On Convex Permutations

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A selection of points drawn from a convex polygon, no two with the same vertical or horizontal coordinate, yields a permutation in a canonical fashion. We characterise and enumerate those permutations which arise in this manner and exhibit some interesting structural properties of the permutation class they form. We conclude with a permutation analogue of the celebrated Happy Ending Problem.

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Figure 1: Convex and standard drawings of the permutation 1243.

1. INTRODUCTION

A set of n points in the plane, no two in the same vertical or horizontal line (we call such a set *generic*) forms a permutation in a canonical manner: label the points from bottom to top using the numbers 1 to n, then read these labels from left to right. Similarly, every permutation of length n may be plotted as a set of points in the plane; indeed, viewed this way permutations serve as class representatives for generic points sets in the plane under the equivalence relation of "stretching and shrinking axes". More formally, two sets S and T of points on the plane are said to be *order isomorphic* if there are strictly increasing functions $f, g : \mathbb{R} \mapsto \mathbb{R}$ such that $\{(f(s_1), g(s_2)) : (s_1, s_2) \in S\} = T$. Every finite generic point set X is order isomorphic to the plot of a unique permutation π. In such a case we say that X is *a plot* of π.

The permutation π is said to contain the permutation σ, written $\sigma \leq \pi$, if a plot of π contains a subset order isomorphic to the plot of σ, or equivalently, if π contains a subsequence of values in the same relative order as the entries of σ; otherwise π is said to *avoid* σ. We refer to a set of permutations closed downward under this ordering (i.e., a downset) as a *permutation class*. That is, $\mathcal{C}$ is a permutation class if and only if whenever $\pi \in \mathcal{C}$ and $\sigma \leq \pi$ then $\sigma \in \mathcal{C}$. Every permutation class can be characterised by the set of minimal permutations not belonging to the class, its *basis*. In other words, β is in the basis of $\mathcal{C}$ if $\beta \notin \mathcal{C}$ but $\alpha \in \mathcal{C}$ for all $\alpha < \beta$.

The class of permutations order isomorphic to finite generic subsets of a circle has been studied by Vatter and Waton [12]. Here we consider a much larger class, the class of all *convex permutations*, i.e., those permutations order isomorphic to a finite convex generic set. For example, the permutation 1243 is convex, as can be seen from Figure 1. Note that the "standard" drawing of 1243 — the drawing with integer coordinates on the right of Figure 1 — is not convex. There are far fewer permutations whose standard drawing is convex than there are convex permutations[1]. It is clear that the set of all convex permutations forms a permutation class, which we call the *convex class*.

[1]The sequence which enumerates those permutations whose standard drawings are convex by length begins

$$1, 2, 6, 20, 66, 188, 466, 1022, 2098, 4032,$$

while the sequence enumerating the convex permutations begins

$$1, 2, 6, 24, 104, 464, 2088, 9392, 42064, 187296.$$

The first of these sequences has been obtained by an exhaustive search; the second follows from the results in Section 3.

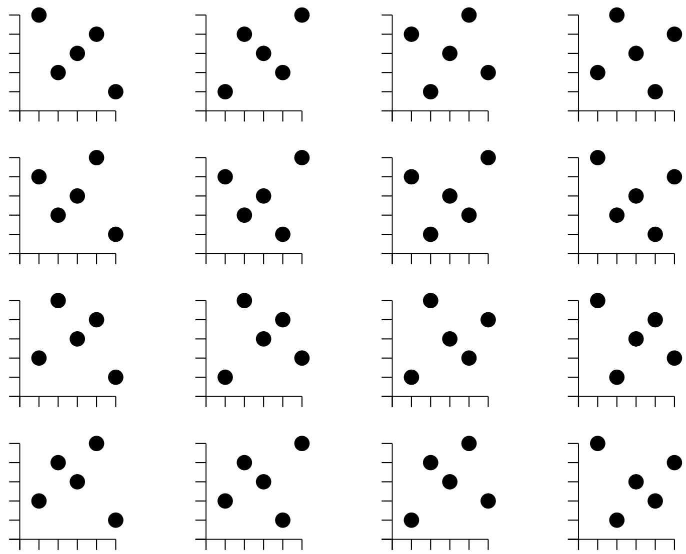

Figure 2: The basis of the convex class.

2. CHARACTERISATION

Our first task is to characterise the convex permutations. We do so by identifying the basis of the class.

Theorem 2.1. *The basis of the convex class consists of all* 16 *permutations* β *of length* 5 *satisfying* $\beta(3) = 3$*, such that the elements* $\beta(1)$ *and* $\beta(2)$ *are neither both smaller than, nor both larger than* 3 *(see Figure 2).*

We can also rephrase Theorem 2.1 in a manner which may be more familiar to students of discrete geometry: a permutation can be drawn in convex position if and only if every five point subpermutation can be drawn in convex position.

Before our proof of Theorem 2.1, we show that the permutations in Figure 2 form the basis for a differently defined class. The entry $\pi(i)$ of the permutation π is said to be a *left-to-right minimum* if there is no lesser entry to left of $\pi(i)$. We similarly define *left-to-right maximum*, *right-to-left minimum* and *right-to-left maximum*. An entry is said to be *extremal* if it is of any of these four types, and an *extremal permutation* is one in which every entry is extremal.

Note that the set of extremal permutations forms a permutation class (which we call the *extremal class*), because removing entries will not affect the status of the left-to-right or right-to-left minima or maxima which remain. Therefore it makes sense to ask for the basis of this class, i.e., the minimal permutations which are not extremal.

Lemma 2.2. *The basis of the extremal class consists of the set of all non-extremal permutations* β *of length* 5*.*

Proof. Clearly no extremal permutation may contain any of these permutations. On the other hand, suppose that the permutation π is not extremal, and thus has a nonextremal point $\pi(i)$. It follows that π must contain a left-to-right minimum below and to the left of $\pi(i)$, a left-to-right maximum above and to the left of $\pi(i)$, a right-to-left minimum below and to the right of $\pi(i)$, and a right-to-left maximum above and to the right of $\pi(i)$. These four extreme points together with $\pi(i)$ give rise to one of the proposed basis elements. □

Any convex drawing of a permutation shows it to be extremal, so to prove Theorem 2.1 we need only establish the converse, that every extremal permutation is convex. We first need to consider some associated constructions.

A permutation is called *unimodal* if it can be written as the concatenation of an increasing followed by a decreasing sequence (either of which might be empty). An equivalent characterization is that such a permutation avoids the patterns 213 and 312.

Lemma 2.3. *Let π be a unimodal permutation of length n and let real numbers $x_1 < x_2 < \cdots < x_n$ be given. Then, there is a convex plot of π such that for each i, $1 \leq i \leq n$, the point corresponding to $\pi(i)$ has first coordinate x_i.*

Proof. We proceed by induction, the case $n = 1$ being trivial. Let π and $x_1 < x_2 < \cdots < x_n$ be given, satisfying the hypotheses above. Either $\pi(1) = 1$ or $\pi(n) = 1$. Without loss of generality suppose that it is the former. By induction, we can plot the subpermutation of π obtained by deleting the first element using points (x_i, y_i) ($2 \leq i \leq n$). Now place the first element at a point (x_1, y_1) with $y_1 < \min\{y_2, \ldots, y_n\}$ and also sufficiently small that the resulting plot is convex. □

By symmetry, a dual result applies to permutations avoiding 132 and 231: given any prescribed sequence $x_1 < x_2 < \cdots < x_n$ such a permutation has a convex plot with $\pi(i)$ being represented by a point (x_i, y_i) for $1 \leq i \leq n$. We also note that in either case, having produced such a plot we may, by compressing the vertical scale (that is, applying the map $(x, y) \mapsto (x, cy)$ for some $0 < c < 1$ arrange that the maximum of the absolute value of the slope of any segment connecting points of the plot can be made as small as we wish.

Lemma 2.4. *Any permutation which avoids 321 can be drawn on any pair of parallel lines of positive slope.*

Proof. Fix two parallel lines of positive slope and take π to be a 321-avoiding permutation. We prove the claim by induction; the base case where π is the empty permutation being trivial, let us suppose that π has at least one point and let $\pi(1) = j$.

As π avoids 321, the subsequence formed by the entries below and to the right of $\pi(1)$ is $1, 2, \ldots, j-1$. If all these entries occur contiguously then it is easy to draw π on the two lines: first draw (by induction) the permutation obtained by removing the entries $\pi(1), 1, 2, \ldots, j-1$ on the lines, and then place points corresponding to these removed entries below and to the left of this drawing.

Otherwise the entries $\pi(1), 1, 2, \ldots, j-1$ do not occur contiguously in π. Denote by $\pi(i)$ the first (reading left to right) entry which breaks up this sequence. Thus π begins

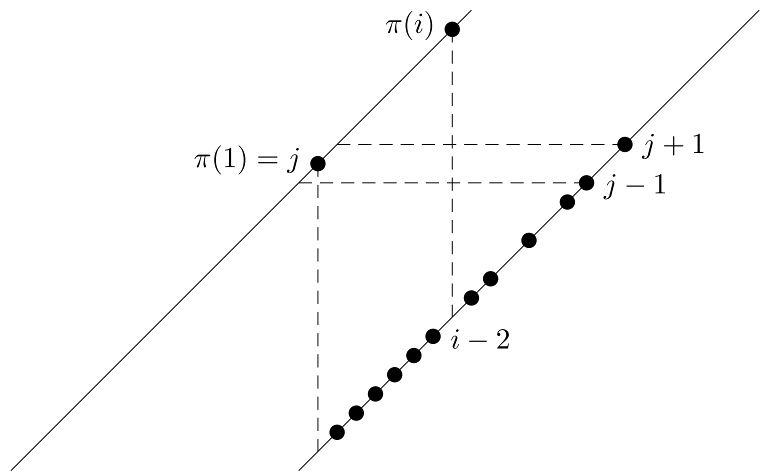

Figure 3: Drawing a 321-avoiding permutation on two parallel lines.

$\pi(1), 1, 2, \ldots, i-2, \pi(i)$. By induction we can draw the permutation obtained by removing the entries $\pi(1), 1, 2, \ldots, i-2$ on our lines. As $\pi(i) > j-1$, the point corresponding to $j-1$ must lie on the bottom line in this drawing, while the point corresponding to $\pi(i)$ must lie to the left and above this point on the top line. To produce a drawing of π we now add, as depicted in Figure 3, a point (which will correspond to $\pi(1)$) lying vertically between the points corresponding to $j-1$ and $j+1$ on the top line (this new point must necessary lie below $\pi(i)$ because $\pi(i) \geq j+1$) and $i-2$ points on the bottom line horizontally between this new point and the point corresponding to $\pi(i)$. $\square$

We remark that Lemma 2.4 can also be deduced from the work of Fanti, Frosini, Grazzini, Pinzani, and Rinaldi [9].

We are now ready to prove the main theorem of the section.

Proof of Theorem 2.1. We need to prove that every extremal permutation, π, can be drawn in convex position. We label the topmost, bottommost, rightmost and leftmost points of such a permutation as T, B, R and L respectively. We shall consider the possible permutations formed by these elements. There are eleven cases in all, {1, 12, 21, 132, 213, 231, 312, 2143, 2413, 3142, 3412}; after symmetry these reduce to five cases:

1: In this case T, B, R and L are identical, so such a permutation has length one and is trivially convex.

12: Here we must have $L = B$ and $R = T$. The set of all points whose values lie between those of L and R inclusive must avoid 321 (for otherwise, a copy of 321 together with L and R would yield a non-extremal subpermutation). Since this accounts for all the points, the permutation may be drawn on two parallel lines.

132: In this case $L = B$ corresponds to the '1', while T corresponds to the '3' and R corresponds to the '2'. In such a permutation the set of all points below R must avoid 321 and so can be drawn on two parallel lines. Choose such a drawing, and then choose points on the horizontal line through R which correctly represent the relative positions of the remaining elements with respect to the elements below R. These remaining elements form a unimodal permutation and so can now be drawn as a convex cap, with a maximum slope smaller than that of the two parallel lines. The resulting set of points is a convex plot of π.

2143: Here all four of the points T, B, R and L are distinct. As in the case '12', the set of all points whose values lie between those of L and R inclusive must avoid 321, so these points can be drawn on two parallel lines. As in the case '132', take the horizontal lines through R and through L and choose a set of points along these lines which correctly represent the positions of the remaining elements with respect to one another, and those already plotted. The elements above R are unimodal and so can now be drawn on a convex cap as above, while those below L can be handled symmetrically.

2413: This case is essentially the same as the preceding one. □

3. ENUMERATION

The convex class has received some previous attention, and has been enumerated independently in at least four different ways. Mansour and Severini [10] give the enumeration of the extremal class (which is the same as the convex class by the results of the previous section), which they call "square permutations". Also, a string of papers — Fanti, Frosini, Grazzini, Pinzani, and Rinaldi [8], Disanto, Frosini, Pinzani, and Rinaldi [4], Bernini, Disanto, Pinzani, and Rinaldi [2], Disanto, Frosini, Rinaldi, and Pinzani [5] — investigate the convex class and its connection to "convex permutominoes"; the enumeration of the convex class occurs in [4]. Finally, Duchi and Poulalhon [6] have given the enumeration using the ECO method.

We include here our independent enumeration of the convex class in order to show how it can be obtained in an essentially automatic fashion from the fact that the convex permutations can be represented as a context-free language with respect to the insertion encoding (which we define briefly below.)

The insertion encoding of Albert, Linton and Ruškuc [1], provides a correspondence between permutation classes and formal languages. This correspondence associates with each permutation a word describing how it is built in a step-by-step process. At each step a new maximum entry n is inserted into an open *slot* (represented by a $\diamond$), until the permutation is complete, at which point no open slots remain. The insertion can occur in one of four ways:

- A slot can be filled (replacing $\diamond$ by n).
- A new maximum can be placed on the left of the slot (replacing $\diamond$ by $n\diamond$).

- A new maximum can be placed on the right of the slot (replacing $\diamond$ by $\diamond n$).
- A new maximum can be placed in the middle of the slot (replacing $\diamond$ by $\diamond n \diamond$).

These operations are called fills (denoted f), lefts (denoted l), rights (denoted r) and middles (denoted m), respectively. In addition these symbols are subscripted by the slot they apply to, where for $i \geq 1$ the subscript i refers to the ith slot from the left and $-i$ refers to the ith slot from the right. Note that two slots will never occur contiguously. For example, the permutation 31254 has insertion encoding $\mathsf{m}_1\mathsf{l}_2\mathsf{f}_1\mathsf{r}_1\mathsf{f}_1$ because its evolution is

$$\begin{gathered} \diamond \\ \diamond 1 \diamond \\ \diamond 12 \diamond \\ 312 \diamond \\ 312 \diamond 4 \\ 31254 \end{gathered}$$

Each intermediate step in this process is called a *configuration*.

In some cases, characterised in [1], the insertion encoding of a permutation class forms a regular language, from which the (rational) generating function for the class can be routinely computed. The convex class does not meet these conditions, but as noted previously the encodings of its elements do form a context-free language and hence procuring its generating function requires little more effort.

Theorem 3.1. *The generating function for the convex class is*

$$\frac{t\left(1-6t+10t^2-4t^2\sqrt{1-4t}\right)}{(1-4t)^2}$$

Proof. Using the insertion encoding to build a convex permutation, consider the insertion of an arbitrary entry in a configuration. If, before the insertion, there were smaller entries on both sides of it and, after the insertion, there are slots on both sides of it, then the entry inserted will inevitably become the '3' in one of the basis entries of the convex class. Thus, no such insertions can be permitted. On the other hand, if every entry is inserted in such a way that there are smaller entries on at most one side of it, or slots remain on at most one side of it after insertion, then no entry can play the role of a '3' in one of the basis entries of the convex class and hence the final permutation produced will be convex. We summarise these conditions with the following four rules, which depend on the number of slots in the configuration and on whether an entry or a slot occurs at each of the two ends of the configuration:

(3+) If there are three or more slots in the configuration then no insertions are possible in any but the outer two slots (recall that two slots never occur contiguously). Therefore all operations in the creation of a convex permutation can be subscripted by either 1 or -1. (For example, in the configuration $\diamond 12 \diamond 3 \diamond$, we are not allowed to operate on the second slot.)

(B) If there are entries at both ends of the configuration then the only permitted operations are at the left-most and right-most slots. (For example, in the configuration $1\diamond 2\diamond 3$, the allowed operations are $\mathsf{f}_{\pm 1}$, l_1, and r_{-1}.)

(S) If there is an entry at a single end of the configuration then at that end we must either fill the slot or insert a new maximum toward that end (e.g., if the entry is on the right, we are allowed to fill the right-most slot or perform a right on it), while at the other end all operations are possible. (For example, in the configuration $\diamond 12\diamond 3$, the allowed operations are $\mathsf{f}_{\pm 1}$, $\mathsf{r}_{\pm 1}$, m_1, and l_1.)

(N) If there are entries at neither end of the configuration then all operations are possible in the two (or one, in the case of the configuration $\diamond$) outermost slots. (For example, in the configuration $\diamond 12\diamond 3\diamond$, the allowed operations are $\mathsf{f}_{\pm 1}$, $\mathsf{l}_{\pm 1}$, $\mathsf{r}_{\pm 1}$, and $\mathsf{m}_{\pm 1}$.)

The enumeration strategy follows this summary (and is essentially mechanical, given the descriptions above). Specifically, we introduce six generating functions. Each counts (with the coefficient of t^k) the number of words of length k which can complete a configuration to a member of the convex class. Three of these count the number of ways to complete configurations with a single slot, depending on if the configuration has entries on Both ends, a Single end, or Neither end; we denote these by $B_1(t)$, $S_1(t)$, and $N_1(t)$. The other three — $B(t,z)$, $S(t,z)$, and $N(t,z)$ — count ways to complete configurations with two or more slots; the coefficient of $t^k z^{s-2}$ in these generating functions is the number of words of length k which can complete a configuration of the given form with s slots to a member of the convex class. Our goal is therefore to compute $N_1(t)$, the number of words of length k which can complete a configuration with a single slot and entries on neither end (in other words, the configuration $\diamond$) to a member of the convex class.

We compute these generating functions, beginning with the Bs which are very straightforward. For B_1, we have small entries on both ends of the configuration and a single slot in the middle, so the allowed operations are l_1, r_1, and f_1 (the last of which terminates the encoding). Thus

$$B_1(t) = 2tB_1(t) + t = \frac{t}{1-2t}.$$

(This generating function should otherwise be clear from the allowed operations: we can choose any word in $\{\mathsf{l}_1, \mathsf{r}_1\}^*$, and then terminate it with f_1.) If there are $s \geq 2$ slots and small entries at both ends, then the allowed operations are l_1, r_{-1} and $\mathsf{f}_{\pm 1}$ and we have

$$B(t,z) = 2tB(t,z) + 2tB_1(t) + 2tzB(t,z) = \frac{2t^2}{(1-2t)(1-2t-2tz)}.$$

Here, the first term corresponds to the result of using l_1 or r_{-1} as this increases the number of symbols in the encoding by one (hence the factor of t) without changing the number of slots. For the $\mathsf{f}_{\pm 1}$ possibilities, we might have started with two slots, and left only one (hence $2tB_1(t)$), or with three or more, and left one fewer (hence $2tzB(t,z)$). Similar justifications apply to all the remaining formulae, and we shall not repeat them.

The situation for the S generating functions is a little more complex. For simplicity we assume that there are entries at the right hand end of the configuration (and hence not at the left). The allowed operations are then f_1 and l_1 (leading to B configurations), m_1 and r_1 (leading to S configurations) and f_{-1} and r_{-1} (also leading to S configurations). These are all distinct if there are at least two slots, but only the operations carrying a 1 subscript can be applied when there is a single slot. Thus:

$$\begin{aligned} S_1(t) &= t + tB_1(t) + tS(t,0) + tS_1(t) \\ S(t,z) &= tS_1(t) + tzS(t,z) + \frac{t(S(t,z) - S(t,0))}{z} + 2tS(t,z) + (t+tz)B(t,z) + tB_1(t). \end{aligned}$$

The $S(t,z)$ terms in the second equation can be collected together in order to apply the kernel method:

$$\left(1 - 2t - tz - \frac{t}{z}\right) S(t,z) = tS_1(t) - \frac{tS(t,0)}{z} + (t+tz)B(t,z) + tB_1(t).$$

Setting the first factor on the left equal to 0, specifically

$$z = \frac{1 - 2t - \sqrt{1-4t}}{2t},$$

and substituting in the right hand side allows us to produce a second equation connecting $S_1(t)$ and $S(t,0)$. Solving this system yields

$$\begin{aligned} S_1(t) &= \frac{t}{\sqrt{1-4t}}, \\ S(t,0) &= \frac{(2t^2 - 3t + 1)\sqrt{1-4t} - 4t^2 + 5t - 1}{(1-2t)(1-4t)}. \end{aligned}$$

In turn, we can now substitute these values in the original equation and solve for $S(t,z)$. The resulting complex expression is of interest only as an ingredient in the next step of the computation so we shall not reproduce it here. Maple code available from the authors can also be used to generate it, or to verify any of the computations and formulas found in this proof.

We repeat this procedure to produce the N generating functions. This time the original equations are

$$\begin{aligned} N_1(t) &= t + tN(t,0) + 2tS_1(t), \\ N(t,z) &= 2tN(t,z) + \frac{2t(N(t,z) - N(t,0))}{z} + (2t + 2tz)B(t,z) + 2tB_1(t). \end{aligned}$$

The kernel in this case is simply $z = 2t/(1-2t)$ and substitution and simplification yields

$$N_1(t) = \frac{t\left(1 - 6t + 10t^2 - 4t^2\sqrt{1-4t}\right)}{(1-4t)^2}.$$

which, as it represents the generating function for permutations formed from the initial configuration $\diamond$, is the generating function for the convex permutations. $\square$

Given the simple form of the generating function it is possible to express the number of permutations of length n in the convex class in closed form (for $n > 1$) as:

$$2(n+2)4^{n-3} - 4(2n-5)\binom{2(n-3)}{n-3}.$$

A possible avenue for further exploration would be to enumerate permutations with a prescribed number of "interior points". From this viewpoint, the formula we have derived counts permutations with 0 interior points.

4. A HAPPY ENDING ANALOGUE

The Happy Ending Problem, due to Klein, first appeared in Erdős and Szekeres [7] as:

> "Can we find for a given number n a number $N(n)$ such that from any set containing at least N points it is possible to select n points forming a convex polygon?"

Here we consider the analogous quantity for permutations: let $f(n)$ denote the largest integer such that every permutation of length $f(n)$ contains a convex permutation of length n.

The Erdős-Szekeres Theorem on monotone subsequences from that same paper gives one upper bound: every permutation of length at least $(n-1)^2+1$ contains a monotone subpermutation of length at least n, and since monotone permutations are convex, $f(n) \leq (n-1)^2+1$. A theorem of Chung [3] gives an improved upper bound. She proved that all permutations of length at least $(n^2+n+1)/3$ contain a unimodal (and hence convex) subpermutation (i.e., in our language, a permutation that can be drawn on a convex cap or cup) of length at least n. For $f(n)$ we are able to give an upper bound of approximately $n^2/4$. Define

$$h(n) = \begin{cases} \left(\frac{n-2}{2}\right)^2 + 2 & \text{for even } n, \\ \left(\frac{n-1}{2}\right)^2 & \text{for odd } n. \end{cases}$$

Proposition 4.1. *Every permutation of length greater than $h(n)$ contains a 123- or 321-avoiding subpermutation (and thus in particular, a convex subpermutation) of length at least n.*

Proof. As is well known, the length of the longest 321 avoiding subpermutation of a permutation π is equal to the sum of the lengths of the first two rows of the tableau produced from it using the Robinson-Knuth-Schensted algorithm (see, e.g., Sagan [11]). Likewise, the length of the longest 123 avoiding subsequence is the sum of the lengths of the first two columns of this tableau. Thus a permutation that avoids such subpermutations of length n must have at most $n-1$ cells in the first two rows, and likewise in the first two columns of its tableau. It is easy to verify that the number of cells in such a tableau is maximized when the first two rows are as nearly equal as possible in length (and symmetrically for the columns) and the tableau is otherwise as full as possible. This provides the stated bound. $\square$

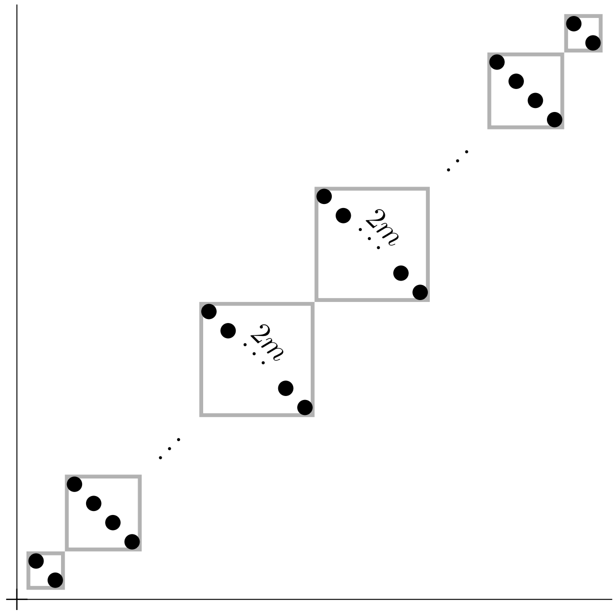

Figure 4: A permutation of length $m(2m+2)$ with longest convex subpermutation of length $4m$.

We provide an explicit construction to establish a lower bound.

Proposition 4.2. *For each positive integer m, there is a permutation of length $m(2m+2)$ that has no convex subpermutation of length $4m+1$.*

Proof. A *layered permutation* is one that can be constructed by replacing each point in an increasing permutation with a decreasing sequence of points, i.e. a layered permutation is a sequence of decreases, each lying above and to the right of the preceding one. Our example is a layered permutation with $2m$ layers. The first m layers have lengths which increase by 2 each time from 2 up to m while the remaining m layers have lengths which decrease by 2 from $2m$ back to 2, see Figure 4. If a convex subpermutation of this permutation contains more than two points from any layer, then it cannot contain points both in layers above and below that one. It follows that the longest convex subpermutation has length $4m$. □

The preceding proposition gives a lower bound for $f(n)$ which is asymptotically $n^2/8$. Thus $\lim_{n\to\infty} f(n)/n^2$ — if it exists — lies between $1/8$ and $1/4$.

REFERENCES

[1] ALBERT, M. H., LINTON, S., AND RUŠKUC, N. The insertion encoding of permutations. *Electron. J. Combin. 12*, 1 (2005), Research paper 47, 31 pp.

[2] BERNINI, A., DISANTO, F., PINZANI, R., AND RINALDI, S. Permutations defining convex permutominoes. *J. Integer Seq. 10*, 9 (2007), Article 07.9.7, 26.

[3] CHUNG, F. R. K. On unimodal subsequences. *J. Combin. Theory Ser. A 29*, 3 (1980), 267–279.

[4] DISANTO, F., FROSINI, A., PINZANI, R., AND RINALDI, S. A closed formula for the number of convex permutominoes. *Electron. J. Combin. 14*, 1 (2007), Research Paper 57, 17 pp.

[5] DISANTO, F., FROSINI, A., RINALDI, S., AND PINZANI, R. The combinatorics of convex permutominoes. *Southeast Asian Bull. Math. 32*, 5 (2008), 883–912.

[6] DUCHI, E., AND POULALHON, D. On square permutations. In *Fifth Colloquium on Mathematics and Computer Science*, Discrete Math. Theor. Comput. Sci. Proc., AI. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2008, pp. 207–222.

[7] ERDŐS, P., AND SZEKERES, G. A combinatorial problem in geometry. *Compos. Math. 2* (1935), 463–470.

[8] FANTI, I., FROSINI, A., GRAZZINI, E., PINZANI, R., AND RINALDI, S. Polyominoes determined by permutations. In *Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities*, Discrete Math. Theor. Comput. Sci. Proc., AG. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2006, pp. 381–389.

[9] FANTI, I., FROSINI, A., GRAZZINI, E., PINZANI, R., AND RINALDI, S. Characterization and enumeration of some classes of permutominoes. *Pure Math. Appl. (PU.M.A.) 18*, 3-4 (2007), 265–290.

[10] MANSOUR, T., AND SEVERINI, S. Grid polygons from permutations and their enumeration by the kernel method. In *Formal power series and algebraic combinatorics (Nankai Univ., 2007)* (2007).

[11] SAGAN, B. E. *The Symmetric Group*, second ed., vol. 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[12] VATTER, V., AND WATON, S. On points drawn from a circle. Preprint.