PATTERN CLASSES OF PERMUTATIONS VIA BIJECTIONS BETWEEN LINEARLY ORDERED SETS

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Abstract. A pattern class is a set of permutations closed under pattern involvement or, equivalently, defined by certain subsequence avoidance conditions. Any pattern class X which is atomic, i.e. indecomposable as a union of proper subclasses, has a representation as the set of subpermutations of a bijection between two countable (or finite) linearly ordered sets A and B. Concentrating on the situation where A is arbitrary and $B = \mathbb{N}$, we demonstrate how the order-theoretic properties of A determine the structure of X and we establish results about independence, contiguousness and subrepresentations for classes admitting multiple representations of this form.

1. Introduction

Pattern classes, sets of permutations defined by certain 'avoided subsequence' conditions, arise naturally in many areas of discrete mathematics. Recently, interest in pattern classes has been heightened by a link with theoretical computer science: for various permuting machines, including stacks, queues, deques and token-passing networks, the set of permutations which may be generated or sorted by a machine M forms such a class (see [6] or [9] for more details). For example, a stack can convert 1234 to 3241 by doing the following sequence of *pushes* (inputs, U) and *pops* (outputs, O): UUUOOUOO, whereas it cannot transform the permutation 1234 into 3124. In fact, it can be shown that a permutation $\pi = p_1 \dots p_n$ cannot be generated by a single stack precisely if $p_j < p_k < p_i$ for some $1 \leq i < j < k \leq n$.

Given two linearly ordered sets $(A, \leq_A)$ and $(B, \leq_B)$, two sequences $\alpha = a_1a_2a_3\dots$ and $\beta = b_1b_2b_3\dots$ ($a_i \in A$, $b_i \in B$) are said to be *order isomorphic* if, for all i, j, we have $a_i \leq_A a_j$ if and only if $b_i \leq_B b_j$. In this paper, a *sequence* will refer to a (generally finite) list of distinct elements of a linearly ordered set, while a *permutation* will mean a rearrangement of the numbers $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. Clearly any (finite) sequence is isomorphic to a unique permutation. A permutation π is said to *involve* a permutation ρ (we write $\rho \preceq \pi$) if π contains a subsequence order isomorphic to ρ; otherwise π *avoids* ρ. A set X of permutations is *closed* if it possesses the following property: if $\pi \in X$, and $\rho \preceq \pi$, then $\rho \in X$. Such sets may be described by specifying permutations avoided by the elements of the given set. For a closed set X, the basis $B(X)$ of X is defined to be the unique set of permutations which are minimal with respect to not lying in X; then $X = \mathcal{A}(B) = \{\sigma : \beta \not\preceq \sigma \text{ for all } \beta \in B\}$, the set of all permutations which *avoid* B.

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The set of all permutations which can be generated by a single stack, for example, is $\mathcal{A}(312)$. For this reason, closed sets are also called *pattern avoidance classes*, or simply *pattern classes*.

This form of representation, while convenient, has certain limitations. In particular, given the basis elements of a pattern class, structural properties of the class are not easily determined. Greater understanding of the structural properties of such classes is highly desirable: papers [1], [2] and [5] have made progress in this area by considering the creation of new classes from old via constructions such as union, intersection, composition, wreath product and juxtaposition. Conversely, viewing an arbitrary closed class as being built from its subclasses using one of these constructions can provide valuable insight into the original class.

A closed set X is called *atomic* if X cannot be expressed as a union of two proper closed subsets. Every pattern class X may be written as the (not necessarily finite) union of atomic classes. In fact, every X may be written as the union of *maximal* atomic classes; note that uniqueness is not guaranteed, so there may be several different expressions of this type for a given X. In the special case when the maximal atomic classes are *independent in union*, i.e. no class is contained in the union of the others, a unique expression is obtained, although such an expression need not exist for every X. For more details, see Chapter 4 of [10].

In their recent paper [3], Atkinson, Murphy and Ruškuc introduce a new way of representing atomic classes. Given a bijection π between two linearly ordered sets A and B, every finite subset $\{c_1, \ldots, c_n\}$ of A, where $c_1 < \ldots < c_n$, maps to a finite sequence $\pi(c_1) \ldots \pi(c_n)$ of elements of B, which is order isomorphic to a permutation. It is clear that the set of all permutations which arise in this way, which we shall denote as $\mathrm{Sub}(\pi : A \to B)$ (or simply $\mathrm{Sub}(\pi)$) is a closed set. (This definition is similar in spirit to the model-theoretic approach to pattern classes taken by Cameron in [8].)

Theorem 1.1 (from Theorem 1.2 of [3])**.** *For a closed set X, the following conditions are equivalent:*

1. *X is atomic;*
2. *$X = \mathrm{Sub}(\pi : A \to B)$ for some linearly ordered sets A, B and bijection π;*
3. *X possesses the* join *property, i.e. for any $\alpha, \beta \in X$, there exists $\gamma \in X$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.*

In fact, we may always assume that A and B in the above theorem are countable (or finite). This follows from the proof of the theorem in [3]. Alternatively, suppose $X = \{\sigma_1, \sigma_2, \sigma_3, \ldots\} = \mathrm{Sub}(\pi' : A' \to B')$, where A', B' are arbitrary. Each σ_i has an embedding as $\pi'|_{A_i}$ for some finite subset A_i of A'. Set $A = \bigcup_{i\in\mathbb{N}} A_i$, $B = \bigcup_{i\in\mathbb{N}} \pi'(A_i)$ and let $\pi = \bigcup_{i\in\mathbb{N}} \pi'|_{A_i}$; clearly A and B are countable and $X = \mathrm{Sub}(\pi : A \to B)$.

The study of atomic sets via the paradigm of bijections can be seen as the first step towards greater understanding of the structure of closed sets in general. We envisage taking a given closed set, expressing it as a union of atomic sets, then using properties of the bijections associated with each of the atomic subsets to determine properties of the original set. This approach is foreshadowed in [1], when properties of the class $\mathcal{A}(321, 2143)$, including its enumeration, are obtained by exploiting its decomposition as the union $\mathcal{A}(321, 2143, 3142) \cup \mathcal{A}(321, 2143, 2413)$. We will show in Section 2 that both of these subclasses are atomic.

A useful way of viewing permutations, introduced in [1], is to consider their *profile* (essentially their shape when represented as a juxtaposition of contiguous increasing segments). Profile classes (consisting of permutations with a fixed finite set of profiles) and the generalised W classes introduced in [4] (where permutations are expressed as (linear) juxtapositions of increasing and decreasing sequences) are both generalized by the work of [11], which expresses permutations in terms of two-dimensional juxtaposition regulated by a $\{0, \pm 1\}$ matrix. Such classes provide further examples of atomic classes.

In this paper, we explore the relationship between the nature of the pattern classes representable as $\mathrm{Sub}(\pi : A \to B)$ and the properties of the ordered sets A and B. For a complete understanding of atomic classes, the ultimate goal must be to deal with the situation when A and B are any linearly ordered sets. In [3], the authors consider the case when the ordinal type of the domain and range of the defining bijection π is that of the natural numbers $\mathbb{N}$. They define a *natural class* to be a closed set of the form $\mathrm{Sub}(\pi : \mathbb{N} \to \mathbb{N})$, and establish various results about such classes. In this paper, we consider the situation where the domain A may be any linearly ordered set and the range is $\mathbb{N}$. We call such classes *supernatural*.

Definition 1.2. Let $(A, \leq_A)$, $(B, \leq_B)$ be two countable linearly ordered sets. Let X be an atomic closed class; we say that X is of type (A, B) if X can be expressed as $X = \mathrm{Sub}(\pi : A \to B)$ for some π.

We will use the notation $\mathcal{T}(A, B)$ to denote the set of all classes of type (A, B). Hence the set of all natural classes, as defined in [3], is $\mathcal{T}(\mathbb{N}, \mathbb{N})$.

We begin our discussion in Section 2 with some basic general results and a case study involving some simple supernatural classes. In Section 3, we set the scene for results about arbitrary classes by presenting a treatment of infinite linearly ordered sets. We show in Section 4 that sets $\mathcal{T}(A, \mathbb{N})$ are independent for all sufficiently 'small' A, while for 'large' A, all $\mathcal{T}(A, \mathbb{N})$ are trivial. In Section 5, we study certain subrepresentation and contiguousness properties for classes which are of both type $(A, \mathbb{N})$ and type $(B, \mathbb{N})$ where A and B are in some way comparable.

2. Examples and a case study

In this section, we consider the relationship between $\mathcal{T}(\mathbb{N}, \mathbb{N})$ and $\mathcal{T}(2\mathbb{N}, \mathbb{N})$, where $2\mathbb{N}$ is the linearly ordered set consisting of two copies of $\mathbb{N}$ one after the other. More formally, we will write $2\mathbb{N}$ as $\{1, 2, 3, \ldots, \omega+1, \omega+2, \omega+3, \ldots\}$, where ω is the order type of the natural numbers; sometimes we will refer to $\{1, 2, 3, \ldots\}$ as $\mathbb{N}_1$ and $\{\omega+1, \omega+2, \omega+3 \ldots\}$ as $\mathbb{N}_2$. This exploration exhibits many of the issues which will subsequently be treated in the general context of arbitrary linearly ordered sets.

It may be tempting to conjecture that, since $2\mathbb{N}$ is an extension of $\mathbb{N}$, every natural class is a member of $\mathcal{T}(2\mathbb{N}, \mathbb{N})$. However, a little reflection shows that this is not the case. Consider the identity bijection on $\mathbb{N}$, $i : \mathbb{N} \to \mathbb{N}$ given by

$$i(n) = n \text{ for all } n \in \mathbb{N}.$$

$X = \mathrm{Sub}(i)$ is the set of all permutations of the form $1\,2\,3\, \ldots n$, and its basis consists of the single permutation $2\,1$, i.e. it is the class of all permutations which avoid descents. Suppose that $X = \mathrm{Sub}(\tau)$, for some bijection $\tau : 2\mathbb{N} \to \mathbb{N}$. Note that both $\mathcal{T}(\mathbb{N}_1)$ and $\mathcal{T}(\mathbb{N}_2)$ are infinite, and hence unbounded, subsets of $\mathrm{im}(\tau) = \mathbb{N}$. In particular, there exists n such that $\tau(n) > \tau(\omega+1)$. But then $\tau(n)\,\tau(\omega+1)$ is

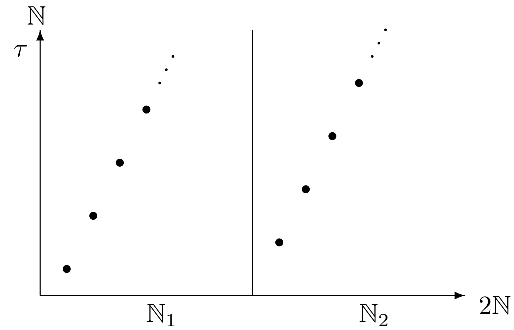

FIGURE 1

an embedding of $2\,1$ in τ, contradicting the fact that $X = \mathcal{A}(2\,1)$. So X is natural but not a member of $\mathcal{T}(2\mathbb{N}, \mathbb{N})$.

This observation is easily generalized to obtain the following result.

Proposition 2.1. *Let A,B be linearly ordered sets.*

(1) *The class $I = \mathcal{A}(21)$ is of type (A, B) if and only if A and B are isomorphic.*
(2) *If $A \ncong B$, then $\mathcal{T}(C, C) \nsubseteq \mathcal{T}(A, B)$ for any linearly ordered set C.*

Next we show that $\mathcal{T}(2\mathbb{N}, \mathbb{N}) \nsubseteq \mathcal{T}(\mathbb{N}, \mathbb{N})$. Define the bijection $\tau : 2\mathbb{N} \to \mathbb{N}$ by

$$\begin{aligned} \tau(n) &= 2n - 1, \\ \tau(\omega + n) &= 2n. \end{aligned}$$

When plotted in the (x, y)-plane, this bijection consists of two strictly increasing infinite sequences side-by-side (see Figure 1).

Let $X = \mathrm{Sub}(\tau)$. Observe that X comprises all permutations consisting of the juxtaposition of two increasing sequences. It may be shown, by standard arguments, that its basis is $\{321, 3142, 2143\}$ (see also [1]). It is asserted in [3] that $\mathcal{A}(321, 3142, 2143)$ is not a natural class; we outline an argument below. Suppose that $X = \mathrm{Sub}(\pi)$, for some bijection $\pi : \mathbb{N} \to \mathbb{N}$. Since $21 \in \mathcal{A}(321, 3142, 2143) = X$, it has an embedding $\pi(k)\,\pi(l)$ $(k < l)$ in π. We may assume without loss of generality that this is the first (i.e. left-most) such embedding of 21. Now consider the permutation $\sigma = 1\,2\,\ldots\,l\,(l+2)\,(l+1) \in X = \mathcal{A}(321, 3142, 2143)$; it must also have an embedding $\pi(j_1)\ldots\pi(j_{l+2})$ in π. Due to the length of the strictly increasing segment, the embedding $\pi(j_{l+1})\,\pi(j_{l+2})$ of the descent must lie to the right of $\pi(l)$. First note that $\pi(j_{l+2})$ cannot lie below $\pi(k)$. If it did, and $\pi(j_{l+1}) < \pi(k)$, then $\pi(k)\pi(j_{l+1})\pi(j_{l+2})$ would be an embedding of 321, while if $\pi(j_{l+1}) > \pi(k)$ then $\pi(k)\pi(l)\pi(j_{l+1})\pi(j_{l+2})$ would be an embedding of 3142. Hence $\pi(j_{l+2}) > \pi(k)$, but then $\pi(k)\pi(l)\pi(j_{l+1})\pi(j_{l+2}) \cong 2143$, a contradiction.

Note that the above discussion has established the claim made in the introduction that $\mathcal{A}(321, 3142, 2143)$ is atomic, by demonstrating that it is of type $(2\mathbb{N}, \mathbb{N})$. Since the basis elements of $\mathcal{A}(321, 2413, 2143)$ are precisely the inverses of those in $\mathcal{A}(321, 3142, 2143)$ and inversion preserves involvement (see Lemma 1 of [13] and [1]), we see that $\mathcal{A}(321, 2413, 2143)$ is also atomic, of type $(\mathbb{N}, 2\mathbb{N})$.

So $\mathcal{T}(2\mathbb{N}, \mathbb{N})$ and $\mathcal{T}(\mathbb{N}, \mathbb{N})$ are independent, in the sense that neither is contained in the other. Having established that $\mathbb{N}$ and $2\mathbb{N}$ give rise to incomparable sets of supernatural classes, we next investigate the intersection $\mathcal{T}(\mathbb{N}, \mathbb{N}) \cap \mathcal{T}(2\mathbb{N}, \mathbb{N})$ of

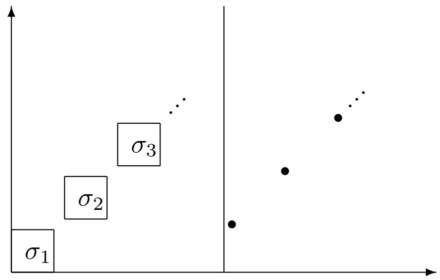

FIGURE 2

these sets. We begin by recalling some terminology from [3]. For two permutations α and β, their *sum* $\alpha \oplus \beta$ is defined to be the permutation $\gamma\delta$ where $\alpha \cong \gamma$, $\beta \cong \delta$, and δ is a rearrangement of $m+1, m+2, \dots$ $(m = |\alpha|)$. For example, $132 \oplus 213 = 132546$. This notation is extended to sets of permutations by defining $X \oplus Y = \{\alpha \oplus \beta : \alpha \in X, \beta \in Y\}$. A permutation is said to be *(sum-)decomposable* if it can be expressed as a sum of two non-empty permutations, *(sum-)indecomposable* otherwise. A set X of permutations is said to be *sum-complete* if, for all $\alpha, \beta \in X$, we have $\alpha \oplus \beta \in X$. It may be easily shown (see [5]) that a class X is sum-complete if and only if its basis $B(X)$ consists entirely of indecomposable elements. It is proved in [3] that any sum-complete class is natural. So for example $\mathcal{A}(21)$ and $\mathcal{A}(321)$ are sum-complete, whereas $\mathcal{A}(321, 2143)$ is not.

The intersection $\mathcal{T}(\mathbb{N}, \mathbb{N}) \cap \mathcal{T}(2\mathbb{N}, \mathbb{N})$ is certainly non-empty since it contains the closed set S of all permutations. S may be represented as a natural class $S = \mathrm{Sub}(\tau)$, where we write $S = \{\sigma_1, \sigma_2, \sigma_3, \dots\}$ and let $\tau = \sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \cdots$. To see that S may be represented as a member of $\mathcal{T}(2\mathbb{N}, \mathbb{N})$, construct a bijection as shown in Figure 2.

Place a copy of σ_1 in the bottom left corner of $\mathbb{N}_1$, then a point above σ_1 at the extreme left of $\mathbb{N}_2$, then a copy of σ_2 above and to the right of σ_1, followed by another point in $\mathbb{N}_2$, and so on.

In fact, the following general result holds.

Proposition 2.2. *The closed class S of all permutations is of type (A, B) for every pair (A, B) of infinite countable linearly ordered sets, and it is the only class with this property; i.e.*

$$\bigcap_{(A,B)} \mathcal{T}(A, B) = \{S\}.$$

Proof. Any infinite linearly ordered set contains a copy of $\mathbb{N}$ or $-\mathbb{N}$ (see Lemma 3.5). Given any A and B, we can find subsets $N_A \subseteq A$ and $N_B \subseteq B$, each isomorphic to one of $\mathbb{N}$ or $-\mathbb{N}$. Moreover, we may choose these subsets to ensure that both $A \setminus N_A$ and $B \setminus N_B$ are infinite. We have seen earlier that $S \in \mathcal{T}(\mathbb{N}, \mathbb{N})$; it may analogously be proved that $S \in \mathcal{T}(C, D)$ where $C, D \in \{\mathbb{N}, -\mathbb{N}\}$. Hence there is a bijection $\pi' : N_A \to N_B$ such that $\mathrm{Sub}(\pi') = S$. Since $A \setminus N_A$ and $B \setminus N_B$ are countably infinite, we can extend π' to a bijection $\pi : A \to B$; clearly $\mathrm{Sub}(\pi) = S$ and hence $S \in \mathcal{T}(A, B)$.

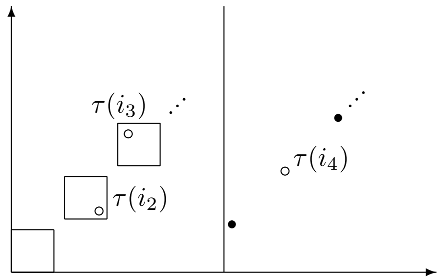

FIGURE 3

That S is the only class of type (A, B) for every pair (A, B) of countable infinite linearly sets follows, for example, from the fact that $\mathcal{T}(\mathbb{Q}, \mathbb{N}) = \{S\}$, which is a special case of Theorem 4.2. □

Our construction for S as a $(2\mathbb{N}, \mathbb{N})$ class may be adapted to obtain $(2\mathbb{N}, \mathbb{N})$ representations of other natural classes. Consider $X = \mathcal{A}(4132)$; since 4132 is indecomposable, X is a sum-complete natural class. Write its elements as $\{\sigma_1, \sigma_2, \sigma_3, \ldots\}$ then apply the above construction to obtain a $(2\mathbb{N}, \mathbb{N})$ class $\mathrm{Sub}(\tau : 2\mathbb{N} \to \mathbb{N})$. We claim that $X = \mathrm{Sub}(\tau)$. Write $\tau_i = \tau|_{\mathbb{N}_i}$ for $i = 1, 2$; then each σ_j is embedded in τ_1 with all points of its embedding less than $\tau(\omega + j)$, hence $X \subseteq \mathrm{Sub}(\tau)$. To check that the reverse containment holds it suffices to check that no basis element of X occurs in $\mathrm{Sub}(\tau)$. Since 4132 is not involved in any of the σ_i and is indecomposable, it cannot possess an embedding lying entirely within τ_1; since τ_2 is an increasing sequence, there can be no embedding of 4132 lying entirely within τ_2. Suppose there is an embedding $\tau(i_1)\tau(i_2)\tau(i_3)\tau(i_4)$ of 4132 in τ (see Figure 3); then $i_4 \in \mathbb{N}_2$, say $i_4 = \omega + m$. Since $\tau(i_3)\tau(i_4)$ is a descent, we must have $i_3 \in \mathbb{N}_1$; then $\tau(i_3)$ lies in the embedding of σ_n in τ_1, where $n > m$. The point $\tau(i_2)$ occurs in the embedding of σ_l where $l \leq m$; but then $\tau(i_1)$ must lie to the left of $\tau(i_2)$ and above $\tau(i_3)$, a contradiction since for all $i < i_2$ we have $\tau(i) < \tau(\omega + l) \leq \tau(\omega + m) < \tau(i_3)$. So $4132 \not\preceq \tau$ and hence $X = \mathrm{Sub}(\tau)$.

One objection which could justifiably be levelled against this construction is that it involves 'cheating', in the sense that all points $\tau(\omega + n)$, $n \in \mathbb{N}$, could be removed from τ without the loss of any sub-permutations of S. Does there exist $X \in \mathcal{T}(\mathbb{N}, \mathbb{N}) \cap \mathcal{T}(2\mathbb{N}, \mathbb{N})$ which can be written as $X = \mathrm{Sub}(\tau : 2\mathbb{N} \to \mathbb{N})$ where $\mathrm{Sub}(\tau|_{\mathbb{N}_1} : \mathbb{N}_1 \to \tau(\mathbb{N}_1)) \neq X$ and $\mathrm{Sub}(\tau|_{\mathbb{N}_2} : \mathbb{N}_2 \to \tau(\mathbb{N}_2)) \neq X$?

If X is a sum-complete class, then the answer to this question is negative. For otherwise, we could find $\sigma \in X$ with no embedding in τ_1, and $\rho \in X$ with no embedding in τ_2; by the sum-completeness of X, $\sigma \oplus \rho$ would be an element of X and hence would have an embedding in τ. Then $\sigma \oplus \rho \preceq \tau_1 \cup \tau_2$, but since σ has no embedding in τ_1, ρ would have to be embedded entirely in τ_2, yielding a contradiction. However, the situation is different in the non-sum-complete case.

Let $X = \mathrm{Sub}(3241) \oplus \mathcal{A}(3241)$. Since 3241 is indecomposable, $\mathcal{A}(3241)$ is sum-complete; by Theorem 3.1 of [3], X is a natural class. Construct the bijection $\tau : 2\mathbb{N} \to \mathbb{N}$ as follows: place 3241 in the bottom left-hand corner of $\mathbb{N}_1$, then arrange the sum-complete part above and to the right of 3241 using the usual

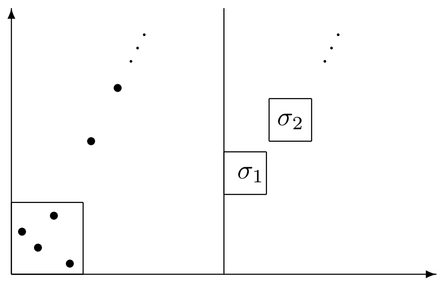

FIGURE 4

process, but this time with the increasing sequence of single points in $\mathbb{N}_1$ and increasing sequence of σ_i in $\mathbb{N}_2$ (see Figure 4).

An analogous argument to the $\mathcal{A}(4132)$ case shows that $\mathrm{Sub}(\tau) = X$. Clearly neither τ_1 nor τ_2 contains all permutations in $\mathrm{Sub}(\tau)$, and hence this construction yields a non-trivial representation of X as a $(2\mathbb{N}, \mathbb{N})$ class.

3. Linearly ordered sets

In this section we establish necessary terminology and results about linearly ordered sets. In addition to using the notation $\mathbb{N}$ to denote the positive integers with their natural ordering, we will use $-\mathbb{N}$ to denote the negative integers with their natural ordering, while C_r $(r \in \mathbb{N})$ will denote a chain of finite length r.

Definition 3.1. Let $(A, \leq_A)$, $(B, \leq_B)$ be two linearly ordered sets. Then we define $A \oplus B$ to be the disjoint union $A \dot{\bigcup} B$ with the ordering:

$$\begin{aligned} x \leq y \quad \Leftrightarrow \quad & x, y \in A, \quad x \leq_A y, \quad \text{or} \\ & x, y \in B, \quad x \leq_B y, \quad \text{or} \\ & x \in A, \quad y \in B. \end{aligned}$$

We will use the notation kA to denote $A \oplus \cdots \oplus A$ with k summands.

Definition 3.2. Let $(A, \leq_A)$, $(B, \leq_B)$ be two linearly ordered sets. Choose pairwise disjoint copies B_a of B, one for each $a \in A$. Then we define AB to be the set $\bigcup\limits_{a \in A} B_a$, with ordering given by

$$\begin{aligned} x \leq y \quad \Leftrightarrow \quad & x, y \in B_a, \quad x \leq y \quad \text{in } B_a, \quad \text{or} \\ & x \in B_{a_1}, \quad y \in B_{a_2} \quad \text{and } a_1 \leq_A a_2. \end{aligned}$$

Informally, AB is the set obtained by replacing each element of A with a copy of B. Observe that kA, as defined in Definition 3.1, and $C_k A$, as defined in Definition 3.2, are isomorphic as linearly ordered sets.

Definition 3.3. Let A be a linearly ordered set. A point $p \in A$ is a *left limit point* if p is not the smallest element and, for every $q \in A$ with $q < p$, the interval $(q, p) = \{x \in A : q < x < p\}$ is non-empty. A point $p \in A$ is a *right limit point* if p is not the largest element and, for every $q \in A$ with $q > p$, the interval $(p, q) = \{x \in A : p < x < q\}$ is non-empty. A *two-sided limit point* is a left and right limit point. A *limit point* is any of the above. If a point in A is not a limit point, it is said to be a *discrete point*.

Clearly, no element p which has an immediate predecessor p^- can be a left limit point (just consider the interval (p^-, p)), and no element p with an immediate successor p^+ can be a right limit point (consider (p, p^+)). Since all elements of $\mathbb{N}$ have immediate successors, and all except the smallest have immediate predecessors, $\mathbb{N}$ has no limit points; all its points are discrete. An example of a left limit point is $\omega + 1$ in $2\mathbb{N}$; $2\mathbb{N}$ has no right limit points. In $\mathbb{Q}$, the set of rational numbers, every element is a limit point.

Note that, in Definition 3.3, 'non-empty' is equivalent to 'infinite': given any interval (q_1, l) where l is a left limit point, we may choose a point $q_2 \in (q_1, l)$ and consider in turn the non-empty interval (q_2, l). Carrying out this process repeatedly yields a sequence $q_1 < q_2 < q_3 < \cdots$ of elements of A lying to the left of l. Similarly, for any interval (r, q_1) where r is a right limit point, we may always find $q_2 \in (r, q_1)$ and proceed to consider (r, q_2). This argument establishes the following result.

Lemma 3.4. *Let A be a linearly ordered set. If A contains a left limit point l, then for any point p to the left of l, a copy of $\mathbb{N}$ can be found in A between p and l. If A contains a right limit point r, then for any point p to the right of r, a copy of $-\mathbb{N}$ can be found in A between r and p.*

The following result is well-known but we include a proof for completeness.

Lemma 3.5. *If A is an infinite linearly ordered set, then A contains a copy of $\mathbb{N}$ or $-\mathbb{N}$.*

Proof. If A contains a limit point p, then we can use Lemma 3.4 to find a copy of $\mathbb{N}$ or $-\mathbb{N}$ in A. Otherwise, all points in A are discrete. Choose an arbitrary $a_1 \in A$; then at least one of $\{b \in A : b < a_1\}$ or $\{b \in A : b > a_1\}$ is infinite. Without loss of generality, assume that $\{b \in A : b > a_1\}$ is infinite. Since A contains no limit points, we may find $a_2 < a_3 < a_4 < \cdots$ with the property that $(a_1, a_2) = \emptyset$, $(a_2, a_3) = \emptyset$ and in general $(a_i, a_{i+1}) = \emptyset$. Then $\{a_1, a_2, a_3, \ldots\} \cong \mathbb{N}$. □

Definition 3.6. Let A be a linearly ordered set.

- If, for all $k \in \mathbb{N}$, A contains a copy of $k\mathbb{N}$ or $k(-\mathbb{N})$ then A is of Type 1.
- If there exist only finitely many k such that A contains a copy of $k\mathbb{N}$ or $k(-\mathbb{N})$, then A is of Type 2.

Lemma 3.7. *Let A, B and C be linearly ordered sets such that $A = B \cup C$ where $B < C$. If A is of Type 1, then at least one of B or C is of Type 1.*

Proof. Without loss of generality, we consider the case when A contains $k\mathbb{N}$ for all k. Also, aiming for a contradiction, suppose that B contains $p\mathbb{N}$ but not $(p+1)\mathbb{N}$, and that C contains $q\mathbb{N}$ but not $(q+1)\mathbb{N}$. Consider a copy of $(p+q+1)\mathbb{N}$ in A; we have $A_1 < A_2 < \cdots < A_{p+q+1}$ where each $A_i \cong \mathbb{N}$. Then A_{p+1} is not contained in B, and so $A_{p+1} \cap C \cong \mathbb{N}$. But then $A_{p+1} \cap C < A_{p+2} < \cdots < A_{p+q+1}$ is a copy of $(q+1)\mathbb{N}$ in C, a contradiction. □

The next result gives an alternative characterization of Type 1 sets.

Lemma 3.8. *Let A be a linearly ordered set. Then A is of Type 1 if and only if A contains a subset isomorphic to BC, where B, $C \in \{\mathbb{N}, -\mathbb{N}\}$.*

Proof. ($\Leftarrow$) Immediate.
($\Rightarrow$) Let A be of Type 1. Consider first the case when A has infinitely many left limit points. The set L of left limit points of A is itself an infinite linearly ordered set and so, by Lemma 3.5, L has a subset isomorphic to one of $\mathbb{N}$ or $-\mathbb{N}$. Assuming for the moment the first of these alternatives, we list these left limit points as $l_1 < l_2 < l_3 < \cdots$. Then by Lemma 3.4, we may obtain a copy of $\mathbb{N}$ between each l_i and l_{i+1}, and hence a copy of $\mathbb{N}\mathbb{N}$ in A. An entirely analogous argument in the case when L contains a copy of $-\mathbb{N}$ yields a copy of $(-\mathbb{N})\mathbb{N}$ in A.

Now consider the case when A has finitely many left limit points. In fact, we may assume that A has no left limit points. For, given A with r left limit points $l_1 < l_2 < \cdots < l_r$, define $A_1 = \{x \in A : x < l_1\}$, $A_j = \{x \in A : l_{j-1} \leq x < l_j\}$ for $j = 2, \ldots, r$, and $A_{r+1} = \{x \in A : x \geq l_r\}$. Then $A = A_1 \cup A_2 \cup \cdots \cup A_{r+1}$, $A_1 < A_2 < \cdots < A_{r+1}$ and, by construction, none of the A_i $(i = 1, \ldots, r+1)$ possess left limit points. By Lemma 3.7, we may find from amongst these a Type 1 set with no left limits; set this to be our new A.

Choose a point $a_{-1} \in A$; then we may choose its predecessor $a_{-2} < a_{-1}$, followed by the predecessor a_{-3} of a_{-2}. Repeating this process yields a copy $M_1 = \ldots a_{-3}a_{-2}a_{-1}$ of $-\mathbb{N}$ consisting of consecutive elements. So $A \cong A_1' \oplus M_1 \oplus A_1''$. By Lemma 3.7, one of A_1' or A_1'' is of Type 1; call this A_1. Repeat the process: inside A_1 we can find $M_2 \cong -\mathbb{N}$ with $A_1 \cong A_2' \oplus M_2 \oplus A_2''$ and A_2 of Type 1, and so on. Thus we create $M_1, M_2, M_3, \ldots$, each isomorphic to $-\mathbb{N}$ and all non-intermingling in the sense that for $i \neq j$, $M_i < M_j$ or $M_j < M_i$. Thus the set of all M_i is an infinite linearly ordered set, and inside this set we can find a copy of $\mathbb{N}$ or $-\mathbb{N}$, i.e. $M_{i_1} < M_{i_2} < M_{i_3} < \cdots$ or $M_{i_1} > M_{i_2} > M_{i_3} > \cdots$. This yields a copy of $\mathbb{N}(-\mathbb{N})$ or $(-\mathbb{N})(-\mathbb{N})$ in A. □

In the following two propositions, we introduce a normal form for Type 2 sets and demonstrate its uniqueness.

Proposition 3.9. *A linearly ordered set of Type 2 has the form $B_1 \oplus \cdots \oplus B_n$ ($n \in \mathbb{N}$), where each member of the sequence $B_1, \ldots, B_n$ is either $\mathbb{N}$, $-\mathbb{N}$ or a chain C_r of length $r \in \mathbb{N}$. Moreover, this sequence can be chosen to be in standard form where no C_r directly precedes any copy of $\mathbb{N}$ or C_s, nor directly follows any copy of $-\mathbb{N}$ or C_s ($s \in \mathbb{N}$).*

Proof. The second part of the proposition is immediate upon observing that $C_r \oplus \mathbb{N} \cong \mathbb{N}$, $-\mathbb{N} \oplus C_r \cong -\mathbb{N}$ and $C_r \oplus C_s \cong C_{r+s}$. If A is finite, then $A \cong C_r$ for some $r \in \mathbb{N}$.

Suppose that A is an infinite set of Type 2. Define $k = k(A)$ to be the largest positive integer such that A contains $k\mathbb{N}$, and $l = l(A)$ to be the largest positive integer such that A contains $l(-\mathbb{N})$. We will prove the result by induction on $k+l$.

If $k + l = 1$, then A contains a copy of either $\mathbb{N}$ or $-\mathbb{N}$. Consider the case when $k = 1$ and $l = 0$, i.e. A contains $\mathbb{N}$ but none of $\{2\mathbb{N}, 3\mathbb{N}, \ldots\}$ and A does not contain $l(-\mathbb{N})$ for any $l \in \mathbb{N}$. We must show that either $A \cong \mathbb{N}$ or $A \cong \mathbb{N} \oplus C_r$ for some r. If A does not possess a smallest element, then A contains a copy of $-\mathbb{N}$, a contradiction. So A has a smallest element a_1. If a_1 is a right limit point, then by Lemma 3.4 a copy of $-\mathbb{N}$ may be found to the right of a_1, again a contradiction. So a_1 has a successor a_2 in A (by the existence of a_2 with $(a_1, a_2) = \emptyset$). Invoking Lemma 3.4 once more, a_2 cannot be a right limit point, and so a_2 has a successor a_3. Repeating this process, we obtain an infinite chain of successive elements in A, all discrete, and so the initial segment of A is a copy $\{a_1, a_2, a_3, \ldots\}$ of $\mathbb{N}$. Suppose $A \neq \mathbb{N}$. Then $A \cong \mathbb{N} \oplus A'$, i.e. $\mathbb{N}$ must be followed in A by some set A'. Note that A' cannot be infinite, else A would contain either $2\mathbb{N}$ or a copy of $-\mathbb{N}$, contradicting $k + l = 1$. So A' must consist of finitely many (r, say) points, and so $A \cong \mathbb{N} \oplus C_r$, as required. An analogous argument establishes the result in the case when $k = 0$ and $l = 1$.

We deal with the general situation by identifying four cases.

Case 1: A has no smallest element and no largest element. Choose an arbitrary point $a \in A$. The sets $A^- = \{x \in A : x < a\}$ and $A^+ = \{x \in A : x \geq a\}$ are both infinite, and so each must contain a copy of $\mathbb{N}$ or $-\mathbb{N}$ by Lemma 3.5. Hence $k(A^-) + l(A^-)$ and $k(A^+) + l(A^+)$ are both less than $k(A) + l(A)$ and so the induction hypothesis applies to each. But $A \cong A^- \oplus A^+$, and the result follows.

Case 2: A has a smallest element and no largest element. Let a_1 be the smallest element of A and select successive elements $a_2, a_3, \ldots$ according to the rule: a_{i+1} is the smallest element of $A \setminus \{a_1, \ldots, a_i\}$, if it exists. Denote the set of all a_i's by A'. Let us first consider the case when A' is infinite, i.e. $A' = \{a_1, a_2, a_3, \ldots\} \cong \mathbb{N}$. Let $A'' = A \setminus A'$; then the induction hypothesis applies to A''. If A'' has expression $D_1 \oplus \cdots \oplus D_r$ in the required form, then $A \cong \mathbb{N} \oplus D_1 \oplus \cdots \oplus D_r$. If A' is finite, say $A' = \{a_1, \ldots, a_k\}$, then $A'' = A \setminus A'$ is of the type considered in Case 1. Applying the result from Case 1 to A'' yields a representation ($D_1 \oplus \cdots \oplus D_r$, say) of the required form; then $A \cong C_k \oplus D_1 \oplus \cdots \oplus D_r$.

Case 3: A has a largest element and no smallest element. This is dual to Case 2.

Case 4: A has a smallest element and a largest element. Let a_1 be the smallest element of A; construct $A' = \{a_1, a_2, \ldots\}$ as in Case 2 via the rule: a_{i+1} is the smallest element of $A \setminus \{a_1, \ldots, a_i\}$ if it exists. Let $A'' = A \setminus A'$. If $A' = \mathbb{N}$ then the induction hypothesis applies to A'' and we proceed as in Case 2. If A' is finite, then A'' is of the type dealt with in Case 3. $\square$

For any Type 2 linearly ordered set A, we will denote by $S(A)$ the standard sequence of $\mathbb{N}$'s, $-\mathbb{N}$'s and C_r's guaranteed by Proposition 3.9. We define $\mathcal{W}$ to be the collection of Type 2 sets whose sequence entries are drawn exclusively from $\{\mathbb{N}, -\mathbb{N}\}$. To move from sequences to ordered sets, we define the operator L which sends a sequence $S = (B_1, B_2, \ldots, B_n)$ in standard form to the linearly ordered set $L(S) = B_1 \oplus \cdots \oplus B_n$. When $L(S) = B_1 \oplus \cdots \oplus B_n$, we will normally write the corresponding disjoint union of sets as $L(S) = B_1' \cup \cdots \cup B_n'$, where $B_i' \cong B_i$ and $B_1' < B_2' < \cdots < B_n'$.

Proposition 3.10. *Let S_1 and S_2 be two non-identical finite sequences of symbols from $\{\mathbb{N}, -\mathbb{N}, C_r (r \in \mathbb{N})\}$ in standard form. Then $L(S_1) \ncong L(S_2)$.*

Proof. Let $S_1 = (B_1, \ldots, B_m)$ and $S_2 = (D_1, \ldots, D_n)$, where all $B_i, D_j \in \{\mathbb{N}, -\mathbb{N}, C_r(r \in \mathbb{N})\}$. Let $A_1 = L(S_1)$ and $A_2 = L(S_2)$. We establish that $A_1 \ncong A_2$ by induction on $m + n$, the total length of S_1 and S_2. We suppose, aiming for a contradiction, that there exists an order-preserving bijection $\phi : A_1 \to A_2$, and consider six possible cases. Case 1 includes an anchor for the induction ($m = n = 1$).

Case 1: $B_1 \neq D_1$. We consider the following three possibilities.

(1) $B_1 \in \{\mathbb{N}, C_r\}$, $D_1 = -\mathbb{N}$: A_1 has a smallest element whereas A_2 does not, so $A_1 \ncong A_2$.

(2) $B_1 = C_r$, $D_1 = C_s$: Without loss of generality, assume that $r < s$. Under

ϕ, B_1' is mapped onto $\bar{D_1'}$, the first r points of D_1'. Then $A_1 \setminus B_1'$ is mapped onto $A_2 \setminus \bar{D_1'}$. Since S_1 is in standard form, we have $B_2 = -\mathbb{N}$ and this case now reduces to (1).

(3) $B_1 = C_r$, $D_1 = \mathbb{N}$: Under ϕ, the r points of B_1' are mapped into $\bar{D_1'}$, the first r points of D_1'. Then $A_1 \setminus B_1'$ must be mapped by ϕ to $A_2 \setminus \bar{D_1'}$. However, since $B_2 = -\mathbb{N}$, the first of these sets has no smallest element whereas the second does, a contradiction.

Case 2: $B_1 = D_1 = \mathbb{N}$. Under ϕ, the elements $b_1, b_2, \dots$ of B_1' are mapped to the corresponding elements $d_1, d_2, \dots$ of D_1'. The result then follows by application of the induction hypothesis to $A_1 \setminus B_1'$ and $A_2 \setminus D_1'$.

Case 3: $B_1 = D_1 = C_r$. Under ϕ, the r points $b_1, \dots, b_r$ of B_1' are mapped to the r points $d_1, \dots, d_r$ of D_1'. The result then follows by application of the induction hypothesis to $A_1 \setminus B_1'$ and $A_2 \setminus D_1'$.

Case 4: $B_1 = D_1 = -\mathbb{N}$, $B_2 = D_2 = \mathbb{N}$. Choose an arbitrary point $z_1 \in B_1' \cup B_2'$ and consider $z_2 = \phi(z_1) \in A_2$. Proceeding predecessor by predecessor, we may identify the sets $\{\dots, b_{-3}, b_{-2}, b_{-1} = z_1\}$ in A_1 and $\{\dots, d_{-3}, d_{-2}, d_{-1} = z_2\}$ in A_2. Proceeding successor by successor, we may identify the sets $\{z_1 = b_1, b_2, b_3, \dots\}$ in A_1 and $\{z_2 = d_1, d_2, d_3, \dots\}$ in A_2. This proves that ϕ maps $B_1' \cup B_2'$ onto $D_1' \cup D_2'$. The result then follows by application of the induction hypothesis to $A_1 \setminus (B_1' \cup B_2')$ and $A_2 \setminus (D_1' \cup D_2')$.

Case 5: $B_1 = D_1 = -\mathbb{N}$, $B_2 = D_2 = -\mathbb{N}$. Consider the leftmost limit points r_1 and r_2 of A_1 and A_2 respectively. These are the largest elements of B_1' and D_1' and are right limit points; we must have $r_1 = \phi(r_2)$. Proceeding predecessor by predecessor, the elements $\{\dots, b_{-3}, b_{-2}, b_{-1} = r_1\}$ of B_1' may be mapped to the corresponding elements $\{\dots, d_{-3}, d_{-2}, d_{-1} = r_2\}$ of D_1'. The result then follows by applying the induction hypothesis to $A_1 \setminus B_1'$ and $A_2 \setminus D_1'$.

Case 6: $B_1 = D_1 = -\mathbb{N}$, $B_2 \neq D_2$. Without loss of generality, we may assume that $B_2 = -\mathbb{N}$ and $D_2 = \mathbb{N}$. Consider the largest element of B_1' and note that it is the leftmost right limit point of A_1 (since $B_2' \cong -\mathbb{N}$). Since ϕ is an order-preserving bijection, r_1 must be mapped under ϕ to some right limit point $r_2 \in A_2$. Such an r_2 can occur no further left than the largest element of D_3', if such exists. But then the set $\{x \in A_2 : x \leq r_2\}$, which contains a copy of $-\mathbb{N} \oplus \mathbb{N} \cong \mathbb{Z}$, is isomorphic under ϕ to $B_1' \cong -\mathbb{N}$, a contradiction. □

4. Independence of sets of supernatural classes

In this section, we investigate the question: can different linearly ordered sets A_1 and A_2 give rise to the same sets $\mathcal{T}(A_1, \mathbb{N})$ and $\mathcal{T}(A_2, \mathbb{N})$ of supernatural classes? We shall see that the answer to this question is radically different for Type 1 and Type 2 sets. In both cases, the following technical result will prove invaluable.

Lemma 4.1. *Let A be a linearly ordered set containing a subset isomorphic to some $B \in \mathcal{W}$, with $S(B) = (B_1, B_2, \dots, B_k)$. Then every bijection $\pi : A \to \mathbb{N}$ involves every juxtaposition of the form $\epsilon_1 \dots \epsilon_k$ where ϵ_i is an increasing sequence if $B_i = \mathbb{N}$ and a decreasing sequence if $B_i = -\mathbb{N}$. Therefore every class $X \in \mathcal{T}(A, \mathbb{N})$ contains every permutation of the above form and, in particular, every permutation of length up to k.*

Proof. Let $\sigma = \epsilon_1 \dots \epsilon_k$ be a juxtaposition of the stated form, let n denote the length of σ, and let the entry n of σ lie in ϵ_i. Consider σ' formed by removing n

from σ, and write it as the juxtaposition $\sigma' = \epsilon_1 \ldots \epsilon_{i-1}\epsilon_i{}'\epsilon_{i+1} \ldots \epsilon_k$, where $\epsilon_i{}'$ is ϵ_i with the entry n removed. Suppose inductively that σ' has an embedding of the type described above, where ϵ_j is embedded in π_j for $j = 1, \ldots, i-1, i+1, \ldots, k$ and $\epsilon_i{}'$ is embedded in π_i. Then it is possible to find a point in π_i to the right (respectively, left) of the embedding of $\epsilon_i{}'$ if $B_i = \mathbb{N}$ (respectively, $B_i = -\mathbb{N}$), which is larger than all points of the embedding of σ'. Taking this point together with the embedding of σ' yields an embedding of σ in π. □

As an immediate consequence, we can show that $\mathcal{T}(A, \mathbb{N})$ is the same for every linearly ordered set A of Type 1.

Theorem 4.2. *If a linearly ordered set A contains a copy of BC for some $B, C \in \{\mathbb{N}, -\mathbb{N}\}$ (i.e. if it is of Type 1), then $\mathcal{T}(A, \mathbb{N})$ is the set of all permutations.*

Proof. Let $X = \text{Sub}(\tau : A \to \mathbb{N})$ be arbitrary in $\mathcal{T}(A, \mathbb{N})$. Let $\sigma = s_1 s_2 \ldots s_r$ be any permutation. Since A is of Type 1, A contains a copy of $r\mathbb{N}$ or $r(-\mathbb{N})$. By Lemma 4.1, $\sigma \in X$, proving that X is the set of all permutations. □

In the remainder of this section, we consider sets of Type 2. Given an arbitrary Type 2 set A, with $S(A) = (X_1, \ldots, X_k)$, as before we will write $A = X_1' \cup \cdots \cup X_k'$ where $X_i' \cong X_i$ and $X_1' < \cdots < X_k'$. For any bijection $\pi : A \to \mathbb{N}$, we define $\pi_i = \pi|_{X_i'}$, and we will call π_i an X_i*-slice*.

An *increase-decrease pattern* is any sequence of letters $\mathbf{i}$ and $\mathbf{d}$. Given an increase-decrease pattern $\rho = (r_1, \ldots, r_k)$ and a permutation $\sigma = s_1 \ldots s_n$, we say that σ contains ρ if there exist subscripts $1 \le i_1 < j_1 \le i_2 < j_2 \le \cdots \le i_k < j_k \le n$ such that $s_{i_t} < s_{j_t}$ if $r_t = \mathbf{i}$ and $s_{i_t} > s_{j_t}$ if $r_t = \mathbf{d}$. A permutation σ may contain more than one increase-decrease pattern ρ; we will refer to the pattern ρ of maximal length $|\sigma| - 1$ as *the* increase-decrease pattern of σ. For example, the increase-decrease pattern of 463125 is $(\mathbf{i}, \mathbf{d}, \mathbf{d}, \mathbf{i}, \mathbf{i})$, but the permutation also contains patterns $(\mathbf{i}, \mathbf{d}, \mathbf{i}, \mathbf{i})$, $(\mathbf{i}, \mathbf{d}, \mathbf{d}, \mathbf{i})$ and $(\mathbf{i}, \mathbf{d}, \mathbf{i})$. Observe that the permutations $21 \oplus 21 \oplus \cdots \oplus 21$ and $12 \ominus 12 \ominus \cdots \ominus 12$ ($k+1$ summands) contain every increase-decrease pattern of length up to k.

For any $A \in \mathcal{W}$ with $S(A) = (X_1, \ldots, X_k)$ we define $\rho(A) = (r_1, \ldots, r_k)$ where $r_i = \mathbf{i}$ if $X_i = -\mathbb{N}$ and $r_i = \mathbf{d}$ if $X_i = \mathbb{N}$. We define $\Gamma(A)$ to be the set of all permutations of minimal length which contain $\rho(A)$. For example, if $A = \mathbb{N} \oplus -\mathbb{N} \oplus \mathbb{N}$, then $\rho(A) = (\mathbf{d}, \mathbf{i}, \mathbf{d})$ and $\Gamma(A) = \{2143, 3142, 3241, 4132, 4231\}$.

Lemma 4.3. *Let $A \in \mathcal{W}$ with $S(A) = (X_1, \ldots, X_k)$, let $\pi : A \to \mathbb{N}$ be a bijection, and let σ be any permutation which contains $\rho(A)$. Then any embedding of σ in π will involve a descent in an $\mathbb{N}$-slice, or an ascent in a $-\mathbb{N}$-slice.*

Proof. Let $\sigma = s_1 \ldots s_n$, let $1 \le i_1 < j_1 \le i_2 < j_2 \le \cdots \le i_k < j_k \le n$ be the subscripts which witness the containment of $\rho(A)$ in σ, and let $\pi(r_1) \ldots \pi(r_n)$ be an embedding of σ in π. For each $l = 1, \ldots, k$, we ask in which π_{t_l} ($t_l \in \{1, \ldots, k\}$) the entry $\pi(r_{i_l})$ occurs. Since $n > k$, we have that for some l, $\pi(r_{i_l})$ will occur in π_{t_l} where $t_l < l$; choose the smallest such l and note that $l \neq 1$. In particular, $l - 1 \le t_{l-1} \le t_l < l$, and so both $\pi(r_{i_{l-1}})$ and $\pi(r_{i_l})$ lie in π_{l-1}. Since $i_{l-1} < j_{l-1} \le i_l$, it follows that $\pi(r_{j_{l-1}})$ lies in π_{l-1} too. By definition of σ containing ρ, we have that $\pi(r_{i_{l-1}})\pi(r_{j_{l-1}})$ is a descent if $X_{l-1} = \mathbb{N}$ and an ascent if $X_{l-1} = -\mathbb{N}$. Hence our embedding of σ possesses a descent in an $\mathbb{N}$-slice or an ascent in a $-\mathbb{N}$-slice, as required. □

As a prelude to our main theorem about independence of Type 2 sets, we define a new property, $\mathcal{ID}$-incompatibility, and prove a series of results about it. For any Type 2 set A, where $S(A) = (X_1, \dots, X_k)$, define $\mathcal{ID}(A)$ to be the set of all bijections $\tau : A \to \mathbb{N}$ such that τ_i is increasing if $X_i = \mathbb{N}$ or C_r (some r) and decreasing if $X_i = -\mathbb{N}$. We shall say that two linearly ordered sets A_1 and A_2 are *$\mathcal{ID}$-incompatible* if $\tau \in \mathcal{ID}(A_1)$ implies $\mathrm{Sub}(\tau) \notin \mathcal{T}(A_2, \mathbb{N})$ and $\tau \in \mathcal{ID}(A_2)$ implies $\mathrm{Sub}(\tau) \notin \mathcal{T}(A_1, \mathbb{N})$.

Proposition 4.4. *Let $A, B \in \mathcal{W}$. If $A \ncong B$, then A and B are $\mathcal{ID}$-incompatible.*

Proof. Let $S(A) = (D_1, \dots, D_k)$ and $S(B) = (E_1, \dots, E_l) \neq S(A)$. Let $\tau \in \mathcal{ID}(A)$ and let $X = \mathrm{Sub}(\tau)$. Suppose, aiming for a contradiction, that $X = \mathrm{Sub}(\pi : B \to \mathbb{N})$.

Case 1: $k < l$. Let $\gamma \in \Gamma(A)$. By Lemma 4.3, $\gamma \notin \mathrm{Sub}(\tau)$, since by construction τ_i contains no descents when $D_i = \mathbb{N}$ and no ascents when $D_i = -\mathbb{N}$. However, γ has length $k+1$ and, since $l \geq k+1$, we have $\gamma \in \mathrm{Sub}(\pi : B \to \mathbb{N})$ by Lemma 4.1. So $\gamma \in X$, a contradiction.

Case 2: $k > l$. Let $\gamma = c_1 \dots c_{l+1} \in \Gamma(B)$. We begin by showing that there are infinitely many embeddings of γ in π.

Define the permutation $\sigma^{(n)}$ as follows:

$$\sigma^{(n)} = \begin{cases} 1 \dots n(n+c_1) \dots (n+c_{l+1}) & \text{if } D_1 = \mathbb{N}, \\ (n+c_1)n \dots 1(n+c_2) \dots (n+c_{l+1}) & \text{if } D_1 = -\mathbb{N}. \end{cases}$$

Since $\sigma^{(n)}$ consists of an increasing/decreasing sequence of length $n+1$ followed by l further points, $\sigma^{(n)}$ may be considered as a juxtaposition of $l + 1 (\leq k)$. increasing/decreasing segments. Hence, by Lemma 4.1, $\sigma^{(n)}$ has an embedding in τ. So $\sigma^{(n)} \in \mathrm{Sub}(\tau) = X$, and hence $\sigma^{(n)}$ has an embedding in π. Each such embedding induces an embedding of γ in π. Letting n grow arbitrarily large, we see that we obtain infinitely many embeddings of γ in π.

By Lemma 4.3, each of these infinitely many embeddings must involve a descent in some π_i where $E_i = \mathbb{N}$, or ascent in some π_i where $E_i = -\mathbb{N}$ $(1 \leq i \leq l)$. Since there are infinitely many embeddings but only a finite number of E_i's, some π_i must contain infinitely many such 'inappropriate' ascents or descents. Assuming without loss of generality that $E_i = \mathbb{N}$, and using standard properties of the natural numbers, π_i contains the permutation $\mu = 2143 \dots (2k+2)(2k+1) = 21 \oplus 21 \oplus \dots \oplus 21$, so $\mu \in \mathrm{Sub}(\tau)$. Since μ contains $\rho(A)$, Lemma 4.3 implies an inappropriate ascent/descent in τ, a contradiction.

Case 3: $k = l$. Let $\gamma = c_1 \dots c_{k+1} \in \Gamma(A)$; by Lemma 4.3, $\gamma \notin \mathrm{Sub}(\tau)$. Let j be the smallest subscript for which $D_j \neq E_j$, and suppose without loss of generality that $D_j = \mathbb{N}$ and $E_j = -\mathbb{N}$. Then $c_j c_{j+1}$ is a descent. Consider $\gamma = c_1 \dots c_{j-1}(c_j c_{j+1}) c_{j+2} \dots c_{k+1}$ as the juxtaposition of $j-1$ one-element segments, followed by a two-element descent, followed by $k-j$ one-element segments. By Lemma 4.1 , since $E_j = -\mathbb{N}$, there is an embedding of γ in π. So $\gamma \in \mathrm{Sub}(\pi) = X$, a contradiction. $\square$

Lemma 4.5. *Let A_1, A_2 be linearly ordered sets of Type 2, and let S_1^* and S_2^* be the sequences obtained by removing all entries C_r ($r \in \mathbb{N}$) from $S(A_1)$ and $S(A_2)$ respectively. If $S_1^* \neq S_2^*$, then A_1 and A_2 are $\mathcal{ID}$-incompatible.*

Proof. Let $\tau \in \mathcal{ID}(A_1)$ and let $X = \mathrm{Sub}(\tau) \in \mathcal{T}(A_1, \mathbb{N})$. We will show that $X \notin \mathcal{T}(A_2, \mathbb{N})$. Aiming for a contradiction, suppose $X = \mathrm{Sub}(\pi : A_2 \to \mathbb{N})$.

Let $A_1^* = L(S_1^*)$ and $A_2^* = L(S_2^*)$; by Proposition 3.10, $A_1^* \not\cong A_2^*$. Clearly $A_1^*, A_2^* \in \mathcal{W}$. Without loss of generality we may assume that $A_1^* \subseteq A_1$ and $A_2^* \subseteq A_2$. Denote by t the maximum value attained by either τ or π on $A_1 \setminus A_1^*$ or $A_2 \setminus A_2^*$ respectively. Note that the removal of a finite number of elements of any $A \in \mathcal{W}$ yields a linearly ordered set isomorphic to A. Consider the mapping τ^* obtained from τ by removing all entries of value at most t. The domain of this mapping is a cofinite subset of A_1^* and hence is isomorphic to A_1^*. Similarly, the range of the mapping is a cofinite subset of $\mathbb{N}$, and hence is isomorphic to $\mathbb{N}$. Thus, after appropriate rescaling, τ^* may be considered as a bijection from A_1^* to $\mathbb{N}$; in fact, since $\tau \in \mathcal{ID}(A_1)$, we have $\tau^* \in \mathcal{ID}(A_1^*)$. We may define $\pi^* : A_2^* \to \mathbb{N}$ likewise. Let $X_\tau^* = \mathrm{Sub}(\tau^*)$ and $X_\pi^* = \mathrm{Sub}(\pi^*)$; from Proposition 4.4 we know that $X_\tau^* \notin \mathcal{T}(A_2^*, \mathbb{N})$. So $X_\tau^* \neq X_\pi^*$, and hence there exists $\rho \in (X_\tau^* \setminus X_\pi^*) \cup (X_\pi^* \setminus X_\tau^*)$. Consider the case when $\rho \in X_\tau^* \setminus X_\pi^*$. Let δ be the subpermutation of τ consisting of all entries of τ of value at most t and an embedding of ρ in τ^*. Then $\delta \in X = \mathrm{Sub}(\tau)$, and so by assumption $\delta \preceq \pi$. But then, by construction of δ, the chosen embedding of ρ at the 'top' of δ must embed into π^*, a contradiction. The case $\rho \in X_\pi^* \setminus X_\tau^*$ is analogous. □

Lemma 4.6. *Let A_1, A_2 be $\mathcal{ID}$-incompatible Type 2 sets. Then $A_1^+ = C_r \oplus k(-\mathbb{N}) \oplus \mathbb{N} \oplus A_1$ and $A_2^+ = C_r \oplus k(-\mathbb{N}) \oplus \mathbb{N} \oplus A_2$, where $r, k \geq 0$, are also $\mathcal{ID}$-incompatible.*

Proof. For notational convenience, in this proof we will identify the summands of A_1^+ and A_2^+ with the corresponding subsets of A_1^+ and A_2^+ isomorphic to them. Let $\tau \in \mathcal{ID}(A_1^+, \mathbb{N})$. We will show that $X = \mathrm{Sub}(\tau) \notin \mathcal{T}(A_2^+, \mathbb{N})$. Suppose on the contrary that $X = \mathrm{Sub}(\pi : A_2^+ \to \mathbb{N})$. Let $Y = \mathrm{Sub}(\tau|_{A_1})$ and $Z = \mathrm{Sub}(\pi|_{A_2})$; then $Y \neq Z$ and we can find some $\rho \in (Y \setminus Z) \cup (Z \setminus Y)$.

Suppose first that $\rho \in Z \setminus Y$. Consider a subsequence $\sigma = \sigma_1\sigma_2 uv\sigma_3$ of π where $\sigma_1 = \pi|_{C_r}$, $\sigma_2 u$ is an increasing subsequence of $\pi|_\mathbb{N}$ of length $k+1$, $v\sigma_3$ is an embedding of ρ in $\pi|_{A_2}$ and $u > v$. By assumption, σ has an embedding in τ. Since $|\sigma_1| = r$ and $\rho \notin Y$, clearly $\sigma_2 uv$ is embedded in $\tau|_{k(-\mathbb{N})\oplus\mathbb{N}}$. But $\sigma_2 uv$ has $\mathcal{ID}$-pattern $\mathbf{i}^k\mathbf{d} = \rho(k(-\mathbb{N}) \oplus \mathbb{N})$, and hence by Lemma 4.3, τ has an increase in a $-\mathbb{N}$-slice or a decrease in an $\mathbb{N}$-slice, a contradiction.

Now suppose that $\rho \in Y \setminus Z$. For an arbitrary $n \geq 1$, consider a subsequence $\sigma^{(n)} = \sigma_1\sigma_2 uv\sigma_3$ of τ where $\sigma_1 = \tau|_{C_r}$, $\sigma_2 u$ is an (increasing) subsequence of $\tau|_\mathbb{N}$ of length n, $v\sigma_3$ is an embedding of ρ in $\tau|_{A_1}$ and $u > v$. Now consider its embedding in π. Reasoning as in the previous case, we see that $\sigma_2 uv$ must be embedded in $\pi|_{k(-\mathbb{N})\oplus\mathbb{N}}$. The $\mathcal{ID}$-pattern of $\sigma_2 uv$ is $\mathbf{i}^{n-1}\mathbf{d}$. Letting $n \to \infty$, we see that either $\pi|_\mathbb{N}$ contains infinitely many descents or one of $\pi|_{-\mathbb{N}}$ contains infinitely many ascents. In the former case, $\pi|_\mathbb{N}$ contains the permutation $21 \oplus \cdots \oplus 21$, where we may choose the number of summands to be arbitrary, while in the latter case some $\pi|_{-\mathbb{N}}$ involves $12 \ominus \cdots \ominus 12$, again with arbitrarily many summands. However, since $\tau \in \mathcal{ID}(A_1^+, \mathbb{N})$, τ does not involve either of these permutations for sufficiently large n. □

Theorem 4.7. *Let A_1, A_2 be linearly ordered sets of Type 2. If $A_1 \not\cong A_2$, then A_1 and A_2 are $\mathcal{ID}$-incompatible.*

Proof. Let $S(A_1) = (B_1, \ldots, B_l)$ and $S(A_2) = (D_1, \ldots, D_m)$. As in Lemma 4.5, define S_1^* and S_2^* by removing all entries C_r from $S(A_1)$ and $S(A_2)$ respectively, and let A_1^* and A_2^* be the corresponding linearly ordered sets $L(S_1^*)$ and $L(S_2^*)$.

Then $A_1^*, A_2^* \in \mathcal{W}$ and it may be assumed without loss of generality that $A_1^* \subseteq A_1$ and $A_2^* \subseteq A_2$.

Let $\tau \in \mathcal{ID}(A_1)$. Let $X = \mathrm{Sub}(\tau) \in \mathcal{T}(A_1, \mathbb{N})$; we will show that $X \notin \mathcal{T}(A_2, \mathbb{N})$. Aiming for a contradiction, suppose $X = \mathrm{Sub}(\pi : A_2 \to \mathbb{N})$.

By Lemma 4.5, we may assume that $S_1^* = S_2^*$ but $S_1 \neq S_2$. Let j be the smallest number such that $B_j \neq D_j$. We begin by considering the case when $j = 1$, i.e. the sequences S_1 and S_2 differ in their first entries.

At least one of B_1, D_1 is a finite chain, and hence the first entry of S_1^* is $-\mathbb{N}$. Suppose $|S_1^*| = k$, and let $\gamma = c_1 \dots c_{k+1} \in \Gamma(A_1^*)$ be chosen so that $c_1 = 1$ (this is always possible since γ must begin with an ascent).

Case 1: $B_1 = -\mathbb{N}$, $D_1 = C_r$. Let t be any number greater than all values of $\tau|_{A_1 \setminus A_1^*}$ and the smallest value of τ_1. Now construct the permutation $\sigma = 1(t+c_2)t(t-1)\dots 2(t+c_3)\dots(t+c_{k+1})$. Then σ has an embedding in π, using D_1' to embed 1, $D_2' \cong -\mathbb{N}$ to embed the descent $(t+c_2)t(t-1)\dots 2$, and the remaining $D_i' \cong \pm\mathbb{N}$, $i \geq 3$, (there are $k-1$ of these) to embed the remaining entries. By assumption, σ also has an embedding in τ. By construction, the induced embedding of $c_2, \dots, c_{k+1}$ in τ must occur entirely within $\pm\mathbb{N}$-slices. Furthermore, at least one entry of τ_1 is smaller than all the entries of this embedding. Since c_1 is the smallest entry in γ and since τ_1 is decreasing, no points of the embedding of $c_2 \dots c_{k+1}$ lie in τ_1. Therefore, the smallest entry of τ_1 together with this embedding of $c_2 \dots c_{k+1}$ forms an embedding of γ in $\tau|_{A_1^*}$, a contradiction.

Case 2: $B_1 = C_r$, $D_1 = -\mathbb{N}$. Let t be the maximum value of π attained on $A_2 \setminus A_2^*$. Define the permutation $\sigma^{(n)} = 1(n+c_2)n(n-1)\dots 2(n+c_3)\dots(n+c_{k+1})$, where $n \geq t$. Arguing as in Case 1, we can show that $\sigma^{(n)}$ is embeddable in τ. Since $X = \mathrm{Sub}(\pi)$, $\sigma^{(n)}$ also has an embedding in π. Consider the induced embedding of γ in π. By construction, all entries $c_2 \dots c_{k+1}$ of γ must be embedded above t, and hence lie in $\pm\mathbb{N}$-slices. Letting $n \to \infty$ yields infinitely many embeddings of γ in π, in all of which $c_2, \dots, c_{k+1}$ are embedded above t. Consider the embedding of c_1. If c_1 is embedded infinitely often in some $\pm\mathbb{N}$-slice, then there are infinitely many embeddings of γ in $\pi|_{A_2^*}$. Since γ has $\mathcal{ID}$-pattern $\rho(A_1^*) = \rho(A_2^*)$, by Lemma 4.3, π must contain infinitely many ascents in some $-\mathbb{N}$-slice or descents in some $\mathbb{N}$-slice, contradicting $\mathrm{Sub}(\tau) = \mathrm{Sub}(\pi)$. Otherwise, c_1 is embedded infinitely often in some C_r-slice π_i. We must have $i > 1$, and so $c_2 \dots c_{k+1}$ must be embedded (infinitely often) in $k-1$ $\pm\mathbb{N}$-slices. Once again, Lemma 4.3 implies the existence of infinitely many inappropriate ascents/descents in π, contrary to the structure of τ.

Case 3: $B_1 = C_r$, $D_1 = C_u$ $(r < u)$. Let t be greater than all entries of $\tau|_{A_1 \setminus A_1^*}$ and the smallest entry of τ_2. Consider the permutation $\sigma = \sigma_1 v w \sigma_2 \sigma_3$, where $\sigma_1 v \cong \pi_1$, $w\sigma_2$ is a descent of length $t-u+1$, $vw\sigma_3 \cong \gamma$ and we have $\sigma_1 v < \sigma_2 < w\sigma_3$. Now $\sigma \in \mathrm{Sub}(\pi)$ since $\sigma_1 v$ may be embedded as π_1, the descent $w\sigma_2$ may be appropriately embedded in π_2, while the $k-1$ entries of σ_3 may be embedded point-by-point in the $k-1$ remaining $\pm\mathbb{N}$-slices. By assumption, $\sigma \in \mathrm{Sub}(\tau)$. For an arbitrary embedding of σ in τ, consider the induced embedding of γ; by construction, the points $c_2 \dots c_{k+1}$ are embedded above t and hence lie entirely in $\pm\mathbb{N}$-slices. Moreover, since $u > r$, c_1 must be embedded in τ_i where $i \geq 2$. If c_1 is embedded in τ_2 then we actually have an embedding of γ in $\tau|_{A_1^*}$, a contradiction. If c_1 is embedded in τ_i, $i > 2$, then it can equally be embedded as the smallest point in τ_2, and again we obtain a contradiction.

Case 4: $B_1 = C_u$, $D_1 = C_r$ $(r < u)$. Let t be the maximum value of π attained on $A_2 \setminus A_2^*$. For $n \geq t$, consider the permutation $\sigma^{(n)} = \sigma_1 v w \sigma_2 \sigma_3$, where $\sigma_1 v = 1 \ldots u$, $w\sigma_2$ is a descent of length $n - u + 1$, $vw\sigma_3 \cong \gamma$, and $\sigma_2 < w\sigma_3$. Arguing as in Case 3, $\sigma^{(n)}$ has an embedding in τ. By assumption, $\sigma^{(n)}$ also has an embedding in π. Consider the induced embedding of γ in π. All entries $c_2, \ldots, c_{k+1}$ of γ are embedded above t and hence lie entirely in $\pi|_{A_2^*}$. Letting $n \to \infty$ yields infinitely many embeddings of γ in π, in all of which $c_2, \ldots, c_{k+1}$ are embedded above t. Consider the embedding of c_1. If c_1 is embedded infinitely often in some $\pm\mathbb{N}$-slice, then there are infinitely many embeddings of γ in $\pi|_{A_2^*}$. Since γ has $\mathcal{ID}$-pattern $\rho(A_2^*)$, π contains infinitely many ascents in some $-\mathbb{N}$-slice or descents in some $\mathbb{N}$-slice, contradicting $\mathrm{Sub}(\pi) = \mathrm{Sub}(\tau)$. Otherwise, c_1 is embedded infinitely often in some C_r-slice π_i. Now, $i > 2$ since $u > r$ and $D_2 = -\mathbb{N}$, hence the k points $c_2 \ldots c_{k+1}$ are embedded (infinitely often) in the remaining $k - 1$ $\pm\mathbb{N}$-slices. Once again, Lemma 4.3 implies the existence of infinitely many inappropriate ascents/descents in π, a contradiction.

This establishes the result when $j = 1$. When $j > 1$ we observe that, for any C_r in S_1, its only possible predecessor is $\mathbb{N}$ and its only possible successor is $-\mathbb{N}$. Hence $B_{j-1} = D_{j-1} = \mathbb{N}$, and we may partition $(B_1, \ldots, B_{j-1}) = (D_1, \ldots, D_{j-1})$ into blocks of the form $C_r, -\mathbb{N}, \ldots, -\mathbb{N}, \mathbb{N}$ (where, if k denotes the number of $-\mathbb{N}$'s in the block, we have $r, k \geq 0$ with $r = 0$ when $k = 0$). It is clear that repeated application of Lemma 4.6 establishes the result. $\square$

Since the $\mathcal{ID}$-incompatibility of two sets implies the independence of their corresponding supernatural classes, the following theorem is immediate.

Theorem 4.8. *Let A_1, A_2 be linearly ordered sets of Type 2. If $A_1 \ncong A_2$, then $\mathcal{T}(A_1, \mathbb{N})$ and $\mathcal{T}(A_2, \mathbb{N})$ are independent, in the sense that neither is contained in the other.*

5. Contiguousness and subrepresentations amongst sets of supernatural classes

In this section, we consider the situation where some class X is representable both as a class of type $(A, \mathbb{N})$ and as a class of type $(B, \mathbb{N})$. When representations exist for X as both an $(A, \mathbb{N})$ class and a $(B, \mathbb{N})$ class where B is in some sense an extension of A, we ask to what extent the larger linearly ordered set offers a genuinely different representation from the smaller. We investigate the question of contiguousness, asking (when the question is meaningful) whether $X \in \mathcal{T}(A, \mathbb{N}) \cap \mathcal{T}(B, \mathbb{N})$ implies that $X \in \mathcal{T}(C, \mathbb{N})$ for all linearly ordered sets C 'between' A and B.

We begin by considering the intersection of $\mathcal{T}(\mathbb{N}, \mathbb{N})$ with other sets of supernatural classes. It is easily shown (for example, using Proposition 2.1 and the analogous result for $\mathcal{A}(12)$) that $\mathcal{T}(\mathbb{N}, \mathbb{N})$ and $\mathcal{T}(-\mathbb{N}, \mathbb{N})$ are independent sets. We now characterize their intersection. The role that, for natural classes, is played by the operation $\oplus$ and the notion of sum-completeness is played for $(-\mathbb{N}, \mathbb{N})$ classes by an operation $\ominus$ and minus-completeness, which are defined as follows. For two permutations α, β, where $|\beta| = m$, $\alpha \ominus \beta$ is the permutation $\gamma\delta$ where $\alpha \cong \gamma$, $\beta \cong \delta$, and γ is a rearrangement of $m+1, m+2, \ldots$. For example, $132 \ominus 213 = 465213$. A set X is *minus-complete* if $\alpha, \beta \in X$ implies $\alpha \ominus \beta \in X$. For every minus-complete class $X = \{\sigma_1, \sigma_2, \sigma_3, \ldots\}$, we have $X = \mathrm{Sub}(\cdots \ominus \sigma_3 \ominus \sigma_2 \ominus \sigma_1) \in \mathcal{T}(-\mathbb{N}, \mathbb{N})$.

Proposition 5.1. *The intersection $\mathcal{T}(\mathbb{N},\mathbb{N}) \cap \mathcal{T}(-\mathbb{N},\mathbb{N})$ is precisely the set of pattern classes which are both sum-complete and minus-complete.*

Proof. Let $X \in \mathcal{T}(\mathbb{N},\mathbb{N}) \cap \mathcal{T}(-\mathbb{N},\mathbb{N})$, say $X = \mathrm{Sub}(\pi : \mathbb{N} \to \mathbb{N}) = \mathrm{Sub}(\tau : -\mathbb{N} \to \mathbb{N})$, and let $\sigma, \rho \in X$. We will show that $\sigma \oplus \rho \in X$; the proof that $\sigma \ominus \rho \in X$ is analogous. Consider some embedding of σ in π; let m be the maximum value attained in this embedding, and let t be such that $\pi(i) > m$ for all $i > t$. Then consider the permutation $t(t-1)\dots 1 \ominus \rho$. This clearly has an embedding in τ and hence in π; by construction, the embedding of ρ in π must lie above and to the right of that of σ. This yields an embedding of $\sigma \oplus \rho$ in π. □

The following result, whilst generally known, is worth mentioning in this context.

Proposition 5.2. *Let α be any permutation. Then $\mathcal{A}(\alpha)$ is either of type $(\mathbb{N},\mathbb{N})$ or of type $(-\mathbb{N},\mathbb{N})$*

Proof. If α is not sum-indecomposable, then it must be minus-indecomposable. Hence its avoidance class is either sum-complete or minus-complete, and so either natural or of type $(-\mathbb{N},\mathbb{N})$. □

For what follows, we require some results about the structure of natural classes. The paper [3] contains two results about the structure of natural classes, the second a partial converse of the first, which we summarize in the following theorem.

Theorem 5.3.
- *Let γ be any (finite) permutation and S be any sum-complete closed set. Then $\mathrm{Sub}(\gamma) \oplus S$ is a natural class.*
- *Let X be a finitely-based natural class. Then either*
  - (i) *$X = \mathrm{Sub}(\gamma) \oplus S$ where S is sum-complete and determined uniquely, or*
  - (ii) *$X = \mathrm{Sub}(\pi : \mathbb{N} \to \mathbb{N})$ where π is unique and ultimately periodic.*

Observe that this offers a characterization only in the case of finitely based natural classes. We introduce an alternative viewpoint.

Lemma 5.4. *Let $X = \mathrm{Sub}(\pi : \mathbb{N} \to \mathbb{N})$, and suppose that every permutation in X has infinitely many embeddings in π. Then for every $\alpha = a_1 \dots a_n \in X$ and every $N \in \mathbb{N}$, there exists an embedding $\pi(i_1)\dots\pi(i_n)$ of α in π such that $i_1 > N$, i.e. there exists an embedding of α starting to the right of N.*

Proof. We begin by establishing that the result holds when α is indecomposable. Suppose not. Then every embedding $\pi(i_1)\dots\pi(i_n)$ of α in π has $i_1 \leq N_1$, for some $N_1 \in \mathbb{N}$.

Consider the sets

$$S_k = \{i_k : \exists \text{ an embedding } \pi(i_1)\dots\pi(i_k)\dots\pi(i_n) \text{ of } \alpha \text{ in } \pi\ \} \quad (1 \leq k \leq n).$$

By our supposition, S_1 is finite. Since α has infinitely many embeddings in π, there must exist some k $(1 < k \leq n)$ such that S_k is infinite. We take the smallest such k. Then for all $1 \leq j < k$, we let $N_j = \max(S_j)$. Clearly $N_1 < N_2 < \dots < N_{k-1}$.

Now let $M = \max_{1 \leq p \leq N_{k-1}} \pi(p)$; we may find some $R \in \mathbb{N}$ such that $\pi(r) > M$ for all $r \geq R$. Since S_k is infinite, we can find an embedding $\pi(i_1)\dots\pi(i_k)\dots\pi(i_n)$ of α in π such that $i_k > R$. Then $\pi(i_1)\dots\pi(i_{k-1}) < \pi(i_k)\dots\pi(i_n)$, since all points in the first sequence lie at or below M by definition of M, while all points in the second set lie above M by construction. But this is a contradiction, since α is indecomposable.

Now consider the case when α is decomposable. Let $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_r$ be its expression as a sum of indecomposable elements of X. Given any embedding $\pi(e_1)\ldots\pi(e_{l_1})$ of α_1 in π (where $l_1 = |\alpha_1|$), let M_1 be the maximum value of π on the interval $[1, e_{l_1}]$. Let $R_1 \in \mathbb{N}$ be any number greater than e_{l_1} with the property that $\pi(r) > M_1$ for all $r \geq R_1$. Then we can find an embedding of α_2 to the right of R_1, thus yielding an embedding of $\alpha_1 \oplus \alpha_2$. Repeating the process, we may obtain an embedding in π of $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_r$. Since our original choice of embedding of α_1 was arbitrary, and (by the first part) there exists an embedding of α_1 which starts to the right of any $N \in \mathbb{N}$, the desired result follows. □

The following result characterizes sum-complete classes.

Proposition 5.5. *Let $X = \mathrm{Sub}(\pi : \mathbb{N} \to \mathbb{N})$. Then X is sum-complete if and only if every permutation of X has infinitely many embeddings in π.*

Proof. ($\Rightarrow$) Let $\sigma \in X$. Then, for any $k \in \mathbb{N}$, the permutation $\sigma_k = \sigma \oplus \cdots \oplus \sigma$ (k summands) is an element of X by the sum-completeness of X. As an element of X, σ_k has an embedding in π, and thus σ has infinitely many embeddings in π.
($\Leftarrow$) Let $\alpha, \beta \in X$. Then we can choose an embedding $\pi(i_1)\ldots\pi(i_n)$ of $\alpha = a_1 \ldots a_n$ in π. Let M be the maximum value attained by π on $[1, i_n]$, and let R be any natural number (greater than i_n) such that $\pi(r) > M$ for all $r \geq R$. Then, by the previous lemma, we can find an embedding of β which starts to the right of R. Hence we have an embedding in π of $\alpha \oplus \beta$, i.e. $\alpha \oplus \beta \in X$ and so X is sum-complete. □

Definition 5.6. Let X be a pattern class. We say that a permutation $\alpha \in X$ is *initial* if the only way that α can be embedded in any other $\gamma \in X$ is as an initial segment of γ. In other words, $\alpha = a_1 \ldots a_m \in X$ is initial if $\gamma = c_1 \ldots c_n \in X$ and $\alpha \cong c_{i_1} \ldots c_{i_m}$ imply $i_j = j$ for all $j = 1, \ldots, m$.

Definition 5.7. Let A be a linearly ordered set. A subset A' of A is an *initial segment of* A if, for each $a \in A'$, $b \in A$ and $b < a$ implies $b \in A'$. For a bijection $\pi : A \to B$ (where B is any set), an *initial segment of* π is a restriction of π to an initial segment of A.

Remark 5.8. If X possesses initial permutations and can be represented as $\mathrm{Sub}(\pi : A \to B)$, then every initial permutation must be embedded as an initial segment of π and have no other embeddings in π. In particular, if X has an initial permutation of length n, then A has an initial segment of size n. Conversely, an initial segment of π with a unique embedding in π defines an initial permutation of X. In the case when $X = \mathrm{Sub}(\pi : \mathbb{N} \to \mathbb{N})$ we observe that, for any $\gamma \in X$ with a unique embedding in π, the shortest initial segment of π containing this embedding of γ has, itself, a unique embedding in π, and therefore γ is involved in an initial permutation of X.

Proposition 5.9. *Let X be a natural class.*

(i) *If X contains no initial permutations, then X is sum-complete.*
(ii) *If X contains a longest initial permutation γ, then γ has a unique embedding in any $\pi : \mathbb{N} \to \mathbb{N}$ such that $X = \mathrm{Sub}(\pi)$, and $X = \mathrm{Sub}(\gamma) \oplus Y$ where Y is sum-complete.*
(iii) *If X contains infinitely many initial permutations, then there exists a unique $\pi : \mathbb{N} \to \mathbb{N}$ such that $X = \mathrm{Sub}(\pi)$.*

Proof. Let $X = \mathrm{Sub}(\pi : \mathbb{N} \to \mathbb{N})$. In case (i), every permutation in X has infinitely many embeddings in π. For, if we could find $\alpha \in X$ with only finitely many embeddings, then by taking the union of all these embeddings we would obtain a permutation with a unique embedding, a contradiction. The assertion now follows by Proposition 5.5.

In case (ii), by Remark 5.8, γ has a unique embedding, necessarily as an initial segment, in π. Let π_γ be this initial segment, and let π' denote the rest of π. Since, for each $\delta \in X$ with only finitely many embeddings in π, the union of these embeddings must be involved in π_γ, the subclass $X' = \mathrm{Sub}(\pi')$ of X contains only $\delta \in X$ with infinitely many embeddings in π (specifically π') and hence is sum-complete. We now show: all points of π' lie above all points of π_γ, i.e. $X = \mathrm{Sub}(\gamma) \oplus X'$. Suppose that some points of π' lie below the maximum value r of π_γ. There can be only finitely many such points, say $\pi(p_1), \ldots, \pi(p_k)$ where $k < r$ and $p_1 < \cdots < p_k$. Consider the initial segment of π, up to and including $\pi(p_k)$. The corresponding permutation has a unique embedding in π, contradicting the definition of γ.

In case (iii), by Remark 5.8, every initial permutation of X is isomorphic to an initial segment of π. Since there are infinitely many initial permutations, π is completely determined by the corresponding initial segments, and hence is unique. □

Using the machinery just developed, we consider the situation when a natural class X is a member of $\mathcal{T}(A, \mathbb{N})$ for an arbitrary linearly ordered set A, i.e. $X \in \mathcal{T}(\mathbb{N}, \mathbb{N}) \cap \mathcal{T}(A, \mathbb{N})$.

For any supernatural class X, a *representation* of X is a bijection $\pi : A \to \mathbb{N}$ such that $X = \mathrm{Sub}(\pi)$. For a subclass Y of X and a linearly ordered set B, we say that π contains a B*-subrepresentation* of Y (or a subrepresentation of Y of type B) if there exists some subset B' of A such that $B' \cong B$ and $Y = \mathrm{Sub}(\pi|_{B'})$. Clearly, Y must be either finite (if B is finite) or a supernatural class of type $(B, \mathbb{N})$.

Proposition 5.10. *Let X be a sum-complete natural class. If $X = \mathrm{Sub}(\tau : A \to \mathbb{N})$ for some Type 2 linearly ordered set A with $S(A) = (B_1, \ldots, B_k)$ then, for some $B_i \in \{\mathbb{N}, -\mathbb{N}\}$ $(1 \le i \le k)$, τ contains a B_i-subrepresentation of X of the form $X = \mathrm{Sub}(\tau|_{B_i'})$.*

Proof. Suppose not. Then for each $i = 1, \ldots, k$, there exists $\alpha_i \in X$ such that $\alpha_i \not\preceq \tau_i$. Consider $\alpha_1 \oplus \cdots \oplus \alpha_k \in X$; it has an embedding in τ. Since $\alpha_i \not\preceq \tau_i$, the final point of the induced embedding of α_i $(i = 1, \ldots, k-1)$ must lie in τ_j where $j > i$. In particular, the final point of the embedding of α_{k-1} occurs in τ_k; but then the induced embedding of α_k must lie entirely in τ_k, a contradiction. □

Proposition 5.11. *Let X be a natural class containing a longest initial permutation γ. Suppose that X is of type $(A, \mathbb{N})$ for some Type 2 linearly ordered set A with $S(A) = (B_1, \ldots, B_k)$. Then*

(i) *$B_1 = \mathbb{N}$ or C_r with $r \ge |\gamma|$;*
(ii) *γ is an initial segment of τ_1;*
(iii) *$Y \subseteq \mathrm{Sub}(\tau_r)$ for some $1 \le r \le k$.*

Proof. By Remark 5.8, γ is an initial segment of τ, and hence of τ_1, proving parts (i) and (ii). We now show that $Y \subseteq \mathrm{Sub}(\tau_r)$ for some r. Suppose not. Then for each $i = 1, \ldots, k$, we can find $\alpha_i \in Y$ such that $\alpha_i \not\preceq \tau_i$. Consider $\gamma \oplus \alpha_1 \oplus \cdots \oplus \alpha_k \in X$;

this must have an embedding in τ. For each α_i ($i = 1, \dots, k-1$), consider its induced embedding; the last point must lie in τ_j where $j > i$. But this forces α_k to be embedded entirely in τ_k, a contradiction. □

Proposition 5.12. *Let X be a natural class containing infinitely many initial permutations. Then $X \notin \mathcal{T}(A, \mathbb{N})$ for any linearly ordered set $A \ncong \mathbb{N}$.*

Proof. Let $X = \mathrm{Sub}(\pi : \mathbb{N} \to \mathbb{N})$ and suppose $X = \mathrm{Sub}(\tau : A \to \mathbb{N})$. Since $X \neq S$, by Theorem 4.2 A must be of Type 2. Say $S(A) = (B_1, \dots, B_k)$; we will show that $B_1 = \mathbb{N}$ and $k = 1$.

Since $X = \mathrm{Sub}(\tau)$ has infinitely many initial permutations, by Remark 5.8, A has initial segments of any size. Hence $B_1 = \mathbb{N}$. Suppose now that $k > 1$, and let $p \in A \setminus B_1'$ be arbitrary. Let Γ be the (infinite) set of all initial permutations in X. Then, as before, every $\gamma \in \Gamma$ is an initial segment of both π and τ_1, and has no other embeddings in either. Choose $\gamma \in \Gamma$ such that $\tau(p)$ is smaller than the largest entry m of the embedding of γ in τ_1. Consider the subpermutation of τ obtained by taking all points of τ not greater than m; call this permutation α. Clearly $\alpha \in X$ with $|\alpha| = m$ and $\gamma \preceq \alpha$. Considering an appropriate embedding of α in π, we see that there exists $\delta \in \Gamma$ such that the initial segment of π corresponding to δ encompasses this embedding of α. But then δ is an initial segment of τ, and so has a unique embedding in τ_1. However, the induced embedding of α in τ_1 together with $\tau(p)$ gives us $m+1$ points in τ smaller than m, a contradiction. □

The next theorem follows upon combining the preceding results with Proposition 5.1.

Theorem 5.13. *Let X be a natural class which is also of type $(A, \mathbb{N})$ for some linearly ordered set $A \ncong \mathbb{N}$. Then X has the form $\mathrm{Sub}(\gamma) \oplus Y$ where Y is sum-complete, γ is either empty or the longest initial permutation of X, and*

- *if Y is not minus-complete, then every representation of X as an $(A, \mathbb{N})$ class contains a natural subrepresentation of X;*
- *if Y is minus-complete, then any representation of X as an $(A, \mathbb{N})$ class contains a subrepresentation of X which is either natural or of type $C_r \oplus -\mathbb{N}$ where $r = |\gamma|$.*

Our next subrepresentation result concerns supernatural classes where the domain "ends" with a copy of C_r.

Proposition 5.14. *Let A be a linearly ordered set with no maximal element. Suppose $X \in \mathcal{T}(A \oplus C_r, \mathbb{N}) \cap \mathcal{T}(A \oplus C_s, \mathbb{N})$ for some $0 \leq r < s \in \mathbb{N}$, say $X = \mathrm{Sub}(\pi : A \oplus C_r \to \mathbb{N}) = \mathrm{Sub}(\tau : A \oplus C_s \to \mathbb{N})$. Then*

(i) $\mathrm{Sub}(\pi|_A) = \mathrm{Sub}(\tau|_A)$;
(ii) *τ contains an $A \oplus C_r$ subrepresentation of X.*

Proof. (i) Let $X' = \mathrm{Sub}(\pi|_A)$ and $X'' = \mathrm{Sub}(\tau|_A)$. To see that $X' \subseteq X''$, let $\sigma \in X'$. Define the permutation σ_1 by $\sigma \oplus 12 \dots s$. Since A has no maximal element, $\sigma_1 \in X'$, and so σ_1 has an embedding in $\mathrm{Sub}(\tau)$. The induced embedding of σ must lie to the left of C_s, i.e. $\sigma \in \mathrm{Sub}(\tau|_A) = X''$. An entirely analogous argument shows that $X'' \subseteq X'$.
(ii) It suffices to prove that there are r distinguished points in C_s such that, for every $\sigma \in X \setminus X'$, there is an embedding of σ in τ which uses no other points of C_s. Suppose the assertion does not hold. Then for each r-element subset S_i

($1 \le i \le b = \binom{s}{r}$) of C_s, we may find a permutation $\sigma_i \in X \setminus X'$ for which the following property holds: for every embedding of σ_i in τ which uses at most r points from C_s, these r points do not come solely from S_i, i.e. $\sigma_i \not\preceq \tau|_{A\cup S_i}$. Take the union $\sigma = \sigma_1 \cup \cdots \cup \sigma_b = s_1 \dots s_n$.

We claim that σ has an embedding in τ which uses at least one and at most r points from C_s. Since $\sigma \in X \setminus X'$, all embeddings of σ in π use some of the points in C_r. Consider such an embedding of σ in π, which uses precisely k points from C_r. Construct the permutation $\sigma' = s_1 \dots s_{n-k}(n+1)\dots(n+s-k)s_{n-k+1}\dots s_n$; this has an embedding in π which may be obtained by inserting an increasing sequence of length $s-k$ into the embedding of σ. (Note that finding such an increasing sequence of length $s-k$ is always possible because A has no maximal element.) Hence $\sigma' \in \mathrm{Sub}(\tau)$ also, by assumption. Consider its embedding in τ; the induced embedding of σ can use at most k points from C_s.

Now take a set S_i of r points of C_s, which contains all points from C_s used in our embedding of σ. Then $\sigma_i \preceq \sigma \preceq \tau|_{A\cup S_i}$, a contradiction. □

As a corollary, we obtain the following contiguity result.

Corollary 5.15. *Let A be any linearly ordered set with no maximal element. If $X \in \mathcal{T}(A \oplus C_r, \mathbb{N}) \cap \mathcal{T}(A \oplus C_s, \mathbb{N})$ for some $0 \le r < s \in \mathbb{N}$, then $X \in \mathcal{T}(A \oplus C_t, \mathbb{N})$ for all $r \le t \le s$.*

Proof. Suppose $X = \mathrm{Sub}(\tau : A \oplus C_s \to \mathbb{N})$. Then by Proposition 5.14, τ contains an $A \oplus C_r$ subrepresentation of X. Supplementing this representation with $t-r$ further (redundant) points of C_s yields an $A \oplus C_t$ subrepresentation of X, as required. □

In particular, if X is a natural class and of type $(\mathbb{N} \oplus C_r, \mathbb{N})$, then X is of type $(\mathbb{N} \oplus C_t, \mathbb{N})$ for all $0 \le t \le r$. However, while r may be arbitrarily large, we cannot replace C_r with a copy of $\mathbb{N}$, as the next example shows.

Example 5.16. Let $X = \mathcal{A}(3241)$. Then $X \in \mathcal{T}(\mathbb{N}, \mathbb{N}) \cap \mathcal{T}(2\mathbb{N}, \mathbb{N})$ but $X \notin \mathcal{T}(\mathbb{N} \oplus C_r, \mathbb{N})$ for any $r \in \mathbb{N}$.

Since 3241 is sum-indecomposable, X is a natural class. In Section 2, we showed that $\mathrm{Sub}(3241) \oplus \mathcal{A}(3241) \in \mathcal{T}(2\mathbb{N}, \mathbb{N})$, and the representation given for that class clearly contains a $(2\mathbb{N}, \mathbb{N})$ subrepresentation of $\mathcal{A}(3241)$. Now suppose that $X = \mathrm{Sub}(\tau : \mathbb{N} \oplus C_r \to \mathbb{N})$. Consider $\tau(\omega + r) = k$, say. By Proposition 5.14, $X = \mathrm{Sub}(\tau|_{\mathbb{N}})$ and, since X is sum-complete, there are infinitely many embeddings of 213 in $\tau|_{\mathbb{N}}$. Choose such an embedding which lies above k. Then this embedding together with $\tau(\omega+r)$ is an embedding of $213 \ominus 1 = 3241$ in τ, a contradiction.

As a consequence of our earlier work on $\mathcal{T}(\mathbb{N}, \mathbb{N}) \cap \mathcal{T}(A, \mathbb{N})$, we obtain the following contiguity result for natural classes which are also of type $(k\mathbb{N}, \mathbb{N})$.

Theorem 5.17. *If $X \in \mathcal{T}(\mathbb{N}, \mathbb{N}) \cap \mathcal{T}(k\mathbb{N}, \mathbb{N})$, then $X \in \mathcal{T}(l\mathbb{N}, \mathbb{N})$ for all l with $1 \le l \le k$.*

Proof. By Theorem 5.13, $X \in \mathcal{T}(\mathbb{N}, \mathbb{N}) \cap \mathcal{T}(k\mathbb{N}, \mathbb{N})$ must be of the form $X = \mathrm{Sub}(\gamma) \oplus Y$ where Y is sum-complete and γ is either empty or the longest initial permutation in X. Let $X = \mathrm{Sub}(\tau : k\mathbb{N} \to \mathbb{N})$ be any representation of X as a $(k\mathbb{N}, \mathbb{N})$ class.

If γ is empty (i.e. if X is sum-complete) then, by Proposition 5.10, there is some $1 \le r \le k$ such that $X = \mathrm{Sub}(\tau_r)$. Taking the union of l of $\{\tau_1, \dots, \tau_k\}$, being sure to include τ_r, yields a representation of X as an $(l\mathbb{N},\mathbb{N})$ class as required. Otherwise, γ is an initial segment τ_γ of τ_1 and all the remaining entries of τ are greater than the entries of τ_γ. Moreover, by Proposition 5.11 there is some $1 \le r \le k$ such that every $Y \subseteq \mathrm{Sub}(\tau_r)$. Taking τ_γ and τ_r together produces an $(\mathbb{N},\mathbb{N})$ subrepresentation of X in τ; adding a further $l-1$ slices τ_i yields an $(l\mathbb{N},\mathbb{N})$ representation of X. $\square$

We have not been able to resolve the contiguity question for all the types $\mathcal{T}(k\mathbb{N},\mathbb{N})$ $(k \in \mathbb{N})$; we state it as an open problem for further investigation:

Open Problem 5.18. If $X \in \mathcal{T}(k\mathbb{N},\mathbb{N}) \cap \mathcal{T}(l\mathbb{N},\mathbb{N})$ for some $1 \le k < l$, is it true that $X \in \mathcal{T}(m\mathbb{N},\mathbb{N})$ for every m with $k \le m \le l$?

6. Concluding remarks

Atomic classes are not only conceptually fundamental in the study of pattern classes but, as witnessed by the results of this paper, are particularly amenable to structural investigation via the bijection paradigm. While many of the most intensively-studied avoidance classes (see [7], [1], [12]) are supernatural, it is clearly desirable to extend the theory from supernatural classes to general atomic classes by allowing the codomain of the bijection to be any (countable) linearly ordered set. Natural questions to ask in this context include: under what conditions on sets A, B, C and D are $\mathcal{T}(A,B)$ and $\mathcal{T}(C,D)$ independent? If L is a complete set of representatives of countable linearly ordered sets, is there some proper subset M of $L \times L$ such that $\cup_{(A,B)\in M}\mathcal{T}(A,B)$ contains all atomic sets, i.e. are there 'unnecessary' linearly ordered sets?

Another priority is to explore how a representation $X = \mathrm{Sub}(\pi : A \to B)$ can be exploited to analyse the standard properties (enumeration, basis, etc) of X. For instance, what properties of a bijection $\pi : \mathbb{N} \to \mathbb{N}$ are sufficient to enable us to compute the enumeration sequence for $X = \mathrm{Sub}(\pi)$ or to ensure that X has a finite basis?

In order that our approach can offer a comprehensive and usable structural framework for pattern classes, it is important to be able to decide whether a given class is atomic and, if not, determine its decomposition into atomic subclasses. For example, is there an algorithmic method of answering the atomic decision problem for a class specified by its basis? For an atomic class, how do we choose appropriate π, A and B? For a non-atomic class, is its decomposition into atomic subclasses finite or infinite? Under what circumstances is such a decomposition unique?

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