\mathbb{R} \\ \operatorname{Lip}(f)\leq 1}} \left| \int f\,d\mu - \int f\,d\nu \right|.$$

We note that it is well-known that L is a metric and that L induces the weak topology. Since $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$ is continuous and $\mathcal{P}(\Sigma^{\mathbb{N}})$ is compact, we conclude that $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$ is uniformly continuous. This implies that we can choose $\delta > 0$ such that all measures $\mu,\nu \in \mathcal{P}(\Sigma^{\mathbb{N}})$ satisfy the following implication:

$$\mathsf{L}(\mu,\nu) \leq \delta \quad \Rightarrow \quad \mathsf{d}(U\mu, U\nu) \leq \tfrac{r}{2}. \tag{6.3}$$

Next, choose a positive integer M_r such that

$$\frac{1}{M_r(1 - s_{\max})} < \delta\,; \tag{6.4}$$

recall, that $s_{\max}$ is defined in (4.1).

If $k \geq M_r$, $\mathbf{u} \in \Sigma^k$ and $\mathbf{k}, \mathbf{l} \in \Sigma^{\mathbb{N}}$, then it follows from (6.4) that

$$\begin{aligned}
\mathsf{L}\big(\, L_k(\mathbf{uk})\,,\, L_k(\mathbf{ul})\,\big) &= \sup_{\substack{f:\Sigma^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}(f)\leq 1}} \left| \int f\, d(L_k(\mathbf{uk})) - \int f\, d(L_k(\mathbf{ul})) \right| \\
&= \sup_{\substack{f:\Sigma^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}(f)\leq 1}} \left| \frac{1}{k}\sum_{i=0}^{k-1} f(S^i(\mathbf{uk})) - \frac{1}{k}\sum_{i=0}^{k-1} f(S^i(\mathbf{ul})) \right| \\
&\leq \sup_{\substack{f:\Sigma^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}(f)\leq 1}} \frac{1}{k}\sum_{i=0}^{k-1} |f(S^i(\mathbf{uk})) - f(S^i(\mathbf{ul}))| \\
&\leq \frac{1}{k}\sum_{i=0}^{k-1} \mathsf{d}_{\Sigma^{\mathbb{N}}}\big(\, S^i(\mathbf{uk})\,,\, S^i(\mathbf{ul})\,\big) \\
&= \frac{1}{k}\sum_{i=0}^{k-1} s_{S^i(\mathbf{uk})\wedge S^i(\mathbf{ul})} \\
&\leq \frac{1}{M_r}\sum_{i=0}^{k-1} s_{\max}^{k-i} \\
&\leq \frac{1}{M_r(1-s_{\max})} \\
&< \delta\,,
\end{aligned}$$

and we therefore conclude from (6.3) that $\mathsf{d}(\, UL_k(\mathbf{uk})\,,\, UL_k(\mathbf{ul})\,) \leq \frac{r}{2}$.
(2) It follows from (1) that there is a positive integer M_r such that if $k \geq M_r$, $\mathbf{u} \in \Sigma^k$ and $\mathbf{k}, \mathbf{l} \in \Sigma^{\mathbb{N}}$, then $\mathsf{d}(\, UL_k(\mathbf{uk})\,,\, UL_k(\mathbf{ul})\,) \leq \frac{r}{2}$.

We now claim that if $m \geq M_r$, then

$$\begin{aligned}
&\Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, UL_k\mathbf{i} \in B(C, \tfrac{r}{2}) \quad \text{for all } k \geq m\Big\} \\
&\qquad\qquad \subseteq \Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, UL_k[\mathbf{i}|k] \subseteq B(C, r) \quad \text{for all } k \geq m\Big\}.
\end{aligned}$$

In order to prove this inclusion, we fix $m \geq M_r$ and $\mathbf{i} \in \Sigma^{\mathbb{N}}$ with $UL_k\mathbf{i} \in B(C, \frac{r}{2})$ for all $k \geq m$. We must now prove that $UL_k[\mathbf{i}|k] \subseteq B(C, r)$ for all $k \geq m$. We therefore fix $k \geq m$ and $\mathbf{j} \in [\mathbf{i}|k]$. We must now prove that $UL_k\mathbf{j} \in B(C, r)$. For brevity write $\mathbf{u} = \mathbf{i}|k$. Since $\mathbf{j} \in [\mathbf{i}|k] = [\mathbf{u}]$, we can now find (unique) $\mathbf{k}, \mathbf{l} \in \Sigma^{\mathbb{N}}$ such that $\mathbf{i} = \mathbf{uk}$ and $\mathbf{j} = \mathbf{ul}$. We now have

$$\begin{aligned}
\mathrm{dist}\,\big(\, UL_k\mathbf{j}\,,\, C\,\big) &\leq \mathsf{d}\big(\, UL_k\mathbf{j}\,,\, UL_k\mathbf{i}\,\big) + \mathrm{dist}\,\big(\, UL_k\mathbf{i}\,,\, C\,\big) \\
&= \mathsf{d}\big(\, UL_k(\mathbf{ul})\,,\, UL_k(\mathbf{uk})\,\big) + \mathrm{dist}\,\big(\, UL_k\mathbf{i}\,,\, C\,\big). \qquad (6.5)
\end{aligned}$$

However, since $k \geq m \geq M_r$ and $\mathbf{u} \in \Sigma^k$, we conclude that $\mathsf{d}(\, UL_k (\mathbf{uk})\,,\, UL_k(\mathbf{ul})\,) \leq \frac{r}{2}$. Also, since $k \geq m$, we deduce that $UL_k\mathbf{i} \in B(C, \frac{r}{2})$, whence $\operatorname{dist}(UL_k\mathbf{i}\,,\,C\,) \leq \frac{r}{2}$. It therefore follows from (6.5) that

$$\begin{aligned}\operatorname{dist}\big(UL_k\mathbf{j}\,,\,C\,\big) &= \mathsf{d}\big(UL_k(\mathbf{ul})\,,\,UL_k(\mathbf{uk})\big) + \operatorname{dist}\big(UL_k\mathbf{i}\,,\,C\,\big)\\ &\leq \frac{r}{2} + \frac{r}{2}\\ &= r.\end{aligned}$$

This completes the proof. □

Lemma 6.2. *Let X be a metric space and let $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a closed subset of X. Then*

$$\dim_{\mathsf{H}}\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\lim_m \operatorname{dist}(\,UL_m\mathbf{i}, C\,) = 0\right\} \leq \underline{f}^{U,\Lambda}(C)\,;$$

recall that $\dim_{\mathsf{H}}$ *denotes the Hausdorff dimension.*

Proof. For a subset Ξ of $\Sigma^{\mathbb{N}}$, we let $\underline{\dim}_{\mathsf{B}}\Xi$ denote the lower box dimension of Ξ; the reader is referred to [10] for the definition of the lower box dimension. We will use the fact that $\dim_{\mathsf{H}}\Xi \leq \underline{\dim}_{\mathsf{B}}\Xi$ for all $\Xi \subseteq \Sigma^{\mathbb{N}}$, see, for example, [8].

We now introduce the following notation. For brevity write

$$\Gamma = \left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\lim_m \operatorname{dist}(\,UL_m\mathbf{i}, C\,) = 0\right\}.$$

Also, for a positive integer m and a positive real number $r > 0$, write

$$\begin{aligned}\Gamma_m(r) &= \left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,UL_k\mathbf{i}\in B(C,r) \text{ for all } k\geq m\right\},\\ \Delta_m(r) &= \left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,UL_k[\mathbf{i}|k]\subseteq B(C,r) \text{ for all } k\geq m\right\}.\end{aligned}$$

Observe that if M is any positive integer, then we clearly have

$$\Gamma \subseteq \bigcup_{m\geq M}\Gamma_m(\tfrac{r}{2}) \tag{6.6}$$

for all $r > 0$. We also observe that it follows from Lemma 6.1 that for each positive number $r > 0$ there is a positive integer M_r such that

$$\Gamma_m(\tfrac{r}{2}) \subseteq \Delta_m(r) \tag{6.7}$$

for all $m \geq M_r$. It follows from (6.6) and (6.7) that

$$\begin{aligned}\Gamma &\subseteq \bigcup_{m\geq M_r}\Gamma_m(\tfrac{r}{2})\\ &\subseteq \bigcup_{m\geq M_r}\Delta_m(r)\,,\end{aligned}$$

whence

$$\begin{aligned}\dim_{\mathsf{H}}\Gamma &\le \dim_{\mathsf{H}}\left(\bigcup_{m\ge M_r}\Delta_m(r)\right)\\ &= \sup_{m\ge M_r}\dim_{\mathsf{H}}\Delta_m(r)\\ &\le \sup_{m\ge M_r}\underline{\dim}_{\mathsf{B}}\Delta_m(r) \qquad (6.8)\end{aligned}$$

for all $r>0$.

Fix a positive integer m. We now prove that

$$\Delta_m(r)\subseteq\bigcup_{\mathbf{i}\in\Pi_\delta^{U,\Lambda}(C,r)}[\mathbf{i}] \qquad (6.9)$$

for all $0<\delta<s_{\min}^m$ and all $r>0$. Indeed, fix $\mathbf{j}\in\Delta_m(r)$. Now, let k_0 denote the unique positive integer such that if we write $\mathbf{j}_0=\mathbf{j}|k_0$, then $s_{\mathbf{j}_0}\le\delta<s_{\widehat{\mathbf{j}_0}}$, i.e. $s_{\mathbf{j}_0}\approx\delta$. Since it follows from Proposition 4.1 that $s_{\min}^{k_0}=s_{\min}^{|\mathbf{j}_0|}\le s_{\mathbf{j}_0}\le\delta<s_{\min}^m$, we conclude that $k_0\ge m$, and the fact that $\mathbf{j}\in\Delta_m(r)$ therefore implies that $UL_{|\mathbf{j}_0|}[\mathbf{j}_0]=UL_{k_0}[\mathbf{j}|k_0]\subseteq B(C,r)$. This shows that $\mathbf{j}_0\in\Pi_\delta^{U,\Lambda}(C,r)$, whence $\mathbf{j}\in[\mathbf{j}|k_0]=[\mathbf{j}_0]\subseteq\cup_{\mathbf{i}\in\Pi_\delta^{U,\Lambda}(C,r)}[\mathbf{i}]$. This proves (6.9).

Inclusion (6.9) shows that for all $0<\delta<s_{\min}^m$, the family $(\,[\mathbf{i}]\,)_{\mathbf{i}\in\Pi_\delta^{U,\Lambda}(C,r)}$ is a covering of $\Delta_m(r)$ of sets $[\mathbf{i}]$ with $\mathbf{i}\in\Pi_\delta^{U,\Lambda}(C,r)$ such that $\operatorname{diam}[\mathbf{i}]=s_{\mathbf{i}}\le\delta$ for all $\mathbf{i}\in\Pi_\delta^{U,\Lambda}(C,r)$. This implies that

$$\underline{\dim}_{\mathsf{B}}\Delta_m(r)\le\liminf_{\delta\searrow0}\frac{\log|\Pi_\delta^{U,\Lambda}(C,r)|}{-\log\delta} \qquad (6.10)$$

for all $r>0$. Since (6.10) holds for all m, we conclude that

$$\sup_{m\ge M_r}\underline{\dim}_{\mathsf{B}}\Delta_n(r)\le\liminf_{\delta\searrow0}\frac{\log|\Pi_\delta^{U,\Lambda}(C,r)|}{-\log\delta} \qquad (6.11)$$

for all $r>0$.

Combining (6.8) and (6.11) now shows that

$$\dim_{\mathsf{H}}\Gamma\le\liminf_{\delta\searrow0}\frac{\log|\Pi_\delta^{U,\Lambda}(C,r)|}{-\log\delta} \qquad (6.12)$$

for all $r>0$. Finally, letting $r\searrow0$ in (6.12) completes the proof. □

In order to state and prove the next lemma we introduce the following notation. Namely, for a Hölder continuous function $\varphi:\Sigma^{\mathbb{N}}\to\mathbb{R}$, we will write

$$P(\varphi)$$

for the topological pressure of φ. We can now state and prove Lemma 6.3.

Lemma 6.3. *Let $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$ with $\operatorname{supp}\mu = \Sigma^{\mathbb{N}}$. (Here $\operatorname{supp}\mu$ denotes the topological support of μ.) Then there exists a sequence $(\mu_n)_n$ of probability measures on $\Sigma^{\mathbb{N}}$ satisfying the following three conditions.*

1. *We have $\mu_n \to \mu$ weakly.*
2. *For each n, the measure μ_n is ergodic.*
3. *We have $h(\mu_n) \to h(\mu)$.*

Proof. Fix a positive integer n. Since $\operatorname{supp}\mu = \Sigma^{\mathbb{N}}$, we deduce that $\mu[\mathbf{i}] > 0$ for all $\mathbf{i} \in \Sigma^*$. Hence, for $m \in \mathbb{N}$ and $i_1 \cdots i_m \in \Sigma^m$, we can define $p_{n,i_1\cdots i_m}$ by

$$p_{n,i_1\cdots i_m} = \begin{cases} \mu[i_1 \cdots i_m] & \text{for } m \le n, \\ \prod_{k=1}^{m-n} \frac{\mu[i_k i_{k+1} \cdots i_{k+(n-1)}]}{\mu[i_{k+1} \cdots i_{k+(n-1)}]} \; \mu[i_{(m-n)+1} \ldots i_m] & \text{for } n < m. \end{cases} \tag{6.13}$$

Since clearly $\sum_i p_{n,i} = 1$ and $\sum_i p_{n,i_1\cdots i_m i} = p_{n,i_1\cdots i_m}$ for all m and all $i_1 \cdots i_m \in \Sigma^m$, there exists a (unique) probability measure μ_n on $\Sigma^{\mathbb{N}}$ such that

$$\mu_n[i_1 \cdots i_m] = p_{n,i_1\cdots i_m}$$

for all m and all $i_1 \cdots i_m \in \Sigma^m$ (cf. [Wa, p. 5]). □

Claim 1. We have $\mu_n \to \mu$ weakly.

Proof of Claim 1. It follows from definition (6.13) that $\mu_n[\mathbf{i}] = \mu[\mathbf{i}]$ for all $\mathbf{i} \in \Sigma^n$. This clearly implies that $\mu_n \to \mu$ weakly. This completes the proof of Claim 1.

□

Claim 2. For each n, there is a Hölder continuous function $\varphi_n : \Sigma^{\mathbb{N}} \to \mathbb{R}$ such that the following conditions hold.

1. *$P(\varphi_n) = 0$,*
2. *The measure μ_n is a Gibbs state of φ_n.*

Proof of Claim 2. We first note that μ_n is shift invariant. Indeed, since μ is shift invariant, a small calculation shows that $\sum_i \mu_n[i\mathbf{i}] = \mu_n[\mathbf{i}]$ for all $\mathbf{i} \in \Sigma^*$. This implies that $\mu_n(S^{-1}[\mathbf{i}]) = \mu_n[\mathbf{i}]$ for all $\mathbf{i} \in \Sigma^*$, whence $\mu_n(S^{-1}B) = \mu_n(B)$ for all Borel sets B.

Next we show that μ_n is a Gibbs state for a Hölder continuous function. Define $\varphi_n : \Sigma^{\mathbb{N}} \to \mathbb{R}$ by

$$\varphi_n(i_1 i_2 \ldots) = \log\left(\frac{\mu[i_1 i_2 \cdots i_n]}{\mu[i_2 \cdots i_n]}\right).$$

The map φ_n is clearly Hölder continuous, and it follows from the definition of μ_n that

$$e^{-n\|\varphi_n\|_\infty}\min_{\mathbf{j}\in\Sigma^n}\mu[\mathbf{j}]\leq\frac{\mu_n[\mathbf{i}|m]}{e^{\sum_{k=0}^{m-1}\varphi_n(S^k\mathbf{i})}}\leq e^{n\|\varphi_n\|_\infty}\max_{\mathbf{j}\in\Sigma^n}\mu[\mathbf{j}]$$

for all $\mathbf{i}\in\Sigma^{\mathbb{N}}$ and all $m>n$. This shows that μ_n is the Gibbs state of φ_n, and that the pressure $P(\varphi_n)$ of φ_n equals 0, i.e. $P(\varphi_n)=0$; cf. [Bo]. This completes the proof of Claim 2.

□

Claim 3. For each n, the measure μ_n is ergodic.

Proof of Claim 3. It follows from Claim 2 that μ_n is the Gibbs state of a Hölder continuous function. This implies that μ_n is ergodic. This completes the proof of Claim 3. □

Claim 4. We have $h(\mu_n)\to h(\mu)$.

Proof of Claim 4. For measurable partitions $\mathcal{A},\mathcal{B}$ of Σ, let $h(\mu;\mathcal{A})$ and $h(\mu;\mathcal{A}|\mathcal{B})$ denote the entropy of $\mathcal{A}$ with respect to μ, and the conditional entropy of $\mathcal{A}$ given $\mathcal{B}$ with respect to μ, respectively. Write $\mathcal{C}=\{\,[i]\,|\,i\in\Sigma\}$ and $\mathcal{C}_n=\vee_{k=0}^{n-1}S^{-k}\mathcal{C}=\{\,[\mathbf{i}]\,|\,\mathbf{i}\in\Sigma^n\}$. It follows from Claim 2 that there is a Hölder continuous function $\varphi_n:\Sigma^{\mathbb{N}}\to\mathbb{R}$ with $P(\varphi_n)=0$ such that μ_n is a Gibbs state of φ_n. Since $P(\varphi_n)=0$ and μ_n is a Gibbs state of φ_n, the Variational Principle now shows that $0=P(\varphi_n)=h(\mu_n)+\int\varphi_n\,d\mu_n$ (cf. [Bo]), whence

$$\begin{aligned}h(\mu_n)&=-\int\varphi_n\,d\mu_n\\&=-\sum_{i_1\cdots i_n}\mu[i_1\cdots i_n]\log\left(\frac{\mu[i_1i_2\cdots i_n]}{\mu[i_2\cdots i_n]}\right)\\&=\;h(\mu;\mathcal{C}_n|\mathcal{C}_{n-1}).\end{aligned}\tag{6.14}$$

Next, we note that it follows from [DGS, 11.4] that $h(\mu;\mathcal{C}_n|\mathcal{C}_{n-1})\to h(\mu;\mathcal{C})$, and we therefore conclude from (6.14) that

$$h(\mu_n)\to h(\mu;\mathcal{C}).\tag{6.15}$$

Finally, it follows immediately from the Kolmogoroff-Sinai theorem that $h(\mu;\mathcal{C})=h(\mu)$. This and (6.15) now show that

$$h(\mu_n)\to h(\mu).$$

This completes the proof of Claim 4. □

The proof now follows from Claim 1, Claim 3 and Claim 4.

The next auxiliary result provides a formula for the upper Hausdorff dimension of a probability measure. If μ is a probability measure on $\Sigma^{\mathbb{N}}$, we define the upper Hausdorff dimension of μ by

$$\overline{\dim}_{\mathsf{H}}\mu = \inf_{\substack{\Xi\subseteq\Sigma^{\mathbb{N}}\\ \mu(\Xi)=1\\ \mathsf{H}}} \Xi.$$

(Recall that $\dim_{\mathsf{H}}$ denotes the Hausdorff dimension.) The next result provides a formula for the upper Hausdorff dimension of an ergodic probability measure on $\Sigma^{\mathbb{N}}$. This result is folklore and follows from the Shannon-MacMillan-Breiman theorem and the ergodic theorem. However, for sake of completeness we have decided to include the short proof.

Proposition 6.4. *Let μ be an ergodic probability measure on $\Sigma^{\mathbb{N}}$. Then* $\overline{\dim}_{\mathsf{H}}\mu = -\frac{h(\mu)}{\int \Lambda\, d\mu}$.

Proof. Since μ is ergodic, it follows from the Shannon-MacMillan-Breiman theorem that

$$\frac{\log \mu([\mathbf{i}|n])}{n} \to -h(\mu) \quad \text{for } \mu\text{-a.a. } \mathbf{i}\in\Sigma^{\mathbb{N}}. \tag{6.16}$$

Also, an application of the ergodic theorem shows that $\frac{\sum_{k=0}^{n-1}\Lambda S^k\mathbf{i}}{n} \to \int \Lambda\, d\mu$ for μ-a.a. $\mathbf{i}\in\Sigma^{\mathbb{N}}$. It follows from this and Proposition 4.1 that

$$\frac{\log s_{\mathbf{i}|n}}{n} \to \int \Lambda\, d\mu \quad \text{for } \mu\text{-a.a. } \mathbf{i}\in\Sigma^{\mathbb{N}}. \tag{6.17}$$

Combining (6.16) and (6.17) now gives

$$\frac{\log \mu([\mathbf{i}|n])}{\log s_{\mathbf{i}|n}} \to -\frac{h(\mu)}{\int \Lambda\, d\mu} \quad \text{for } \mu\text{-a.a. } \mathbf{i}\in\Sigma^{\mathbb{N}}. \tag{6.18}$$

Next, for each $\mathbf{i}\in\Sigma^{\mathbb{N}}$ and $r>0$, let $n_{\mathbf{i},r}$ denote the unique integer such that $s_{\mathbf{i}|n_{\mathbf{i},r}} < r \le s_{\widehat{\mathbf{i}|n_{\mathbf{i},r}}}$. It follows from the definition of the metric $\mathsf{d}_{\Sigma^{\mathbb{N}}}$ on $\Sigma^{\mathbb{N}}$ (see (6.1) and (6.2)) that $B(\mathbf{i},r) = [\mathbf{i}|n_{\mathbf{i},r}]$. Also, if we let c denote the constant from Condition (C3) in Sect. 2.1, then it follows from Proposition 4.1 that $s_{\mathbf{i}|n_{\mathbf{i},r}} < r \le s_{\widehat{\mathbf{i}|n_{\mathbf{i},r}}} \le \frac{c}{s_{\min}} s_{\mathbf{i}|n_{\mathbf{i},r}}$. Combining these facts, we now deduce from (6.18) that

$$\begin{aligned}\lim_{r\searrow 0}\frac{\log \mu(B(\mathbf{i},r))}{\log r} &= \lim_{r\searrow 0}\frac{\log \mu([\mathbf{i}|n_{\mathbf{i},r}])}{\log s_{\mathbf{i}|n_{\mathbf{i},r}}}\\ &= \lim_{n}\frac{\log \mu([\mathbf{i}|n])}{\log s_{\mathbf{i}|n}}\\ &= -\frac{h(\mu)}{\int \Lambda\, d\mu} \quad \text{for } \mu\text{-a.a. } \mathbf{i}\in\Sigma^{\mathbb{N}},\end{aligned}$$

whence

$$\mu\text{-ess}\sup_{\mathbf{i}} \liminf_{r\searrow 0} \frac{\log \mu(B(\mathbf{i},r))}{\log r} = -\frac{h(\mu)}{\int \Lambda\, d\mu}\,, \tag{6.19}$$

where μ-ess sup denotes the μ essential supremum.

Finally, we note that it is well-known that $\overline{\dim}_{\mathsf{H}}\mu = \mu\text{-ess}\sup_{\mathbf{i}} \liminf_{r\searrow 0} \frac{\log \mu(B(\mathbf{i},r))}{\log r}$ (see, for example, [11]), and it therefore follows immediately from (6.19) that $\overline{\dim}_{\mathsf{H}}\mu = \mu\text{-ess}\sup_{\mathbf{i}} \liminf_{r\searrow 0} \frac{\log \mu(B(\mathbf{i},r))}{\log r} = -\frac{h(\mu)}{\int \Lambda\, d\mu}$. □

The final auxiliary result says that the map $C \to \underline{f}^{U,\Lambda}(C)$ is upper semi-continuous. In order to state this result we introduce the following notation. For a metric space X, we write

$$\mathcal{F}(X) = \Big\{ F \subseteq X \,\Big|\, F \text{ is closed and non-empty} \Big\} \tag{6.20}$$

and we equip $\mathcal{F}(X)$ with the Hausdorff metric D; recall, that since X may be unbounded, the Hausdorff distance D is defined as follows, namely, for $E, F \in \mathcal{F}(X)$, write

$$\Delta(E,F) = \min\left(\sup_{x\in E} \operatorname{dist}(x,F)\,,\, \sup_{y\in F} \operatorname{dist}(y,E) \right) \tag{6.21}$$

and define D by

$$\mathsf{D} = \min(1, \Delta). \tag{6.22}$$

Lemma 6.5. *Let X be a metric space and let $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Equip $\mathcal{F}(X)$ with the Hausdorff metric D. Then the function $\underline{f}^{U,\Lambda} : \mathcal{F}(X) \to \mathbb{R}$ is upper semicontinuous, i.e. for each $C \in \mathcal{F}(X)$ and each $\varepsilon > 0$, there exists a real number $\rho > 0$ such that if $F \in \mathcal{F}(X)$ and $\mathsf{D}(F,C) < \rho$, then*

$$\underline{f}^{U,\Lambda}(F) \le \underline{f}^{U,\Lambda}(C) + \varepsilon.$$

Proof. Let $C \in \mathcal{F}(X)$ and $\varepsilon > 0$. Next, it follows from the definition of $\underline{f}^{U,\Lambda}(C)$ that we can choose a real number r_0 with $0 < r_0 < 1$ such that

$$\underline{f}^{U,\Lambda}(C, r_0) < \underline{f}^{U,\Lambda}(C) + \varepsilon. \tag{6.23}$$

Let $\rho = \frac{r_0}{2}$. We now prove the following claim.

Claim 1. Let $F \in \mathcal{F}(X)$ with $\mathsf{D}(F,C) < \rho$. For all $0 < r < \rho$ and all $\delta > 0$, we have

$$N_\delta^{U,\Lambda}(F,r) \le N_\delta^{U,\Lambda}(C, r_0).$$

Proof of Claim 1. Fix $0 < r < \rho$ and $\delta > 0$. Since $\mathsf{D}(F,C) < \rho = \frac{r_0}{2}$ and $r_0 < 1$, we first conclude that $B(F, \frac{r_0}{2}) \subseteq B(C, r_0)$. Hence, if $\mathbf{i} \in \Pi_\delta^{U,\Lambda}(F,r)$, then this and the fact that $0 < r < \rho = \frac{r_0}{2}$ imply that $UL_{|\mathbf{i}|}[\mathbf{i}] \subseteq B(F,r) \subseteq B(F,\rho) =$

$B(F,\frac{r_0}{2}) \subseteq B(C,r_0)$ and so $\mathbf{i} \in \Pi_\delta^{U,\Lambda}(C,r_0)$. This shows that $\Pi_\delta^{U,\Lambda}(F,r) \subseteq \Pi_\delta^{U,\Lambda}(C,r_0)$, whence $N_\delta^{U,\Lambda}(F,r) \leq N_\delta^{U,\Lambda}(C,r_0)$. This completes the proof of Claim 1.

We now claim that if $F \in \mathcal{F}(X)$ and $\mathsf{D}(F,C) < \rho$, then

$$\underline{f}^{U,\Lambda}(F) \leq \underline{f}^{U,\Lambda}(C) + \varepsilon. \tag{6.24}$$

To prove this, let $F \in \mathcal{F}(X)$ with $\mathsf{D}(F,C) < \rho$. It follows from Claim 1 and (6.23) that if $0 < r < \rho$, then

$$\begin{aligned}\underline{f}^{U,\Lambda}(F,r) &= \liminf_{\delta\searrow 0} \frac{\log N_\delta^{U,\Lambda}(F,r)}{-\log\delta}\\ &\leq \liminf_{\delta\searrow 0} \frac{\log N_\delta^{U,\Lambda}(C,r_0)}{-\log\delta}\\ &= \underline{f}^{U,\Lambda}(C,r_0)\\ &< \underline{f}^{U,\Lambda}(C) + \varepsilon.\end{aligned}$$

Since this inequality holds for all $0 < r < \rho$, we finally conclude that $\underline{f}^{U,\Lambda}(F) = \lim_{r\searrow 0} \underline{f}^{U,\Lambda}(F,r) \leq \underline{f}^{U,\Lambda}(C) + \varepsilon$. □

We can now state and prove the main result in this section, namely, Theorem 6.6 providing a proof of inequality (2.2).

Theorem 6.6. *Let X be a metric space and let $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a closed subset of X. We have*

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in C}} -\frac{h(\mu)}{\int \Lambda\, d\mu} \leq \underline{f}^{U,\Lambda}(C).$$

Proof. Let $\varepsilon > 0$. Next, fix $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$ with $U\mu \in C$. We will now prove that

$$-\frac{h(\mu)}{\int \Lambda\, d\mu} \leq \underline{f}^{U,\Lambda}(C) + \varepsilon. \tag{6.25}$$

Let $\mathcal{F}(X)$ be defined as in (6.20), i.e. $\mathcal{F}(X) = \{F \subseteq X \,|\, F$ is closed and non-empty $\}$, and equip $\mathcal{F}(X)$ with the Hausdorff metric D, see (6.21) and (6.22). It follows from Lemma 6.5 that the function $\underline{f}^{U,\Lambda} : \mathcal{F}(X) \to \mathbb{R}$ is upper semi-continuous, and we can therefore choose $\rho_\varepsilon > 0$ such that:

$$\begin{aligned}&\text{if } F \in \mathcal{F}(X) \text{ and } \mathsf{D}(F,C) < \rho_\varepsilon, \text{ then}\\ &\underline{f}^{U,\Lambda}(F) \leq \underline{f}^{U,\Lambda}(C) + \varepsilon.\end{aligned} \tag{6.26}$$

Next, observe that we can choose an S-invariant probability measure γ on $\Sigma^{\mathbb{N}}$ such that $\operatorname{supp}\gamma = \Sigma^{\mathbb{N}}$. For $t \in (0,1)$, we now write $\mu_t = (1-t)\mu + t\gamma \in$

$\mathcal{P}_S(\Sigma^{\mathbb{N}})$. As U is continuous with $U\mu \in C$ and $\mu_t \to \mu$ weakly as $t \searrow 0$, there exists $0 < t_\varepsilon < 1$ such that for all $0 < t < t_\varepsilon$, we have

$$\operatorname{dist}(U\mu_t, C) < \rho_\varepsilon. \tag{6.27}$$

Fix $0 < t < t_\varepsilon$. Since U is continuous and $\operatorname{dist}(U\mu_t, C) < \rho_\varepsilon$ (by (6.27)), it follows from Lemma 6.3 that we may choose a sequence $(\mu_{t,n})_n$ of S-invariant probability measures on $\Sigma^{\mathbb{N}}$ such that

$$\mu_{t,n} \to \mu_t \quad \text{weakly}\,, \tag{6.28}$$

$$\mu_{t,n} \text{ is ergodic,} \tag{6.29}$$

$$h(\mu_{t,n}) \to h(\mu_t) \tag{6.30}$$

and

$$\operatorname{dist}(U\mu_{t,n}, C) < \rho_\varepsilon \tag{6.31}$$

for all n. Observe that it follows from (6.31) that $\mathsf{D}(\,C \cup \{\mu_{t,n}\}\,,\,C\,) < \rho_\varepsilon$, and we therefore conclude from (6.31) that

$$\underline{f}^{U,\Lambda}(\,C \cup \{\mu_{t,n}\}\,) \le \underline{f}^{U,\Lambda}(C) + \varepsilon \tag{6.32}$$

for all n. We now prove the following two claims.

Claim 1. For all $0 < t < t_\varepsilon$*, we have*

$$-\frac{(1-t)h(\mu) + th(\gamma)}{(1-t)\int \Lambda\, d\mu + t\int \Lambda\, d\gamma} \le \lim_n \overline{\dim}_{\mathsf{H}}\, \mu_{t,n}.$$

Proof of Claim 1. Using the fact that the entropy map $h : \mathcal{P}_S(\Sigma) \to \mathbb{R}$ is affine (cf. [47]) we conclude that

$$\begin{aligned}-\frac{(1-t)h(\mu) + th(\gamma)}{(1-t)\int \Lambda\, d\mu + t\int \Lambda\, d\gamma} &\le -\frac{h((1-t)\mu + t\gamma)}{\int \Lambda\, d((1-t)\mu + t\gamma)}\\ &= -\frac{h(\mu_t)}{\int \Lambda\, d\mu_t}.\end{aligned} \tag{6.33}$$

However, since Λ is continuous and $\mu_{t,n} \to \mu_t$ weakly (by (6.28)), we conclude that $\int \Lambda\, d\mu_{t,n} \to \int \Lambda\, d\mu_t$. We deduce from this and the fact that $h(\mu_{t,n}) \to h(\mu_t)$ (by (6.30)) that

$$-\frac{h(\mu_t)}{\int \Lambda\, d\mu_t} = \lim_n -\frac{h(\mu_{t,n})}{\int \Lambda\, d\mu_{t,n}}. \tag{6.34}$$

Combining (6.33) and (6.34) now yields

$$-\frac{(1-t)h(\mu) + th(\gamma)}{(1-t)\int \Lambda\, d\mu + t\int \Lambda\, d\gamma} \le \lim_n -\frac{h(\mu_{t,n})}{\int \Lambda\, d\mu_{t,n}}. \tag{6.35}$$

Also, since $\mu_{t,n}$ is ergodic (by (6.29)), it follows from Proposition 6.4 that $\overline{\dim}_{\mathsf{H}}\,\mu_{t,n} = -\frac{h(\mu_{t,n})}{\log N}$, and we therefore conclude from (6.35) that

$$-\frac{(1-t)h(\mu)+th(\gamma)}{(1-t)\int\Lambda\,d\mu+t\int\Lambda\,d\gamma} \leq \lim_n -\frac{h(\mu_{t,n})}{\int\Lambda\,d\mu_{t,n}} = \lim_n \overline{\dim}_{\mathsf{H}}\,\mu_{t,n}.$$

This completes the proof of Claim 1.

Claim 2. For all $0<t<t_\varepsilon$, *we have*

$$\lim_n \overline{\dim}_{\mathsf{H}}\,\mu_{t,n} \leq \underline{f}^{U,\Lambda}(C)+\varepsilon.$$

Proof of Claim 2. It follows immediately from the ergodicity of $\mu_{t,n}$ and the ergodic theorem that $\mu_{t,n}(\{\mathbf{i}\in\Sigma^{\mathbb{N}}\mid \lim_m L_m\mathbf{i}=\mu_{t,n}\})=1$. Hence

$$\begin{aligned}\overline{\dim}_{\mathsf{H}}\,\mu_{t,n} &\leq \dim_{\mathsf{H}}\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\lim_m L_m\mathbf{i}=\mu_{t,n}\right\}\\ &\leq \dim_{\mathsf{H}}\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\lim_m UL_m\mathbf{i}=U\mu_{t,n}\right\}\\ &\leq \dim_{\mathsf{H}}\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\lim_m \operatorname{dist}\big(UL_m\mathbf{i}\,,\,C\cup\{U\mu_{t,n}\}\big)=0\right\}. \qquad (6.36)\end{aligned}$$

Next, it follows from (6.36) using Lemma 6.2 and (6.32) that

$$\begin{aligned}\overline{\dim}_{\mathsf{H}}\,\mu_{t,n} &\leq \dim_{\mathsf{H}}\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\lim_m \operatorname{dist}\big(UL_m\mathbf{i}\,,\,C\cup\{U\mu_{t,n}\}\big)=0\right\} && \text{[by ((6.36)]}\\ &\leq \underline{f}^{U,\Lambda}(\,C\cup\{\mu_{t,n}\}\,) && \text{[by Lemma 6.2]}\\ &\leq \underline{f}^{U,\Lambda}(C)+\varepsilon. && \text{[by (6.32)]}\end{aligned}$$

This completes the proof of Claim 2.

Combining Claims 1 and 2 shows that for all $0<t<t_\varepsilon$, we have

$$-\frac{(1-t)h(\mu)+th(\gamma)}{(1-t)\int\Lambda\,d\mu+t\int\Lambda\,d\gamma} \leq \underline{f}^{U,\Lambda}(C)+\varepsilon. \qquad (6.37)$$

Letting $t\searrow 0$ in (6.37) now gives $-\frac{h(\mu)}{\int\Lambda\,d\mu}\leq \underline{f}^{U,\Lambda}(C)+\varepsilon$. This proves (6.25).

Since $\mu\in\mathcal{P}_S(X)$ with $U\mu\in C$ was arbitrary, it follows immediately from (6.25) that

$$\sup_{\substack{\mu\in\mathcal{P}(\Sigma^{\mathbb{N}})\\ U\mu\in C}} -\frac{h(\mu)}{\int\Lambda\,d\mu} \leq \underline{f}^{U,\Lambda}(C)+\varepsilon.$$

Finally, letting $\varepsilon\searrow 0$ gives the desired result. □

7. Proof of inequality (2.3)

The purpose of this section is to prove Theorem 7.1 providing a proof of inequality (2.3).

Theorem 7.1. *Let X be a metric space and let $U:\mathcal{P}(\Sigma^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Let $C\subseteq X$ be a closed subset of X and $r>0$.*

(1) *We have*

$$\underline{f}^{U,\Lambda}(C,r)\leq\ \sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}(\cdot;r)\,\big).$$

(2) *We have*

$$\underline{f}^{U,\Lambda}(C)\leq\liminf_{r\searrow 0}\ \sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}(\cdot;r)\,\big).$$

Proof. (1) Fix $\varepsilon>0$. For brevity write $t=\underline{f}^{U,\Lambda}(C,r)-\varepsilon$. Since $t=\underline{f}^{U,\Lambda}(C,r)-\varepsilon<\underline{f}^{U,\Lambda}(C,r)=\liminf_{\delta\searrow 0}\frac{\log N_\delta^{U,\Lambda}(C,r)}{-\log\delta}$, we can find δ_ε with $0<\delta_\varepsilon<1$ such that

$$t<\frac{\log N_\delta^{U,\Lambda}(C,r)}{-\log\delta}$$

for all $0<\delta<\delta_\varepsilon$. Consequently, for all $0<\delta<\delta_\varepsilon$, we have

$$\delta^{-t}\leq N_\delta^{U,\Lambda}(C,r).\tag{7.1}$$

Next, let c denote the constant from Condition (C3) in Sect. 2.1 and fix $\rho>0$ with $\rho<\min(\,\frac{s_{\min}}{c}\,,\,\delta_\varepsilon\,))$. We now prove the following two claims.
Claim 1. For $\in\mathbb{N}$ and $\mathbf{i}\in\Sigma^*$, the following implication holds:

$$s_{\mathbf{i}}\approx\rho^n\quad\Rightarrow\quad\rho^{n+1}<s_{\mathbf{i}}\leq\rho^n\,;$$

recall, that for $\delta>0$, we write $s_{\mathbf{i}}\approx\delta$ if $s_{\mathbf{i}}\leq\delta<s_{\hat{\mathbf{i}}}$, see Sect. 2.1.
Proof of Claim 1. Indeed, if $\mathbf{i}=i_1\cdots i_m\in\Sigma^m$ with $s_{\mathbf{i}}\approx\rho^n$, then $s_{\mathbf{i}}\leq\rho^n<s_{\hat{\mathbf{i}}}$, whence $s_{\mathbf{i}}\leq\rho^n$. It also follows from Proposition 4.1 that $s_{\mathbf{i}}=s_{\hat{\mathbf{i}}i_m}\geq\frac{1}{c}s_{\hat{\mathbf{i}}}s_{i_m}>\frac{1}{c}\rho^n s_{\min}=\frac{s_{\min}}{c\rho}\rho^{n+1}\geq\rho^{n+1}$ where the last inequality is due to the fact that $\frac{s_{\min}}{c\rho}\geq 1$ because $\rho<\min(\,\frac{s_{\min}}{c}\,,\,\delta_\varepsilon\,)\leq\frac{s_{\min}}{c}$. This completes the proof of Claim 1.
Claim 2. We have

$$\sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t=\infty.$$

Proof of Claim 2. It is clear that

$$\sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t = \sum_n \sum_{\substack{\mathbf{i}\\ \rho^{n+1}<s_{\mathbf{i}}\le\rho^n\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t + \sum_{\substack{\mathbf{i}\\ \rho<s_{\mathbf{i}}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t$$

$$\ge \sum_n \sum_{\substack{\mathbf{i}\\ \rho^{n+1}<s_{\mathbf{i}}\le\rho^n\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t. \tag{7.2}$$

Also, for $n\in\mathbb{N}$ and $\mathbf{i}\in\Sigma^*$, the following implication follows from Claim 1:

$$s_{\mathbf{i}}\approx\rho^n \quad\Rightarrow\quad \rho^{n+1}<s_{\mathbf{i}}\le\rho^n. \tag{7.3}$$

We conclude immediately from (7.3) that

$$\sum_n \sum_{\substack{\mathbf{i}\\ \rho^{n+1}<s_{\mathbf{i}}\le\rho^n\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t \ge \sum_n \sum_{\substack{\mathbf{i}\\ s_{\mathbf{i}}\approx\rho^n\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t. \tag{7.4}$$

Combining (7.2) and (7.4) shows that

$$\sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t \ge \sum_n \sum_{\substack{\mathbf{i}\\ s_{\mathbf{i}}\approx\rho^n\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t$$

$$= \sum_n \sum_{\mathbf{i}\in\Pi^U_{\mathbf{s},\rho^n}(C,r)} s_{\mathbf{i}}^t. \tag{7.5}$$

However, if $\mathbf{i}\in\Pi^U_{\mathbf{s},\rho^n}(C,r)$, then $s_{\mathbf{i}}\approx\rho^n$, and it therefore follows from Claim 1 that $\rho^{n+1}<s_{\mathbf{i}}\le\rho^n$, whence $s_{\mathbf{i}}\ge\rho^{nt}\rho^{|t|}$. We conclude from this and (7.5) that

$$\begin{aligned}\sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t &\ge \sum_n \sum_{\mathbf{i}\in\Pi^U_{\mathbf{s},\rho^n}(C,r)} s_{\mathbf{i}}^t\\ &\ge \rho^{|t|}\sum_n \sum_{\mathbf{i}\in\Pi^U_{\mathbf{s},\rho^n}(C,r)} \rho^{nt}\\ &= \rho^{|t|}\sum_n \left|\Pi^U_{\mathbf{s},\rho^n}(C,r)\right| \rho^{nt}\\ &= \rho^{|t|}\sum_n N^U_{\mathbf{s},\rho^n}(C,r)\,\rho^{nt}.\end{aligned} \tag{7.6}$$

Finally, since $\rho^n \leq \rho < \min(\frac{s_{\min}}{c}, \delta_\varepsilon) \leq \delta_\varepsilon$, we deduce from (7.1) that $\rho^{-nt} = (\rho^n)^{-t} \leq N^U_{\mathbf{s},\rho^n}(C,r)$. This and (7.6) now implies that

$$\begin{aligned}\sum_{\substack{\mathbf{i} \\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(C,r)}} s_{\mathbf{i}}^t &\geq \rho^{|t|}\sum_n \rho^{-nt}\rho^{nt}\\ &= \rho^{|t|}\sum_n 1\\ &= \infty.\end{aligned}$$

This completes the proof of Claim 2.

We conclude immediately from Claim 2 that $\underline{f}^{U,\Lambda}(C,r) - \varepsilon = t \leq \sigma_{\mathsf{ab}}\big(\zeta_C^{U,\Lambda}(\cdot;r)\big)$. Finally, letting $\varepsilon \searrow 0$ completes the proof.

(2) This follows immediately from (1). ☐

8. Proof of Theorem 2.2

For $x, y \in \mathbb{R}^M$, write

$$[\![x,y]\!] = \Big\{(1-t)x + ty \,\Big|\, t \in [0,1]\Big\},$$

i.e. $[\![x,y]\!]$ denotes the line-segment joining x and y.

Lemma 8.1. *Let $E \subseteq \mathbb{R}^M$ and let $x \in E$ and $y \in \mathbb{R}^M \setminus E$. Then $[\![x,y]\!] \cap \partial E \neq \varnothing$.*

Proof. Let $t_0 = \sup\{t \in [0,1] \,|\, (1-t)x + ty \in E\}$. Then $(1-t_0)x + t_0 y \in [\![x,y]\!]$, and since $x \in E$ and $y \in \mathbb{R}^M \setminus E$, it is easily seen that $(1-t_0)x + t_0 y \in \partial E$. ☐

Lemma 8.2. *Let $C \subseteq \mathbb{R}^M$ be a closed subset of $\mathbb{R}^M$ and let $r, \varepsilon > 0$ with $r < \varepsilon$. Then $B\big(I(C,\varepsilon), r\big) \subseteq C$; recall, that $I(C,\varepsilon) = \{x \in C \,|\, \operatorname{dist}(x, \partial C) \geq \varepsilon\}$, see Sect. 2.3.*

Proof. Let $y \in B\big(I(C,\varepsilon), r\big)$. We must now prove that $y \in C$. Assume, in order to reach a contradiction, that $y \notin C$. Since $I(C,\varepsilon)$ is closed, it follows that we can find $x \in I(C,\varepsilon)$ such that $|y - x| = \operatorname{dist}\big(y, I(C,\varepsilon)\big)$. Also, since $x \in I(C,\varepsilon) \subseteq C$ and $y \notin C$, it follows from Lemma 8.2 that there is $v \in [\![x,y]\!] \cap \partial C$. We now conclude that

$$\begin{aligned} r &\geq \operatorname{dist}\big(y, I(C,\varepsilon)\big) && [\text{since } y \in B\big(I(C,\varepsilon), r\big)]\\ &= |y - x| \\ &\geq |v - x| && [\text{since } v \in [\![x,y]\!]]\\ &\geq \operatorname{dist}\big(x, \partial C\big) && [\text{since } v \in \partial C]\\ &\geq \varepsilon. && [\text{since } x \in I(C,\varepsilon)]\end{aligned}$$

However, this inequality contradicts the fact that $r < \varepsilon$. ☐

Proof of Theorem 2.2. We first note that it follows from Theorem 2.1 that

$$\underline{f}^{U,\Lambda}(C) = \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in C}} -\frac{h(\mu)}{\int \Lambda\, d\mu}.$$

Hence it suffices to prove that

$$\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}\,\big) = \underline{f}^{U,\Lambda}(C).$$

We first show that

$$\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}\,\big) \le \underline{f}^{U,\Lambda}(C). \tag{8.1}$$

Indeed, it follows immediately from the definitions of the zeta-functions $\zeta_C^{U,\Lambda}$ and $\zeta_C^{U,\Lambda}(\cdot;r)$ that if $r>0$, then $\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}\,\big) \le \sigma_{\mathsf{ab}}\big(\,\zeta_{B(C,r)}^{U,\Lambda}\,\big) = \sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}(\cdot;r)\,\big)$, whence $\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}\,\big) \le \liminf_{r\searrow}\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}(\cdot;r)\,\big)$. We conclude from this and Theorem 2.1 that $\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}\,\big) \le \liminf_{r\searrow 0}\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}(\cdot;r)\,\big) = \underline{f}^{U,\Lambda}(C)$. This proves (8.1).

Next, we show that

$$\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}\,\big) \ge \underline{f}^{U,\Lambda}(C). \tag{8.2}$$

Observe that if $r,\varepsilon>0$ with $r<\varepsilon$, then it follows from Lemma 8.2 that $B\big(\,I(C,\varepsilon)\,,r\,\big)\subseteq C$, and the definitions of the zeta-functions $\zeta_C^{U,\Lambda}$ and $\zeta_C^{U,\Lambda}(\cdot;r)$ therefore imply that $\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}\,\big) \ge \sigma_{\mathsf{ab}}\big(\,\zeta_{B(\,I(C,\varepsilon)\,,r\,)}^{U,\Lambda}\,\big) = \sigma_{\mathsf{ab}}\big(\,\zeta_{I(C,\varepsilon)}^{U,\Lambda}(\cdot;r)\,\big)$ for all $r,\varepsilon>0$ with $r<\varepsilon$. Hence, for all $\varepsilon>0$ we have

$$\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}\,\big) \ge \liminf_{r\searrow 0}\sigma_{\mathsf{ab}}\big(\,\zeta_{I(C,\varepsilon)}^{U,\Lambda}(\cdot;r)\,\big). \tag{8.3}$$

Also, since $I(C,\varepsilon)$ is closed, it follows from Theorem 2.1 that $\liminf_{r\searrow 0}\sigma_{\mathsf{ab}}$ $\big(\,\zeta_{I(C,\varepsilon)}^{U,\Lambda}(\cdot;r)\,\big) = \underline{f}^{U,\Lambda}(\,I(C,\varepsilon)\,)$. We conclude from this and (8.3) that

$$\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}\,\big) \ge \underline{f}^{U,\Lambda}\big(\,I(C,\varepsilon)\,\big), \tag{8.4}$$

for all $\varepsilon>0$. Finally, using inner continuity at C and letting $\varepsilon\searrow 0$, it follows from (8.4) that $\sigma_{\mathsf{ab}}\big(\,\zeta_C^{U,\Lambda}\,\big) \ge \lim_{\varepsilon\searrow 0}\underline{f}^{U,\Lambda}(\,I(C,\varepsilon)\,) = \underline{f}^{U,\Lambda}(C)$. This proves (8.2). □

9. Proof of Theorem 2.3

The purpose of this section is to prove Theorem 2.3.

Proof of Theorem 2.3. For brevity write $G=\{s\in\mathbb{C}\,|\,\mathrm{Re}(s)>\sigma_{\mathsf{ab}}(\,\zeta_C^{U,\Lambda}\,)\}$. Since $\sup_{|\mathbf{i}|=n}\frac{1}{\log s_{\mathbf{i}}}\to 0$ as $n\to\infty$ (because $\sup_{|\mathbf{i}|=n}s_{\mathbf{i}}\to 0$ as $n\to\infty$), we conclude that the series $Z_C^{U,\Lambda}(s)=\sum_{\mathbf{i}\,,\,UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}\frac{1}{\log s_{\mathbf{i}}}s_{\mathbf{i}}^s$ converges uniformly in the variable s on all compact subsets of G.

Since the series $Z_C^{U,\Lambda}(s)=\sum_{\mathbf{i}\,,\,UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}\frac{1}{\log s_{\mathbf{i}}}s_{\mathbf{i}}^s$ converges uniformly in the variable s on all compact subsets of G, we conclude that the formal calculations below are justified, namely, if $s\in G$, then we have

$$\begin{aligned}
\exp Z_C^{U,\Lambda}(s) &= \exp \sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}} \frac{1}{\log s_{\mathbf{i}}} s_{\mathbf{i}}^s \\
&= \exp \sum_{\substack{\mathbf{i}\\ \mathbf{i}\text{ is prime}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}} \sum_n \frac{1}{\log \underbrace{s_{\mathbf{i}\cdots\mathbf{i}}}_{n\text{ times}}} \underbrace{s_{\mathbf{i}\cdots\mathbf{i}}^s}_{n\text{ times}} \\
&= \exp \sum_{\substack{\mathbf{i}\\ \mathbf{i}\text{ is prime}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}} \sum_n \frac{1}{n\log s_{\mathbf{i}}} s_{\mathbf{i}}^{sn} \\
&= \prod_{\substack{\mathbf{i}\\ \mathbf{i}\text{ is prime}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}} \exp\left(\frac{1}{\log s_{\mathbf{i}}}\sum_n \frac{1}{n} s_{\mathbf{i}}^{sn}\right) \\
&= \prod_{\substack{\mathbf{i}\\ \mathbf{i}\text{ is prime}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}} \exp\left(\frac{1}{\log s_{\mathbf{i}}}\log\left(\frac{1}{1-s_{\mathbf{i}}^s}\right)\right) \\
&= \prod_{\substack{\mathbf{i}\\ \mathbf{i}\text{ is prime}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}} \left(\frac{1}{1-s_{\mathbf{i}}^s}\right)^{\frac{1}{\log s_{\mathbf{i}}}} \\
&= Q_C^{U,\Lambda}(s). \qquad (9.1)
\end{aligned}$$

It follows from the calculations involved in establishing (9.1) that the product $Q_C^{U,\Lambda}(s)$ converges and that $Q_C^{U,\Lambda}(s)\neq 0$ for all $s\in G$. In addition, we deduce from (9.1) that for all $s\in G$, we have $\frac{d}{ds}Q_C^{U,\Lambda}(s)=\frac{d}{ds}\exp Z_C^{U,\Lambda}(s)=(\exp Z_C^{U,\Lambda}(s))\,\frac{d}{ds}Z_C^{U,\Lambda}(s)=Q_C^{U,\Lambda}(s)\,\frac{d}{ds}Z_C^{U,\Lambda}(s)$, whence

$$\frac{d}{ds} Z_C^{U,\Lambda}(s) = \frac{\frac{d}{ds} Q_C^{U,\Lambda}(s)}{Q_C^{U,\Lambda}(s)} = L\, Q_C^{U,\Lambda}(s). \tag{9.2}$$

Once again using the fact that the series $Z_C^{U,\Lambda}(s) = \sum_{\mathbf{i}\,,\, UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C} \frac{1}{\log s_{\mathbf{i}}} s_{\mathbf{i}}^s$ converges uniformly in the variable s on all compact subsets of G, we deduce that if $s \in G$, then we have

$$\begin{aligned}
\frac{d}{ds} Z_C^{U,\Lambda}(s) &= \frac{d}{ds} \sum_{\substack{\mathbf{i} \\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}} \frac{1}{\log s_{\mathbf{i}}} s_{\mathbf{i}}^s \\
&= \sum_{\substack{\mathbf{i} \\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}} \frac{1}{\log s_{\mathbf{i}}} \frac{d}{ds} s_{\mathbf{i}}^s \\
&= \sum_{\substack{\mathbf{i} \\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq C}} s_{\mathbf{i}}^s \\
&= \zeta_C^{U,\Lambda}(s).
\end{aligned} \tag{9.3}$$

Finally, combining (9.2) and (9.3) gives $\zeta_C^{U,\Lambda}(s) = \frac{d}{ds} Z_C^{U,\Lambda}(s) = L\, Q_C^{U,\Lambda}(s)$ for all $s \in G$. □

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