

Automatic presentations for semigroups

Alan J. Cain^a, Graham Oliver^b, Nik Ruškuc^a,
Richard M. Thomas^{b,*}

^a*School of Mathematics and Statistics, University of St Andrews, North Haugh,
St Andrews, Fife KY16 9SS, United Kingdom*

^b*Department of Computer Science, University of Leicester, University Road,
Leicester, LE1 7RH, United Kingdom*

Abstract

This paper applies the concept of FA-presentable structures to semigroups. We give a complete classification of the finitely generated FA-presentable cancellative semigroups: namely, a finitely generated cancellative semigroup is FA-presentable if and only if it is a subsemigroup of a virtually abelian group. We prove that all finitely generated commutative semigroups are FA-presentable. We give a complete list of FA-presentable one-relation semigroups and compare the classes of FA-presentable semigroups and automatic semigroups.

Key words: Automatic presentation, FA-presentable, cancellative semigroup, virtually abelian group

1991 MSC: 20F10, 20M35, 03D45

1 Introduction

Automatic presentations were introduced by Khoussainov and Nerode [23] to fulfill a need to extend finite model theory to infinite structures in such a way that interesting decision problems remain soluble; the present paper applies this concept to semigroups. We give definitions and examples, survey some previously published results, and establish some new ones, most importantly

* Corresponding author

Email addresses: `alanc@mcs.st-andrews.ac.uk` (Alan J. Cain),
`G.Oliver@mcs.le.ac.uk` (Graham Oliver), `nik@mcs.st-andrews.ac.uk` (Nik
Ruškuc), `rmt@mcs.le.ac.uk` (Richard M. Thomas).

a complete characterization of the finitely generated cancellative semigroups admitting automatic presentations.

Recall that a *structure* \mathcal{A} is a tuple (A, R_1, \dots, R_n) where:

- A is a set called the *domain* of \mathcal{A} ;
- for each i with $1 \leq i \leq n$, there is an integer $r_i \geq 1$ such that R_i is a subset of A^{r_i} ; r_i is called the *arity* of R_i .

An obvious instance of a structure is a relational database. However, there are many other natural examples; for instance, a semigroup is a structure (S, \circ) , where \circ has arity 3, and a group is a structure $(G, \circ, e, {}^{-1})$, where \circ has arity 3, e has arity 1, and ${}^{-1}$ has arity 2.

Informally, an automatic presentation for the structure (A, R_1, \dots, R_n) consists of a regular language of abstract representatives for the elements of A such that the relations R_i are all recognizable by synchronous finite automata; see Definition 2.3. A structure that admits an automatic presentation is said to be FA-presentable.

One important field of research has been the attempt to classify FA-presentable structures with specific classes of structures. As any finite structure is FA-presentable, we are really only interested in infinite structures here. In some cases this means that we have no real examples (for example, any FA-presentable integral domain is finite [24]). Essentially the only cases where we have a complete classification are those of:

- Boolean algebras [24];
- ordinals [9];
- finitely generated groups [28].

(For a number of partial results for FA-presentable groups, see [27]; for some necessary conditions for trees and linear orders to be FA-presentable, see [25].)

As far as groups are concerned, we also have the notion of an ‘automatic group’ in the sense of [13]. This has been generalized to semigroups (as in [7,29,21]). The considerable success of the theory of automatic groups was another motivation to have a general notion of FA-presentable structures; see also [30,32]. We note that a structure admitting an automatic presentation is often called an ‘automatic structure’; although we will avoid that term, the reader should be aware of the terminological clash with the different notion of an automatic structure for a group or semigroup in the sense of [13,7].

In this paper we will be particularly concerned with FA-presentable semigroups. When one moves from groups to semigroups, it appears that the problem becomes significantly more difficult. For example, if one has an undirected

graph Γ with vertices V and edges E , then we have a semigroup with elements $S = V \cup \{e, 0\}$, where we have the following products:

$$uv = \begin{cases} e & \text{if } u, v \in V \text{ and } \{u, v\} \in E; \\ 0 & \text{if } u, v \in V \text{ and } \{u, v\} \notin E; \end{cases}$$

$$ue = eu = u0 = 0u = 0 \text{ for } u \in V \cup \{e, 0\}.$$

Moreover, if we form the semigroup S from the graph Γ in this way, then S is FA-presentable if and only if Γ is FA-presentable. It is known [24] that the isomorphism problem for FA-presentable graphs is Σ_1^1 -complete (and hence undecidable); hence the isomorphism problem for FA-presentable semigroups is also Σ_1^1 -complete.

Given this, it seems sensible to restrict oneself to some naturally occurring classes of semigroups. Given the classification of the FA-presentable finitely generated groups referred to above, a natural class to consider is that of the FA-presentable finitely generated cancellative semigroups. In this paper we give a complete classification of these structures: a finitely generated cancellative semigroup is FA-presentable if and only if it embeds into a virtually abelian group (Theorem 10.1).

We remark that there are many examples of *non-cancellative* finitely generated FA-presentable semigroups. It is easy to see that adjoining a zero to a semigroup always preserves FA-presentability and destroys cancellativity. All finite semigroups, whether cancellative or not, are FA-presentable. Another example is the bicyclic monoid; see Example 3.2.

In Section 6, we prove that all finitely generated commutative semigroups are FA-presentable (Theorem 6.1). We also classify the FA-presentable one-relation semigroups (Proposition 9.1).

Finally, in Section 11, we consider the relationship between the classes of FA-presentable semigroups and automatic semigroups.

2 Automatic presentations

A semigroup is a set equipped with an associative binary operation \circ , although the operation symbol is often suppressed, so that $s \circ t$ is denoted st . We recall the idea of a “convolution mapping” which we will need throughout this paper:

Definition 2.1 *Let L be a regular language over a finite alphabet A . Define, for $n \in \mathbb{N}$,*

$$L^n = \{(w_1, \dots, w_n) : w_i \in L \text{ for } i = 1, \dots, n\}.$$

Let $\$$ be a new symbol not in A . The mapping $\text{conv} : (A^*)^n \rightarrow ((A \cup \{\$\})^n)^*$ is defined as follows. Suppose

$$w_1 = w_{1,1}w_{1,2} \cdots w_{1,m_1}, \quad w_2 = w_{2,1}w_{2,2} \cdots w_{2,m_2}, \quad \dots, \quad w_n = w_{n,1}w_{n,2} \cdots w_{n,m_n},$$

where $w_{i,j} \in A$. Then $\text{conv}(w_1, \dots, w_n)$ is defined to be

$$(w_{1,1}, w_{2,1}, \dots, w_{n,1})(w_{1,2}, w_{2,2}, \dots, w_{n,2}) \cdots (w_{1,m}, w_{2,m}, \dots, w_{n,m}),$$

where $m = \max\{m_i : i = 1, \dots, n\}$ and with $w_{i,j} = \$$ whenever $j > m_i$.

Observe that the map conv sends an n -tuple of words to a word of n -tuples. We then have:

Definition 2.2 Let A be a finite alphabet, and let $R \subseteq (A^*)^n$ be a relation on A^* . Then R is said to be regular if

$$\{\text{conv}(w_1, \dots, w_n) : (w_1, \dots, w_n) \in R\}$$

is a regular language over $(A \cup \{\$\})^n$.

Having done this, we can now define the concept of an ‘automatic presentation’ for a structure:

Definition 2.3 Let $\mathcal{S} = (S, R_1, \dots, R_n)$ be a relational structure. Let L be a regular language over a finite alphabet A , and let $\phi : L \rightarrow S$ be a surjective mapping. Then (L, ϕ) is an automatic presentation for \mathcal{S} if:

- (1) the relation $L_= = \{(w_1, w_2) \in L^2 : \phi(w_1) = \phi(w_2)\}$ is regular, and
- (2) for each relation R_i of arity r_i , the relation

$$L_{R_i} = \{(w_1, w_2, \dots, w_{r_i}) \in L^{r_i} : (\phi(w_1), \dots, \phi(w_{r_i})) \in R_i\}$$

is regular.

A structure with an automatic presentation is said to be FA-presentable.

As noted in Section 1, a semigroup can be viewed as a relational structure in which the binary operation \circ becomes a ternary relation. The following definition simply restates the preceding one in the special case where the structure is a semigroup:

Definition 2.4 Let S be a semigroup. Let L be a regular language over a finite alphabet A , and let $\phi : L \rightarrow S$ be a surjective mapping. Then (L, ϕ) is an automatic presentation for S if the relations

$$L_= = \{(w_1, w_2) \in L^2 : \phi(w_1) = \phi(w_2)\},$$

$$L_\circ = \{(w_1, w_2, w_3) \in L^3 : \phi(w_1)\phi(w_2) = \phi(w_3) \text{ in } S\}$$

are both regular.

3 Examples

In this section, we give some examples of FA-presentable semigroups. We first exhibit a well-known example:

Example 3.1 *The natural numbers under addition are FA-presentable: let $L = \{0, 1\}^*\{1\} \cup \{0\}$ and define $\phi : L \rightarrow \mathbb{N}$ by letting $\phi(w)$ be the natural number expressed by w in reverse binary notation. The equality relation $L_=$ is the diagonal relation $\{(w, w) : w \in L\}$, for every natural number has a unique representative of L . A finite automaton can recognize the relation $L_+ = \{(u, v, w) : \phi(u) + \phi(v) = \phi(w)\}$ because it can add u to v digit by digit and compare the result with w , storing the carry in its internal state. So (L, ϕ) is an automatic presentation for $(\mathbb{N}, +)$.*

Example 3.2 *The bicyclic monoid B , which is presented by $\langle b, c \mid bc = 1 \rangle$, is FA-presentable. Notice that every element of the bicyclic monoid has a normal form $c^i b^j$ and that*

$$c^i b^j \circ c^k b^l = \begin{cases} c^i b^{l+(j-k)} & \text{if } j \geq k \\ c^{i+(k-j)} b^l & \text{if } j < k \end{cases}.$$

Retain the language L and the mapping ϕ from the previous example. Let $K = \{\text{conv}(x, y) : x, y \in L\}$, where $\psi : K \rightarrow B$ is given by

$$\text{conv}(x, y) \mapsto c^{\phi(x)} b^{\phi(y)}.$$

Then (K, ψ) is an automatic presentation for B : the equality relation $K_=$ is the diagonal relation, and the multiplication relation K_\circ is easily seen to be automatic, since addition of natural numbers (in reverse binary notation) can be carried out by an automaton, as can subtraction and comparison.

4 Basic results

The following notions and proposition will be useful in what follows:

Definition 4.1 *Let (L, ϕ) be an automatic presentation for a structure. Then (L, ϕ) is a binary automatic presentation if the language L is over a two-letter alphabet; it is an injective automatic presentation if the mapping ϕ is injective (so that every element of the structure has exactly one representative in L).*

Proposition 4.2 ([23, Corollary 4.3] & [2, Lemma 3.3]) *Any structure that admits an automatic presentation admits an injective binary automatic presentation.*

An *interpretation* of one structure inside another is, loosely speaking, a copy of the former inside the latter. The following definition is restricted to an interpretation of one semigroup inside another.

Definition 4.3 *Let S and T be semigroups. Let $n \in \mathbb{N}$. An (n -dimensional) interpretation of T in S consists of the following:*

- *a first-order formula $\psi(x_1, \dots, x_n)$, called the domain formula, which specifies those n -tuples of elements of S used in the interpretation;*
- *a surjective map $f : \psi(S^n) \rightarrow T$, called the co-ordinate map (where $\psi(S^n)$ denotes the set of n -tuples of elements of S satisfying the formula ψ);*
- *a first-order formula $\theta_=(x_1, \dots, x_n; y_1, \dots, y_n)$ that is satisfied by*

$$(a_1, \dots, a_n; b_1, \dots, b_n)$$

in the semigroup S if and only if $f(a_1, \dots, a_n) = f(b_1, \dots, b_n)$ in the semigroup T ;

- *a first-order formula $\theta_\circ(x_1, \dots, x_n; y_1, \dots, y_n; z_1, \dots, z_n)$ that is satisfied by*

$$(a_1, \dots, a_n; b_1, \dots, b_n; c_1, \dots, c_n)$$

in the semigroup S if and only if $f(a_1, \dots, a_n)f(b_1, \dots, b_n) = f(c_1, \dots, c_n)$ in the semigroup T .

The following result, although here stated only for semigroups, is true for structures generally:

Proposition 4.4 ([2, Proposition 3.13]) *Let S and T be semigroups. If S has an automatic presentation and there is an interpretation of T in S , then T has an automatic presentation.*

The fact that a tuple of elements (a_1, \dots, a_n) of a structure S satisfies a first-order formula $\theta(x_1, \dots, x_n)$ is denoted $S \models \theta(a_1, \dots, a_n)$. We then have:

Proposition 4.5 ([23]) *Let \mathcal{S} be a structure with an automatic presentation. For every first-order formula $\theta(x_1, \dots, x_n)$ over the structure there is an automaton which accepts $\text{conv}(w_1, \dots, w_n)$ if and only if $S \models \theta(\phi(w_1), \dots, \phi(w_n))$. Moreover, there is an algorithm which effectively constructs such an automaton from such a formula.*

(Proposition 4.4 is actually a consequence of Proposition 4.5.)

As a consequence of Proposition 4.5, FA-presentable structures have decidable first-order theories. In the context of semigroups, this means that any first-order definable property or relation of semigroups is decidable. For example, Green’s relations and cancellativity are both decidable for FA-presentable semigroups. This contrasts the situation for automatic semigroups, where Green’s relation \mathcal{R} [22] and cancellativity [5] are undecidable.

5 Finitely generated FA-presentable groups

As mentioned in Section 1, a classification of the finitely generated groups with an automatic presentation was given in [28]. For convenience, we state the result here (along with some extra details from [28] that we will need later). Recall that a group G is said to be *virtually abelian* if it has an abelian subgroup A of finite index. If G is finitely generated, then the subgroup A is finitely generated as well. Using the fact that any finitely generated abelian group is the direct sum of finitely many cyclic groups, we may assume that A is of the form \mathbb{Z}^n for some $n \geq 0$.

Theorem 5.1 ([28]) *A finitely generated group admits an automatic presentation if and only if it is virtually abelian. In particular, a group G with a subgroup \mathbb{Z}^n of index ℓ admits an automatic presentation (L, ϕ) , where L is the language of words*

$$g_i \text{conv}(\varepsilon_1 z_1, \dots, \varepsilon_n z_n),$$

where $\varepsilon_i \in \{+, -\}$, z_i is a natural number in reverse binary notation, g_1, \dots, g_ℓ are representatives of the cosets of \mathbb{Z}^n in G , with $\phi : L \rightarrow G$ being defined in the natural way:

$$\phi(g_i \text{conv}(\varepsilon_1 z_1, \dots, \varepsilon_n z_n)) = g_i(\varepsilon_1 z_1, \dots, \varepsilon_n z_n).$$

6 Commutative semigroups

Commutative semigroups often have pleasant properties with regard to finite ‘descriptions’. For example, Rédei’s Theorem shows that all finitely generated commutative semigroups are finitely presented [31], and finitely generated commutative monoids are presented by finite confluent Noetherian rewriting systems [10]. The following result is thus perhaps unsurprising:

Theorem 6.1 *Every finitely generated commutative semigroup admits an automatic presentation.*

To prove this result, we will need the following lemma:

Lemma 6.2 ([2, Corollary 3.14]) *The class of FA-presentable structures is closed under forming quotients by first-order definable congruences, under forming finitary direct products, and under passing to first-order definable substructures. Moreover, in each case an automatic presentation is effectively constructable.*

We now proceed with the proof of Theorem 6.1:

PROOF. Finitely generated free commutative semigroups are isomorphic to $(\mathbb{N}, +)^n - \{(0, \dots, 0)\}$ for some n and are thus FA-presentable, since the class of FA-presentable structure is closed under direct products, the exclusion of a single element gives a first-order definable substructure.

Every commutative semigroup is a quotient of a free commutative semigroup and, by [33], the corresponding congruence is first-order definable; the result then follows from Lemma 6.2.

We observe that not all countable commutative semigroups are FA-presentable: for example, any monoid which contains (\mathbb{N}, \times) is not FA-presentable [24, Theorem 3.6].

7 Growth

In the proof of Theorem 5.1 above, given in [28], one essential ingredient was the notion of growth. Before defining the growth of a semigroup, we first establish notation for and state a basic property of lengths of the words representing the elements of the domain of the structure.

Definition 7.1 *Let S be a semigroup with an injective automatic presentation (L, ϕ) . For any $s \in S$, denote by $l(s)$ the length of the unique word in L representing s .*

Proposition 7.2 ([2, Proposition 5.1]) *Let S be a semigroup with an injective automatic presentation; then there is a constant $N \in \mathbb{N}$ such that, for all $s, t \in S$,*

$$l(st) \leq \max\{l(s), l(t)\} + N.$$

We now turn to the concept of *growth*:

Definition 7.3 *Let S be a semigroup generated by a finite set X . Define $\delta(s)$*

to be the length of the shortest product of elements of X that equals s , i.e.

$$\delta(s) = \min\{n \in \mathbb{N} : s = x_1 \cdots x_n \text{ for some } x_i \in X\}.$$

The growth function $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ of S is given by

$$\gamma(n) = |\{s \in S : \delta(s) \leq n\}|.$$

If the function γ is bounded above by a polynomial function (that is, if there exists a polynomial function β and some $N \in \mathbb{N}$ such that $\beta(n) > \gamma(n)$ for $n > N$), then S is said to have polynomial growth.

Note that whether a semigroup has polynomial growth or not is independent of the choice of finite generating set [17]. We now have the following result:

Theorem 7.4 *Any finitely generated subsemigroup of a semigroup admitting an automatic presentation has polynomial growth.*

Before embarking on a proof of this result, we pause to emphasize that polynomial growth is dependent on the structures in question being semigroups: general algebras admitting automatic presentations are only guaranteed to have at most exponential growth [23, Lemma 4.5].

PROOF. Let S be a semigroup, finitely generated by X , that admits an automatic presentation. By Proposition 4.2, assume without loss of generality that this automatic presentation is injective and binary. This proof follows that in [28], which dealt with groups. The main ingredient is provided by the following lemma:

Lemma 7.5 *Let $R = \max\{l(a) : a \in X\}$. There is a constant N such that, for all $m \in \mathbb{N}$,*

$$\max\{l(a_1 \cdots a_m) : a_i \in X\} \leq R + \lceil \log_2 m \rceil N. \quad (1)$$

PROOF. Let N be the constant of Proposition 7.2. We proceed by induction on m .

For $m = 1$, the inequality (1) holds, since

$$\max\{l(a_1) : a_1 \in X\} = R = R + \lceil \log_2 1 \rceil N.$$

Now assume that (1) is true for $1 \leq m \leq k$. The cases of k being odd or even must be considered separately:

(1) Suppose k is odd, with $k = 2r - 1$. Then, by Proposition 7.2:

$$\begin{aligned}
& \max\{l(a_1 \cdots a_{k+1}) : a_i \in X\} \\
&= \max\{l(a_1 \cdots a_{2r}) : a_i \in X\} \\
&\leq \max\{l(a_1 \cdots a_r), l(a_{r+1} \cdots a_{2r}) : a_i \in X\} + N \\
&\leq \max\{R + \lceil \log_2 r \rceil N, R + \lceil \log_2 r \rceil N\} + N \\
&= R + \lceil \log_2 r \rceil N + N \\
&= R + (\lceil \log_2 r + 1 \rceil)N \\
&= R + \lceil \log_2 2r \rceil N \\
&= R + \lceil \log_2(k + 1) \rceil N,
\end{aligned}$$

as required.

(2) Suppose k is even, say $k = 2r$. Then:

$$\begin{aligned}
& \max\{l(a_1 \cdots a_{k+1}) : a_i \in X\} \\
&= \max\{l(a_1 \cdots a_{2r+1}) : a_i \in X\} \\
&\leq \max\{l(a_1 \cdots a_r), l(a_{r+1} \cdots a_{2r+1}) : a_i \in X\} + N \\
&\leq \max\{R + \lceil \log_2 r \rceil N, R + \lceil \log_2(r + 1) \rceil N\} + N \\
&= R + \lceil \log_2(r + 1) \rceil N + N.
\end{aligned}$$

At this point, two subcases are required, depending on whether r is a power of two:

(a) Suppose that r is not a power of 2. Since the function $\lceil \log_2 y \rceil$ on the set $\{y \in \mathbb{N} : y > 0\}$ takes the same value on y and $y + 1$ except when y is a power of 2, $\lceil \log_2(r + 1) \rceil = \lceil \log_2 r \rceil$. Therefore, by the reasoning in part (1),

$$R + \lceil \log_2(r + 1) \rceil N + N = R + \lceil \log_2 r \rceil N + N = R + \lceil \log_2(k + 1) \rceil N,$$

as required.

(b) Suppose that $r = 2^x$, where $x \in \mathbb{N}$. Observe that

$$\lceil \log_2(k + 1) \rceil = \lceil \log_2(2r + 1) \rceil = \lceil \log_2(2^{x+1} + 1) \rceil = x + 2.$$

Consequently,

$$\begin{aligned}
& R + \lceil \log_2(r + 1) \rceil N + N \\
&= R + \lceil \log_2(r + 1) + 1 \rceil N \\
&= R + \lceil \log_2(r + 1) + \log_2 2 \rceil N \\
&= R + \lceil \log_2 2(r + 1) \rceil N \\
&= R + \lceil \log_2 2(2^x + 1) \rceil N \\
&= R + \lceil \log_2(2^{x+1} + 2) \rceil N \\
&= R + (x + 2)N \\
&= R + \lceil \log_2(k + 1) \rceil N,
\end{aligned}$$

as required.

We now return to the proof of Theorem 7.4. By Lemma 7.5, the number of possible words in L for elements of the form $x_1 \cdots x_m$, where $x_i \in X$, is no greater than

$$2^{(R+\lceil \log_2 m \rceil N)+1} = 2^{R+1}(2^{\lceil \log_2 m \rceil})^N \leq 2^{R+1}(2^{1+\log_2 m})^N = km^N,$$

where $k = 2^{R+1}2^N$ is a constant. So there are at most km^N elements $s \in S$ with $\delta(s) = m$. Consequently,

$$\gamma(n) = |\{s \in S : \delta(s) \leq n\}| \leq k \cdot 1^N + k \cdot 2^N + \dots + k \cdot n^N \leq kn^{N+1}.$$

So S has polynomial growth. This establishes Theorem 7.4.

8 Maximum Group Homomorphic Image

Given the classification of FA-presentable finitely generated groups (see Theorem 5.1 above), it makes sense to investigate (finitely generated) groups related to semigroups. The *maximum group homomorphic image* of a semigroup S , if it exists, is the largest group G such that there is a surjective homomorphism from S onto G , in the sense that there is a homomorphism from this group G onto any group H that is a homomorphic image of S . The congruence associated to this homomorphic image is called the *minimum group congruence*. (For further background information, see [20, Section 5.3].)

Definition 8.1 *Let S be a semigroup. A subset K of S is:*

- unitary if for all $s \in S$ and $k \in K$, we have $(sk \in K \vee ks \in K) \implies s \in K$;
- dense if for all $s \in S$ there exists $x, y \in S$ such that $sx \in K$ and $ys \in K$;
- reflexive if for all $a, b \in S$, we have $ab \in K \implies ba \in K$.

The subsemigroup generated by K is denoted $\langle K \rangle$.

Definition 8.2 *Let S be a semigroup, with E its set of idempotents. Then S is:*

- regular if for every $s \in S$ there exists $s' \in S$ such that $ss's = s$;
- π -regular if for every $s \in S$, there exists $n \in \mathbb{N}$ and $s' \in S$ such that $s^n s' s^n = s^n$;
- strongly π -inverse if it is π -regular and E is commutative;
- a unitary dense E -semigroup if E is a subsemigroup, and E is unitary and dense;

- a strongly $\langle E \rangle$ -unitary dense monoid if it is a monoid and $\langle E \rangle$ is reflexive, unitary and dense.

Using a variety of results from the literature, we obtain the following result:

Proposition 8.3 *If S is FA-presentable and either*

- a regular semigroup,
- a strongly π -inverse semigroup,
- a unitary dense E -semigroup, or
- a strongly $\langle E \rangle$ unitary dense monoid,

then the maximum group homomorphic image of S exists and is FA-presentable.

PROOF. For each of the given species of semigroup, the minimum group congruence exists and is first-order definable [14,35]. So the maximum group homomorphic image of any such semigroup will be FA-presentable by Lemma 6.2.

Corollary 8.4 *Let S be the free inverse monoid on the set A ; then, S has an automatic presentation if and only if $|A| = 1$.*

PROOF. The monoid S is regular and its maximum group homomorphic image is the free group on A . Thus if S is FA-presentable, then so is the free group with on A , whence $|A| = 1$ since otherwise it contains a free subsemigroup on two generators, which does not have polynomial growth, which would contradict Theorem 7.4.

Conversely, if $|A| = 1$, then free inverse monoid on A is isomorphic to the semigroup formed by the set

$$\{(r, s, t) \in \mathbb{Z}^3 : r \geq 0, s \geq 0, -s \leq t \leq r\}$$

under the operation

$$(r, s, t)(r', s', t') = (\max\{r, r' + t\}, \max\{s, s' - t\}, s + s');$$

see [20, p.219]. Since a finite automaton can add, subtract, and compare integers in reverse binary notation, it is clear that this semigroup is FA-presentable.

9 One-relation semigroups

In this section, we characterize those one-relation semigroup presentations that define FA-presentable semigroups.

Proposition 9.1 *A semigroup S with one defining relation has an automatic presentation if and only if either S is monogenic, or S is generated by two elements, say a and b , and the defining relation is one of:*

$$\begin{aligned} a = b^k; \quad ab = ba; \quad ab = b^k; \quad ba = b^k; \quad ab = aba; \\ ba = aba; \quad ab = bab^2; \quad ba = b^2ab; \quad a = bab; \quad a^2 = b^2. \end{aligned}$$

PROOF. Vazhenin [34] proved that these semigroups are precisely the one-relation semigroups with decidable first-order theory. The proof involves an interpretation of each of these semigroups in $(\mathbb{N}, +)^k$ for some $k \in \mathbb{N}$. The semigroup $(\mathbb{N}, +)^k$ is FA-presentable by Theorem 6.1; thus each of these semigroups is FA-presentable by Proposition 4.4.

10 Characterization of FA-presentable cancellative semigroups

The present section is dedicated to proving the following characterization theorem:

Theorem 10.1 *A finitely generated cancellative semigroup is FA-presentable if and only if it embeds into a virtually abelian group.*

Recall that a semigroup S has a group of left (respectively, right) quotients G if S embeds into G and every element of G is of the form $t^{-1}s$ (respectively, st^{-1}) for $s, t \in S$. If a semigroup S has a group of left (respectively, right) quotients, then this group is unique up to isomorphism. For further information on groups of left and right quotients, see [8, Section 1.10].

The following result, due to Grigorchuk, generalizes the result of Gromov [18] that a finitely generated group of polynomial growth is virtually nilpotent (i.e. it has a nilpotent subgroup of finite index):

Theorem 10.2 ([16]) *A finitely generated cancellative semigroup has polynomial growth if and only if it has a virtually nilpotent group of left quotients.*

We then have the following immediate consequence of Theorems 10.2 and 7.4:

Corollary 10.3 *Let S be a finitely generated cancellative semigroup that admits an automatic presentation. Then the group of left quotients of S exists and is virtually nilpotent.*

Note that the groups of left and right quotients of subsemigroups of virtually nilpotent groups coincide (see [26] or [4, Sections 5.2–5.3]). We now have:

Proposition 10.4 *Let S be a finitely generated cancellative semigroup that admits an automatic presentation. Then the [necessarily virtually nilpotent] group of left (and right) quotients of S admits an automatic presentation.*

PROOF. Let G be the group of left (and right) quotients of S . The strategy is to show that G has a 2-dimensional interpretation in S .

- The domain formula is tautological: $\phi(x_1, x_2) := x_1 = x_1$. Thus all pairs of elements of S are used.
- The co-ordinate map is $f(x_1, x_2) = x_1^{-1}x_2$. Since G is the group of left quotients of S , the mapping f is surjective as required.
- The formula $\theta_=_$ is given by

$$\theta_=(x_1, x_2; y_1, y_2) := (\exists a, b)(x_1a = x_2b \wedge y_1a = y_2b),$$

since

$$\begin{aligned} f(x_1, x_2) &= f(y_1, y_2) \\ \iff (\exists a, b)(f(x_1, x_2) = ab^{-1} \wedge f(y_1, y_2) = ab^{-1}) \\ \iff (\exists a, b)(x_1^{-1}x_2 = ab^{-1} \wedge y_1^{-1}y_2 = ab^{-1}) \\ \iff (\exists a, b)(x_1a = x_2b \wedge y_1a = y_2b). \end{aligned}$$

- The formula θ_0 is given by

$$\begin{aligned} \theta_0(x_1, x_2; y_1, y_2; z_1, z_2) &:= \\ &(\exists a, b, c, d)(cx_1a = dy_2b \wedge cx_2 = dy_1 \wedge z_2b = z_1a), \end{aligned}$$

since

$$\begin{aligned} f(x_1, x_2)f(y_1, y_2) &= f(z_1, z_2) \\ \iff (\exists a, b)(f(x_1, x_2)f(y_1, y_2) = ab^{-1} \wedge f(z_1, z_2) = ab^{-1}) \\ \iff (\exists a, b)(x_1^{-1}x_2y_1^{-1}y_2 = ab^{-1} \wedge z_1^{-1}z_2 = ab^{-1}) \\ \iff (\exists a, b, c, d)(c^{-1}d = x_2y_1^{-1} \wedge x_1^{-1}c^{-1}dy_2 = ab^{-1} \wedge z_1^{-1}z_2 = ab^{-1}) \\ \iff (\exists a, b, c, d)(cx_2 = dy_1 \wedge dy_2b = cx_1a \wedge z_2b = z_1a). \quad \square \end{aligned}$$

We are now in a position to prove one direction of Theorem 10.1:

Proposition 10.5 *A finitely generated cancellative semigroup admitting an automatic presentation embeds into a finitely generated virtually abelian group.*

PROOF. Let S be a finitely generated cancellative semigroup with an automatic presentation. By Proposition 10.4, its group of left quotients G has an automatic presentation. Since S is finitely generated, G is also. Theorem 5.1 then shows that G is virtually abelian.

The other direction is provided by:

Proposition 10.6 *Every finitely generated subsemigroup of a virtually abelian group admits an automatic presentation.*

PROOF. Let G be a virtually abelian group. Let \mathbb{Z}^n be a finite-index abelian subgroup of G . By replacing \mathbb{Z}^n by its core (the maximal normal subgroup of G contained in \mathbb{Z}^n) if necessary, we may assume that \mathbb{Z}^n is normal in G . Let k be the index of \mathbb{Z}^n in G . Let A be a finite alphabet representing a subset of G , and let S be the semigroup generated by this subset. Throughout this proof, denote by \bar{w} the element of S represented by the word w over an alphabet representing a generating set. This notational distinction is necessary to avoid confusion when there are several representatives for the same element.

Let $B = \{a \in A : \bar{a} \in \mathbb{Z}^n\}$ and let $C = A - B$. So B consists of all letters in A representing elements of the abelian subgroup \mathbb{Z}^n and C consists of letters representing elements of other cosets of \mathbb{Z}^n .

Introduce a new alphabet D representing the set

$$\{\bar{w} : w \in C^{\leq k}, \bar{w} \in \mathbb{Z}^n\},$$

where $C^{\leq k}$ denotes the set of words over C of length at most k . Notice that since the set $C^{\leq k}$ is finite, so is D . Furthermore, the semigroup S is generated by $\bar{B} \cup \bar{C} \cup \bar{D}$. We next observe the following lemma:

Lemma 10.7 *Every element of the semigroup S is represented by a word over $B \cup C \cup D$ that contains at most $k^2 - 1$ letters from C .*

PROOF. Let $s \in S$, and let $w \in (B \cup C \cup D)^+$ with $\bar{w} = s$. Then w is of the form

$$u_0 c_1 u_1 c_2 \cdots u_{m-1} c_m u_m, \tag{2}$$

where each u_i lies in $(B \cup D)^*$ and each c_i in C . The aim is to show that such a word w can be transformed into one that still represents $s \in S$ but contains at most $k^2 - 1$ letters from C

First stage. For any word w of the form (2) and for $i = 0, \dots, m-1$, let $\psi_w(i)$ be maximal such that $\overline{c_{i+1}u_{i+1} \cdots c_m u_m}$ and $\overline{c_{\psi_w(i)+1}u_{\psi_w(i)+1} \cdots c_m u_m}$ lie in the same coset of \mathbb{Z}^n in G . It is clear that $\psi_w(i)$ is always defined and is not less than i . Notice that since there are k distinct cosets of \mathbb{Z}^n in G , $\psi_w(i)$ can take at most k distinct values as i ranges from 0 to $m-1$. Furthermore, for each i , $\overline{c_{i+1}u_{i+1} \cdots c_{\psi_w(i)}u_{\psi_w(i)}}$ lies in \mathbb{Z}^n and so commutes with $\overline{u_i}$.

Define a mapping $\beta' : (B \cup C \cup D)^+ \rightarrow (B \cup C \cup D)^+$ as follows: for w of the form (2), $\beta'(w)$ is defined to be

$$u_0 c_1 u_1 c_2 \cdots c_i c_{i+1} u_{i+1} \cdots c_{\psi_w(i)} u_{\psi_w(i)} u_i c_{\psi_w(i)+1} \cdots u_{m-1} c_m u_m,$$

where i is minimal with $\psi_w(i) \neq i$, and $\beta'(w) = w$ if $\psi_w(i) = i$ for all i . By the remark at the end of the last paragraph, $\overline{w} = \overline{\beta'(w)}$.

The mapping $\beta : (B \cup C \cup D)^+ \rightarrow (B \cup C \cup D)^+$ is defined by $\beta(w) = (\beta')^p(w)$, where p is minimal with $(\beta')^p(w) = (\beta')^{p+1}(w)$. Again, $\overline{w} = \overline{\beta(w)}$.

So $\beta(w)$ is the word obtained from w by shifting each u_i rightwards to one of at most k distinct positions between the various letters c_j . Thus $\beta(w)$ has the form (2) with at most k of the words u_i being non-empty.

Second stage. Define a mapping $\gamma' : (B \cup C \cup D)^+ \rightarrow (B \cup C \cup D)^+$ as follows: if $w \in (B \cup C \cup D)^+$ has a subword $v \in C^{\leq k}$ with $\overline{v} \in \mathbb{Z}^n$, then choose the leftmost, shortest such subword and replace it with the letter of D representing the same element of S . (Such a letter exists by the definition of D .)

The mapping $\gamma : (B \cup C \cup D)^+ \rightarrow (B \cup C \cup D)^+$ is defined by $\gamma(w) = (\gamma')^p(w)$, where p is minimal with $(\gamma')^p(w) = (\gamma')^{p+1}(w)$. Since each application of γ' that results in a different word decreases the number of letters from C present, such a p must exist. Observe that $\overline{w} = \overline{\gamma(w)}$ and that $\gamma(w)$ cannot contain a subword of k letters from C , for such a string must contain a subword representing an element of \mathbb{Z}^n .

Third stage. The final mapping $\delta : (B \cup C \cup D)^+ \rightarrow (B \cup C \cup D)^+$ is given by $\delta(w) = (\gamma\beta)^p(w)$, where p is minimal with $(\gamma\beta)^p(w) = (\gamma\beta)^{p+1}(w)$. Observe that $\overline{w} = \overline{\delta(w)}$. Now, $\delta(w)$ is of the form (2) with at most k words u_i being nonempty and does not contain k consecutive letters from C . So separated by the k nonempty words u_i are strings of at most $k-1$ letters from C . So the total number of letters from C in $\delta(w)$ is at most $(k-1) \times (k+1) = k^2 - 1$.

We now return to the proof of Proposition 10.6. Choose a set of representatives g_1, \dots, g_k for the cosets of \mathbb{Z}^n in G . Suppose $B \cup D = \{b_1, \dots, b_q\}$.

For $c_1, \dots, c_m \in C$ with $0 \leq m \leq k^2 - 1$, define

$$P_{c_1 \dots c_m} = \{u_0 c_1 u_1 c_2 \dots u_{m-1} c_m u_m : u_i = b_1^{\alpha_{i,1}} \dots b_q^{\alpha_{i,q}}, \alpha_{i,j} \in \mathbb{N} \cup \{0\}\}.$$

By Lemma 10.7 and the fact that the elements $\overline{b_j}$ commute, every element of S is represented by an element in at least one of the sets $P_{c_1 \dots c_m}$. That is,

$$S = \bigcup_{\substack{c_1, \dots, c_m \in C \\ 0 \leq m \leq k^2 - 1}} \overline{P_{c_1 \dots c_m}}. \quad (3)$$

By Theorem 5.1, the virtually abelian group G has an automatic presentation (L, ϕ) , where L is the language of words

$$g_h \text{conv}(\varepsilon_1 z_1, \dots, \varepsilon_n z_n), \quad (4)$$

where $\varepsilon_i \in \{+, -\}$ and z_i is a natural number in reverse binary notation. (In L , the coset representative g_h functions simply as a *symbol*.) The aim is now to show that the subset of L representing elements of S is regular. To do so, it suffices to show that the set of words in L representing elements of $\overline{P_{c_1 \dots c_m}}$ is regular, since (3) is a finite union.

To this end, fix c_1, \dots, c_m and write P for $P_{c_1 \dots c_m}$. Let $z_{i,j} \in \mathbb{Z}^n$ be such that $\overline{b_j c_{i+1} \dots c_m} = \overline{c_{i+1} \dots c_m} z_{i,j}$. Let $u_0 c_1 u_1 \dots c_m u_m \in P$ with $u_i = b_1^{\alpha_{i,1}} \dots b_q^{\alpha_{i,q}}$. Then

$$\overline{u_0 c_1 u_1 \dots c_m u_m} = \overline{c_1 \dots c_m} \prod_{i=0}^m \prod_{j=1}^q z_{i,j}^{\alpha_{i,j}},$$

or, switching to additive notation and supposing $\overline{c_1 \dots c_m} = g_h(z'_1, \dots, z'_n)$ and $z_{i,j} = (z_{i,j,1}, \dots, z_{i,j,n})$ for all i, j :

$$\overline{u_0 c_1 u_1 \dots c_m u_m} = g_h(z'_1, \dots, z'_n) \sum_{i=0}^m \sum_{j=1}^q \alpha_{i,j} (z_{i,j,1}, \dots, z_{i,j,n}).$$

Therefore define $\theta(z_1, \dots, z_n)$ to be

$$\begin{aligned} & (\exists \alpha_{0,1}, \dots, \alpha_{m,q}) \left((\alpha_{0,1} \geq 0) \wedge \dots \wedge (\alpha_{m,q} \geq 0) \right. \\ & \quad \wedge \left(z_1 = z'_1 + \sum_{i=0}^m \sum_{j=1}^m \alpha_{i,j} z_{i,j,1} \right) \\ & \quad \wedge \left(z_2 = z'_2 + \sum_{i=0}^m \sum_{j=1}^m \alpha_{i,j} z_{i,j,2} \right) \\ & \quad \vdots \\ & \quad \left. \wedge \left(z_n = z'_n + \sum_{i=0}^m \sum_{j=1}^m \alpha_{i,j} z_{i,j,n} \right) \right), \end{aligned}$$

where $\alpha_{i,j} z_{i,j,k}$ is understood to be shorthand for

$$\underbrace{\alpha_{i,j} + \dots + \alpha_{i,j}}_{z_{i,j,k} \text{ times}}.$$

By a special case of Theorem 5.1, the structure $(\mathbb{Z}, +)$ admits an automatic presentation (M, ψ) , where M is the set of words ϵz , where $\epsilon \in \{+, -\}$ and z is in reverse binary notation. Furthermore, it is clear that, in this presentation, the relation \geq is regular. That is, (M, ψ) is an automatic presentation for $(\mathbb{Z}, +, \geq)$.

The set of words in L representing elements of \bar{P} is then

$$\{g_h \text{conv}(z_1, \dots, z_n) : (\mathbb{Z}, +, \geq) \models \theta(\psi(z_1), \dots, \psi(z_n))\}.$$

(Recall that g_h is the representative of the coset in which $\overline{c_1 \cdots c_m}$ lies.) By Proposition 4.5, this set is a regular subset of L .

Union together the [finitely many] regular subsets of L obtained for the various c_1, \dots, c_m to see that the set L_S consisting of those words in L representing elements of S is regular. So S admits the automatic presentation $(L_S, \phi|_{L_S})$.

Propositions 10.6 and 10.5 together yield Theorem 10.1.

11 FA-presentability, automaticity, and Cayley graphs

We recall the definition of an automatic semigroup; see [7] for further background information:

Definition 11.1 *A semigroup S is automatic if there exists a finite generating set A for S and a regular language L over A such that every element of S is represented by at least one element of L , and, for all $a \in A \cup \{\varepsilon\}$, the relation*

$$L_a = \{(u, v) : ua = v \text{ in } S\}$$

is regular.

If S is an automatic semigroup, then the Cayley graph of S (viewed as a labelled graph) is FA-presentable: the language L (as in the definition of ‘automatic’) is a regular language of representatives for the vertices of the Cayley graph (the elements of S), and the adjacency relations (the relations L_a for $a \in A$) and the equality relation (the relation L_ε) are all regular.

The converse of this does not hold: let H be the discrete Heisenberg group — that is, the multiplicative group of matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } x, y, z \in \mathbb{Z}.$$

The Cayley graph of H is FA-presentable, but H is not automatic [3, p. 651].

Observe that whether the Cayley graph of a semigroup is FA-presentable is not dependent on the choice of generating set:

Proposition 11.2 *Let S be a semigroup and suppose the Cayley graph of S with respect to some finite generating set X is FA-presentable. Let Y be any finite generating set for S . Then the Cayley graph of S with respect to Y is also FA-presentable.*

PROOF. Let (L, ϕ) be an automatic presentation for the Cayley graph of S with respect to X . Let $y \in Y$. Since X generates S , there exists a word $w = w_1 \cdots w_k$ with $w_i \in X$ with $y = w$ in S . Then the adjacency relation L_y is given by

$$\begin{aligned} L_y &= L_{w_1} \circ L_{w_2} \circ \cdots \circ L_{w_k} \\ &= \{(u, v) : (\exists t_1, \dots, t_{k-1}) \\ &\quad ((u, t_1) \in L_{w_1} \wedge (t_2, t_3) \in L_{w_2} \wedge \dots \wedge (t_{k-1}, v) \in L_{w_k})\}. \end{aligned}$$

So the relations L_y are first-order definable and thus regular. So (L, ϕ) is also an automatic presentation for the Cayley graph of S with respect to Y .

Let \mathcal{S} , \mathcal{C} , and \mathcal{G} be respectively the classes of finitely generated semigroups, finitely generated cancellative semigroups, and finitely generated groups. Let \mathcal{F} be the class of FA-presentable semigroups, \mathcal{A} the class of automatic semigroups, and \mathcal{T} the class of semigroups whose Cayley graphs are FA-presentable.

With this notation, the discussion above can be summarized by the following result:

Proposition 11.3 $\mathcal{S} \cap \mathcal{A} \subsetneq \mathcal{S} \cap \mathcal{T}$ and $\mathcal{C} \cap \mathcal{A} \subsetneq \mathcal{C} \cap \mathcal{T}$.

Within the class of finitely generated groups \mathcal{G} , we can say more:

Proposition 11.4 $\mathcal{G} \cap \mathcal{F} \subsetneq \mathcal{G} \cap \mathcal{A} \subsetneq \mathcal{G} \cap \mathcal{T}$.

PROOF. By Theorem 5.1, the finitely generated FA-presentable groups are precisely the virtually abelian groups, which are known to be automatic [13, Section 4.1]. Free groups are automatic but not FA-presentable. This establishes the first proper inclusion. For the second proper inclusion, recall that every automatic group has an FA-presentable Cayley graph, but that the non-automatic group H defined above has an FA-presentable Cayley graph.

However, this does not generalize to semigroups:

Proposition 11.5 *The classes $\mathcal{C} \cap \mathcal{F}$ and $\mathcal{C} \cap \mathcal{A}$ are incomparable; thus the classes $\mathcal{S} \cap \mathcal{F}$ and $\mathcal{S} \cap \mathcal{A}$ are also.*

PROOF. The non-inclusion of $\mathcal{C} \cap \mathcal{A}$ in $\mathcal{C} \cap \mathcal{F}$ follows from the non-inclusion of $\mathcal{G} \cap \mathcal{A}$ in $\mathcal{G} \cap \mathcal{F}$. The first author has previously exhibited an example of a non-automatic finitely generated subsemigroup S of a virtually abelian group [6]. This semigroup S must admit an automatic presentation by Theorem 10.1. This establishes the non-inclusion of $\mathcal{C} \cap \mathcal{A}$ in $\mathcal{C} \cap \mathcal{F}$.

12 Unary automatic presentations & Word problems

An automatic presentation (L, ϕ) is *unary* if the language L consists of words over a one-letter alphabet. This section considers unary automatic presentations and connections to word problems for semigroups and groups.

Theorem 12.1 *Let S be a cancellative semigroup that admits a unary automatic presentation. Then S is finite.*

PROOF. The proof of [2, Theorem 7.19], which asserts that groups admitting unary automatic presentations are finite, holds in the more general setting of cancellative semigroups.

However, there do exist infinite non-cancellative semigroups admitting unary automatic presentations: for example, a countable semigroup of right zeros $Z = \{z_i : i \in \mathbb{N}\}$ (with $z_i z_j = z_j$ for all $i, j \in \mathbb{N}$) admits the automatic presentation (L, ϕ) , where $L = a^*$ and $\phi : L \rightarrow Z$ is defined by $a^k \mapsto z_k$. The multiplication relation is then

$$\{(a^i, a^j, a^j) : i, j \in \mathbb{N}\},$$

which is clearly regular. Note that Z is left-cancellative, so even one-sided cancellative unary FA-presentable semigroups can be infinite.

Recall that the word problem of a group G with respect to a [semigroup] generating set X is the set of words over X that are equal to 1_G . The word problem for a group is said to be *one-counter* if it is accepted by a one-counter automaton. (See [1] for background information on one-counter automata.) The word problem for a semigroup S with respect to a generating set X , as defined by Duncan and Gilman [11], is the set $\{u\#v^r : u, v \in X^*, u = v \text{ in } S\}$, where v^r denotes the reverse of the word v and $\#$ is a new symbol not in X .

Blumensath [2, Proposition 7.22] proved that the Cayley graph of a finitely generated group G is virtually cyclic if and only if the Cayley graph of G admits a unary automatic presentation. This, together with Herbst's [19] result that finitely generated groups with one-counter word problem are precisely the virtually cyclic groups, yields the following corollary:

Corollary 12.2 *A finitely generated group has a one-counter word problem if and only if its Cayley graph has a unary automatic presentation.*

Finitely generated virtually abelian groups — which are precisely the finitely generated FA-presentable groups — are characterized by having word problems recognizable by blind one-counter automata [12, Theorem 1]. This fact, together with the preceding corollary, suggests the following question:

Problem 12.3 *Are finitely generated FA-presentable semigroups classifiable by having word problems recognizable by some 'natural' class of automata?*

Acknowledgements. Part of the work described in this paper was undertaken whilst the fourth author was on study leave from the University of Leicester and the support of the University in this regard is appreciated. The fourth author would also like to thank Hilary Craig for all her help and encouragement.

References

- [1] J. Berstel, Transductions and context-free languages, vol. 38 of Leitfäden der Angewandten Mathematik und Mechanik [Guides to Applied Mathematics and Mechanics], B. G. Teubner, Stuttgart, 1979.
- [2] A. Blumensath, Automatic structures, Diploma thesis, RWTH Aachen (1999).
- [3] A. Blumensath, E. Grädel, Finite presentations of infinite structures: automata and interpretations, Theory Comput. Syst. 37 (6) (2004) 641–674.

- [4] A. J. Cain, Presentations for subsemigroups of groups, Ph.D. Thesis, University of St Andrews (2005).
- [5] A. J. Cain, Cancellativity is undecidable for automatic semigroups, *Q. J. Math.* 57 (3) (2006) 285–295.
- [6] A. J. Cain, A group-embeddable non-automatic semigroup whose universal group is automatic, *Glasg. Math. J.* 48 (2) (2006) 337–342.
- [7] C. M. Campbell, E. F. Robertson, N. Ruškuc, R. M. Thomas, Automatic semigroups, *Theoret. Comput. Sci.* 250 (1–2) (2001) 365–391.
- [8] A. H. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups (Vol. I)*, No. 7 in *Mathematical Surveys*, American Mathematical Society, Providence, R.I., 1961.
- [9] C. Delhommé, Automaticité des ordinaux et des graphes homogènes, *C. R. Math. Acad. Sci. Paris* 339 (1) (2004) 5–10.
- [10] V. Diekert, Commutative monoids have complete presentations by free (noncommutative) monoids, *Theoret. Comput. Sci.* 46 (2-3) (1986) 319–327.
- [11] A. Duncan, R. H. Gilman, Word hyperbolic semigroups, *Math. Proc. Cambridge Philos. Soc.* 136 (3) (2004) 513–524.
- [12] M. Elder, M. Kambites, G. Ostheimer, On groups and counter automata, Submitted.Preprint: arXiv:math/0611188v1.
- [13] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, W. P. Thurston, *Word Processing in Groups*, Jones & Bartlett, Boston, Mass., 1992.
- [14] G. M. S. Gomes, A characterization of the group congruences on a semigroup, *Semigroup Forum* 46 (1) (1993) 48–53.
- [15] R. I. Grigorchuk, Cancellative semigroups of power growth, *Math. Notes* 43 (3) (1988) 175–183.
- [16] R. I. Grigorchuk, Semigroups with cancellations of degree growth, *Mat. Zametki* 43 (3) (1988) 305–319, 428, [In Russian. See [15] for a translation.].
- [17] R. I. Grigorchuk, On growth in group theory, in: *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, Math. Soc. Japan, Tokyo, 1991.
- [18] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.* (53) (1981) 53–73.
- [19] T. Herbst, On a subclass of context-free groups, *RAIRO Inform. Théor. Appl.* 25 (3) (1991) 255–272.
- [20] J. M. Howie, *Fundamentals of Semigroup Theory*, vol. 12 of *London Mathematical Society Monographs (New Series)*, Clarendon Press, Oxford University Press, New York, 1995.

- [21] J. F. P. Hudson, Regular rewrite systems and automatic structures, in: *Semigroups, automata and languages (Porto, 1994)*, World Sci. Publ., River Edge, NJ, 1996, pp. 145–152.
- [22] M. Kambites, F. Otto, Uniform decision problems for automatic semigroups, *J. Algebra* 303 (2) (2006) 789–809.
- [23] B. Khossainov, A. Nerode, Automatic presentations of structures, in: *Logic and computational complexity (Indianapolis, IN, 1994)*, vol. 960 of *Lecture Notes in Computer Science*, Springer, Berlin, 1995, pp. 367–392.
- [24] B. Khossainov, A. Nies, S. Rubin, F. Stephan, Automatic structures: richness and limitations, in: *Proceedings of the 19th IEEE Symposium on Logic in Computer Science*, IEEE Computer Society, 2004, pp. 110–119.
- [25] B. Khossainov, S. Rubin, F. Stephan, Automatic partial orders, in: *Proceedings of the 18th IEEE Symposium on Logic in Computer Science*, IEEE Computer Society, 2003, pp. 168–177.
- [26] B. H. Neumann, T. Taylor, Subsemigroups of nilpotent groups, *Proc. Roy. Soc. Ser. A* 274 (1963) 1–4.
- [27] A. Nies, Describing groups, *Bull. Symbolic Logic* 13 (3) (2007) 305–339.
- [28] G. P. Oliver, R. M. Thomas, Automatic presentations for finitely generated groups, in: V. Diekert, B. Durand (eds.), *22nd Annual Symposium on Theoretical Aspects of Computer Science (STACS'05)*, Stuttgart, Germany, vol. 3404 of *Lecture Notes in Comput. Sci.*, Springer, Berlin, 2005.
- [29] F. Otto, A. Sattler-Klein, K. Madlener, Automatic monoids versus monoids with finite convergent presentations, in: *Rewriting techniques and applications (Tsukuba, 1998)*, vol. 1379 of *Lecture Notes in Comput. Sci.*, Springer, Berlin, 1998, pp. 32–46.
- [30] L. Pelecq, *Isomorphismes et automorphismes des graphes context-free, équationnels et automatiques*, Ph.D. thesis, Bordeaux 1 University (1997).
- [31] L. Rédei, *Theorie der Endlich Erzeugbaren Kommutativen Halbgruppen*, vol. 41 of *Hamburger Mathematische Einzelschriften*, Physica-Verlag, Würzburg, 1963, [In German].
- [32] G. Sénizergues, Definability in weak monadic second-order logic of some infinite graphs, in: K. Compton, J.-E. Pin, W. Thomas (eds.), *Automata Theory: Infinite Computations*, vol. 9202, *Dagstuhl Seminar*, Wadern, Germany, 1992, p. 16.
- [33] M. A. Taïclin, Algorithmic problems for commutative semigroups, *Dokl. Akad. Nauk SSSR* 178 (1968) 786–789.
- [34] Y. M. Vazhenin, Semigroups with one defining relation whose elementary theories are decidable, *Sibirsk. Mat. Zh.* 24 (1) (1983) 40–49, 191.
- [35] Y. Zhang, S. Li, D. Wang, Semidirect products and wreath products of strongly π -inverse monoids, *Georgian Math. J.* 3 (3) (1996) 293–300.