ON THE RELATIONSHIP BETWEEN HYPERSEQUENT CALCULI AND LABELLED SEQUENT CALCULI FOR INTERMEDIATE LOGICS WITH GEOMETRIC KRIPKE SEMANTICS

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On the Relationship between Hypersequent Calculi and Labelled Sequent Calculi for Intermediate Logics with Geometric Kripke Semantics

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Abstract

In this thesis we examine the relationship between hypersequent and some types of labelled sequent calculi for a subset of intermediate logics—logics between intuitionistic (**Int**), and classical logics—that have geometric Kripke semantics, which we call **Int***/`Geo`.

We introduce a novel calculus for a fragment of first-order classical logic, which we call partially-shielded formulae (or `PSF` for short), that is adequate for expressing the semantic validity of formulae in **Int***/`Geo`, and apply techniques from correspondence theory to provide translations of hypersequents, simply labelled sequents and relational sequents (simply labelled sequents with relational formulae) into `PSF`. Using these translations, we show that hypersequents and simply labelled sequents for calculi in **Int***/`Geo` share the same models. We also use these translations to justify various techniques that we introduce for translating simply labelled sequents into relational sequents and vice versa. In particular, we introduce a technique called "transitive unfolding" for translating relational sequents into simply labelled sequents (and by extension, hypersequents) which preserves linear models in **Int***/`Geo`.

We introduce syntactic translations between hypersequent calculi and simply labelled sequent calculi. We apply these translations to a novel hypersequent framework **HG3ipm*** for some logics in **Int***/`Geo` to obtain a corresponding simply labelled sequent framework **LG3ipm***, and to an existing simply labelled calculus for **Int** from the literature to obtain a novel hypersequent calculus for **Int**.

We introduce methods for translating a simply labelled sequent calculus into a corresponding relational calculus, and apply these methods to **LG3ipm*** to obtain a novel relational framework **RG3ipm*** that bears similarities to existing calculi from the literature. We use transitive unfolding to translate proofs in **RG3ipm*** into proofs in **LG3ipm*** and **HG3ipm*** with the communication rule, which corresponds to the semantic restriction to linear models.

Declaration

I, Robert Rothenberg, certify that this thesis, which is approximately 68.000 words in length (excluding appendices), has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

Date .... Signature of candidate ....

I was admitted as a research student in October 2005 and as a candidate for the degree of Doctor of Philosophy in August 2006; the higher study for which this is a record was carried out in the University of St Andrews between 2005 and 2010.

Date .... Signature of candidate ....

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Doctor of Philosophy in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

Date .... Signature of supervisor ....

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Dedicated to Pegine, Ella and Harry.

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CHAPTER 1

Introduction

1.1. Overview

1.1.1. Sequent Calculi. Sequent calculi were introduced in the 1930s by Gentzen [Gen35] as systems that were amenable to studying the *formal* properties of classical and intuitionistic logics. A **sequent** is an expression $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite lists of logical formulae. The meaning of a sequent is (approximately) that it is true in an interpretation $\Im$ if the truth of all of the formulae in Γ (called the **antecedent**) in $\Im$ imply the truth of some of the formulae in Δ (called the **succedent**) in $\Im$. A **sequent calculus** is a finite set of schematic axioms and rules. A **proof** in a sequent calculus is a (constructed) tree where the nodes are decorated by sequents and the relationship between a node and its children is given as an instance of one of the rules in that calculus. The root sequent of a proof is considered **valid** (that is, true in all interpretations). For example, a proof in a sequent calculus of Peirce's Law is given in Example 1.1 below:

EXAMPLE 1.1. *A proof of Peirce's Law in the calculus* **G1c** *(Figure A.1 on page 233):*

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A, B}\,\mathsf{RW}}{\Rightarrow A, A \supset B}\,\mathsf{R}\supset \quad A \Rightarrow A}{(A \supset B) \supset A \Rightarrow A}\,\mathsf{L}\supset}{\Rightarrow ((A \supset B) \supset A) \supset A}\,\mathsf{R}\supset$$

In some cases, a proof of a single-succedent sequent $\Gamma \Rightarrow A$ corresponds to a natural deduction proof from the formulae in Γ to A. (A proof was given for the sequent and natural deduction systems for intuitionistic logic in [Gen35], where rules in a calculus that affect the left side of a sequent correspond to elimination rules, and rules that affect the right side of a sequent correspond to introduction rules.) For example, the sequent inference

$$\dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A}\,\mathsf{L}\wedge_1$$

corresponds to the natural deduction inference

$$\frac{A \wedge B}{A}\ \wedge\mathsf{E}$$

Such sequent calculi can be used as a means of studying natural deduction proofs.

A desirable feature of a sequent calculus is the cut-admissibility property—that is, adding the cut rule,

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

(which corresponds to the application of a lemma in a proof) to a calculus does not increase the strength of the calculus. One shows cut is admissible by showing that a sequent proven with the use of the cut rule can be proven without the use of the cut rule. This can be done proof-theoretically by showing that a proof which contains instances of cut can be rewritten as a proof without instances of cut. Generally this involves permuting instances of cut over other rules and eliminating cuts from the axioms.

The admissibility of the cut rule can be used to show that a calculus is *complete* with respect to a Hilbert-style system for that logic by showing that the schematic axioms of that system are derivable in the calculus, and using the modus ponens rule,

$$\frac{\Rightarrow A \quad A \Rightarrow B}{\Rightarrow B}\ \mathsf{MP}$$

(which is clearly a special case of cut). Because instances of cut can be eliminated, the system without cut is also complete. (Recall that a Hilbert-style system for a logic consists of schematic axioms, modus ponens and substitution.)

There are other advantages of having cut-free systems. Depending on the properties of the calculus, it may be more amenable to mechanical proof search. It may also allow one to show that the calculus is consistent, e.g., by showing that $\Rightarrow A \wedge \neg A$ is not derivable.

Sequent calculi are also useful formalisms for developing the proof theory of non-classical logics that do not have an obvious notion of natural deduction. However, for some logics (such as modal or superintuitionistic logics), sequent calculi appear inadequate for developing cut-free systems, or do not allow for other desired features, such as having permutable rules for all logical connectives, e.g. see [Avr96, Wan98a].

1.1.2. Extensions to Sequent Calculi. Since the 1950s, various extensions to Gentzen-style sequent calculi for non-classical logics have been proposed. These were generally due to considerations stemming from the development of relational semantics by Beth [Bet56], Kripke [Kri59a, Kri59b], Hintikka [Hin62], and others.[1] Recall that relational models consist (informally) of a set of points (also called "states" or "worlds") and a set of relations between points. The truth of a formula at a point may be dependent on the related points in the model. (Formal descriptions of relational models for some intermediate logics are given in Chapter 3.)

The literature contains various extensions of Gentzen-style sequent calculi. These extensions can be grouped into the categories outlined below (similar to those given in [Pao02], although expanded to include multi-arrow sequents and deep inference), with brief informal descriptions of specific extensions that we will examine in this thesis. These categories are determined by the syntactic part of the sequent calculus that is extended:

Higher-Order Systems: Higher-order extensions contain *collections* of sequents, where the location of the sequent often corresponds to a point in the relational semantics. These include hypersequents (lists of sequents, e.g. [Avr91a], briefly introduced below) and higher-order sequents (sequents of sequents, e.g. [Doš85a]). Generally the relationships between points are implicit, although some formalisms such as [Cer93, Pog08a] explicitly structure the collections as trees.

Hypersequents are non-empty, finite multisets of sequents (called **components**), and are written as

$$\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

where the pipe operator "|" denotes a meta-level disjunction. Informally, a hypersequent is true in some interpretation if at least one of its components is true in that interpretation. Depending on the rules of a hypersequent calculus, formulae can be exchanged between components, for example, the characteristic rule

[1] Chagrov and Zakharyaschev [CZ97, p. 57] give a brief historical bibliography of the development of relational semantics.

for Jankov-De Morgan logic from [Avr96]:

$$\frac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow}{\mathcal{H} \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow} \ \mathsf{LQ}$$

Hypersequents are discussed in detail in Chapter 4.

Labelled Systems: Labelled systems (e.g. [Vig00, Neg07]) are extensions of sequent calculi where formulae are annotated with terms that correspond to a point in the relational semantics for the corresponding logic. The sequents may also include additional kinds of formulae that explicitly indicate the relationships between points. We note the kinds of labelled systems that are discussed in this thesis:

Simply labelled sequents are sequents consisting of formulae annotated with atomic labels, e.g. A^x, but are otherwise written as normal sequents. Schematic sequents are written so that variables that denote lists of arbitrarily labelled formulae are underlined, e.g. $\underline{\Gamma}$, and variables that denote lists of formulae with that label, e.g. Γ^x. Depending on the rules of a simply labelled calculus, formulae can be relabelled, in much the same way that formulae can be exchanged between components of a hypersequent calculus, for example, the simply labelled form of the LQ rule (introduced in Section 7.2.1 later):

$$\frac{\Gamma_1^z, \Gamma_2^z, \underline{\Gamma'} \Rightarrow \underline{\Delta}}{\Gamma_1^x, \Gamma_2^y, \underline{\Gamma'} \Rightarrow \underline{\Delta}} \ \mathsf{LQ}$$

where x, y, z are distinct and do not occur in $\underline{\Gamma'}, \underline{\Delta}$.

Simple relational sequents (or "relational sequents," for short) are simply labelled sequents extended with a multiset of **relational formulae** of the form $x \leq y$, where x, y are atomic labels. Relational sequents are written as $\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$, where Σ is a multiset of relational formulae. A **relational calculus** is a calculus for relational sequents, and may also contain rules that affect the relational formulae, e.g. the directedness rule from [DN10]:

$$\frac{w \leq x, w \leq y, x \leq z, y \leq z, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{w \leq x, w \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}} \ \mathsf{dir}$$

Simply labelled and simple relational sequents are discussed in detail in Chapter 5.

An alternative notation for relational sequents is to incorporate the relations into the labels, so that the relation between two labels can be determined by comparing them, rather than maintaining a separate multiset of relational formulae. For example, formulae in a **prefix sequent** are labelled with lists of atoms, and the relation $x \leq xy$ holds because x is a prefix of xy. (In prefix calculi, pure relational rules may be omitted, as the relationships are implicit in the labels.)

Another alternative notation is an **indexed sequent**, which is represented as a hypersequent with prefix calculus-style labels annotating the sequent arrows of the components, rather than individual formulae. Indexed sequents can be seen as forms of prefix sequent or simply labelled sequent.

The syntax of labelled sequents can be extended further, to allow relational formulae in the succedent, compound terms with operators as labels, higher-arity relational formulae, multiple kinds of relational formulae, or variables in labels. (Various extensions will also be discussed later in Chapter 5.)

Multi-sided Sequent Systems: By considering the two sides of a sequent as corresponding to the truth values of bivalent logics, one can extend sequents for finitely-many-valued logics such as $\mathbf{G_k}$ or $\mathbf{Ł_k}$, e.g. [Rou67, Tak70]. In many ways, these systems resemble higher-order systems such as hypersequents, although they have different motivations.

Meta-Rule Systems: One may incorporate the meta-rules of a logic into the object language of the calculus, e.g. Dunn-Mints systems, [Dun73, Min76, Pao02] or display calculi [Bel82, Wan94]. [Pao02] considers these as extensions of the comma in sequents with multiple types of comma.

Multi-arrow Sequent Systems: One can also extend sequent systems by allowing multiple types of sequent arrows. For example, [BF99, BCF01, BCF03b, CFM04] introduce "sequents of relations" for fuzzy logics that use $\leq$ and $<$ in lieu of the sequent arrow $\Rightarrow$.

Deep Inference Systems: Another extension is to allow rules to be applied at arbitrary places inside the structure of a sequent, e.g. [Brü04, Pog08a].

Many systems incorporate features of more than one extension. For example, [CFM04] can be considered a multi-arrow higher-order extension, and [Pog08a] can be considered a deep-inference higher-order extension. (Note that we consider tableau systems to be inverted forms of sequent calculi and their extensions, and thus an alternative notation rather than as a distinct formalism—an issue noted by [Fit99]).

We can consider these formalisms such as rules for manipulating *data structures*. From this, we need to consider consequence relations between more complex data structures than sets of formulae or sequents—a point noted by various authors, such as Gabbay [Gab96], Restall [Res00] and Paoli [Pao02]. (The topic is beyond the scope of this thesis.)

The distinction between logical and structural rules is one between rules which (mostly) manipulate the logical formulae and rules which (mostly) manipulate the data structures. Depending on the formalism, further distinctions of structural rules can be made. For example, hypersequents allow one to distinguish between rules which affect one or more components (internal and external rules), and labelled sequents allow one to distinguish between rules which only affect labels or relational formulae (labelling and relational rules), and other structural rules.

Likewise, the corresponding notion of what a logic is needs to be extended to include the data structures derived in these formalisms, particularly when it is not clear how to translate arbitrary structures into sets of formulae, or when the data structures may contain non-logical formulae, such as relational sequents.

1.1.3. Relationship between Formalisms. The question of how these extended "logics" relate to one another is an open one. Can one translate formulae or structures in one logic into those of another, particularly when the formulae of one logic is of a form that does not occur in another? Can a formalism express concepts that are not expressible in other formalisms, or can it not express concepts that are expressible in others? Similarly, how are these different data structures used for the same, or similar logics, related to one another?

This question helps to motivate the need for a translation between formalisms. For example, the sequent $x \leq y; (A \vee B)^x \Rightarrow A^x, B^y$ is derivable in the relational sequent calculus **G3I** [DN10] for the $\mathbf{Int}_{\leq}$ extension of **Int**, but it is not at all obvious that it has a corresponding formula in **Int**. Note that the relational formula $x \leq y$ refers to the relational semantics of **Int**, an in itself does not correspond to a particular logical formula in **Int**, so it is difficult to give a “good” translation from arbitrary relational sequents to sequents or hypersequents that does not incorporate the semantics of the logic. (This problem is addressed in Chapters 8 and 9.)

Although the the above noted extensions to Gentzen-style calculi have different motivations, the underlying embedding of relational semantics for many of the logics that these proof systems are designed for (particularly when they are for the same logics, such as the logics discussed in this thesis) suggests a formalisable relationship between these kinds of proof systems. A formal model can be considered to be an alternative notation for reasoning in a particular logic, so it follows that different logical calculi for the same kind of logic (and model) can be considered alternative notations for one another, and that the model can be used as an intermediate construction for translating between these systems. Our conjecture is that some forms of labelled calculi can be considered as an equivalent *alternative notation* for these extensions (being close representations of the corresponding relational models). Labelled calculi can be seen as incorporating the names of locations and their relationships into the object language. While this can be seen as “polluting” a calculus with semantic notation (see Section 2.3), this can also been seen as enabling one to reason about a formalism that a labelled calculus is considered an alternative notation for. Thus, labelled systems may be useful as intermediate formalisms for reasoning about or translating between other formalisms.

The larger picture (to which this thesis is a small contribution) is to show that these extensions to sequent calculi can be *mechanically* translated between each other. A mechanical translation between formalisms is of interest with respect to proof assistant software, as it would allow the user of such tools to choose the kind of formalism to work in, independent of the underlying implementation. In one sense, this is in keeping with the Model-View-Controller [Ree79] technique of separating the user interface from the implementation details. Having such a translation motivates research into implementation techniques for various formalisms

A mechanical translation may also be of use by allowing one to work in systems that are more suited to proving certain kinds of properties, such as interpolation or cut-elimination, and then translate the proofs into other formalisms. Proofs of interpolation are worthwhile candidates for translation, because generally they are given using single-succedent sequent calculi, e.g. in [TS00]. The only exception to this that we are aware of are proofs of the interpolation property for some modal logics using deep inference systems in [Bíl09].

Having translations between formalisms gives formal comparisons between them, and can be used to formalise much of the criteria on judging what makes a "good" formalism. (The literature gives many such criteria, which often seems more ad hoc or philosophical than based on measurable rationale. Much of the terminology is discussed later in Chapter 2.)

1.2. Synopsis of this Thesis

1.2.1. Motivation. The specific aim of this thesis will be to formally describe the relationship between two kinds of extensions of sequent calculi, hypersequents and some kinds of labelled calculi, that have been used to provide frameworks for some intermediate logics (logics between intuitionistic and classical logic, inclusive).

The intuition behind this thesis is explained by supposing we have the following hypersequent:

$$B, C \Rightarrow D \mid A, B, C \Rightarrow D \mid A, B, B, C => A \tag{1}$$

Rather than using the standard hypersequent notation above, we could use names for the components that formulae occur in:

$$A^y, A^z, B^x, B^y, B^z, B^z, C^x, C^y, C^z \Rightarrow D^x, D^y, A^z \tag{2}$$

This is a reasonable choice with regards to implementation: if the rules of a hypersequent calculus generally do not require manipulating components as a whole, then it may be more efficient to implement a theorem prover for a hypersequent calculus by simplifying the data structure, and associating each formula in the hypersequent with a label that corresponds to the component, rather than maintain distinct lists of sequents which must be handled separately. (Indeed, this came about as part of the original research project behind this work, which was to explore ways of utilising hypersequents as a basis for automated reasoning.)

However, we can consider the sequent in (2) to be a simply labelled sequent. So it appears that hypersequents and simply labelled sequents are a different notation for the same abstract structure. A comparison of hypersequent and simply labelled rules from the frameworks from **HG3ipm*** (introduced in Section 4.4.4) and **LG3ipm*** (introduced in Section 7.2.1), respectively, is given in Figure 1.1. (Note that the hypersequent rules are multisuccedent variants of the rules from [Avr91a].)

Logic	Hypersequent Rule	Labelling Rule
Jan	$\dfrac{\mathcal{H}\vert\Gamma_1,\Gamma_2 \Rightarrow}{\mathcal{H}\vert\Gamma_1 \Rightarrow \vert\Gamma_2 \Rightarrow}\ \mathsf{LQ}$	$\dfrac{\Gamma_1^z,\Gamma_2^z,\underline{\Gamma'} \Rightarrow \underline{\Delta}}{\Gamma_1^x,\Gamma_2^y,\underline{\Gamma'} \Rightarrow \underline{\Delta}}\ \mathsf{LQ}$
GD	$\dfrac{\mathcal{H}\vert\Gamma_1 \Rightarrow \Delta_1,\Delta_2\vert\Gamma_1,\Gamma_2 \Rightarrow \Delta_1 \quad \mathcal{H}\vert\Gamma_1,\Gamma_2 \Rightarrow \Delta_1\vert\Gamma_2 \Rightarrow \Delta_1,\Delta_2}{\mathcal{H}\vert\Gamma_1 \Rightarrow \Delta_2\vert\Gamma_2 \Rightarrow \Delta_2}\ \mathsf{Com_m}$	
		$\dfrac{\Gamma_1^x,\Gamma_1^y,\Gamma_2^y,\underline{\Gamma'} \Rightarrow \underline{\Delta}',\Delta_1^x,\Delta_2^x,\Delta_2^y \quad \Gamma_1^x,\Gamma_2^x,\Gamma_2^y,\underline{\Gamma'} \Rightarrow \underline{\Delta}',\Delta_1^x,\Delta_1^y,\Delta_2^y}{\Gamma_1^x,\Gamma_2^y,\underline{\Gamma'} \Rightarrow \underline{\Delta}',\Delta_1^x,\Delta_2^y}\ \mathsf{Com_m}$
Cl	$\dfrac{\mathcal{H}\vert\Gamma_1,\Gamma_2 \Rightarrow \Delta_1,\Delta_2}{\mathcal{H}\vert\Gamma_1 \Rightarrow \Delta_1\vert\Gamma_2 \Rightarrow \Delta_2}\ \mathsf{S}$	$\dfrac{\Gamma_1^z,\Gamma_2^z,\underline{\Gamma'} \Rightarrow \underline{\Delta}',\Delta_1^z,\Delta_2^z}{\Gamma_1^x,\Gamma_2^y,\underline{\Gamma'} \Rightarrow \underline{\Delta}',\Delta_1^x,\Delta_2^y}\ \mathsf{S}$

FIGURE 1.1. A comparison of multisuccedent hypersequent and labelling rules.

By observing the relationship between labelling rules from **LG3ipm*** and the corresponding relational rules given by Dyckhoff and Negri in [DN10] from **G3I*** (Figure 5.4 on page 122) for the intermediate logics in Figure 1.2 on the next page, we note that the relational formulae in the labelling rules can be considered as abbreviations that incorporate the persistence property of intuitionistic Kripke frames: for all propositional variables P and points x,y in the frame, Rxy and $x \Vdash P$ imply $y \Vdash P$. Indeed, we chose to investigate the correspondence between calculi for intermediate logics in part because they have the persistence property in common, which motivates the translation. (Another motivation is the existence of hypersequent and labelled frameworks for an overlapping subset of intermediate logics that can be used as objects of study.)

Note that the correspondence between rules in Figure 1.2 on the following page is not exact: the simply labelled LQ rules actually corresponds to a variant of the dir rule, which does not require a common root.

Logic	Labelling Rule	Relational Rule
Jan	$\dfrac{\Gamma_1^z,\Gamma_2^z,\underline{\Gamma}' \Rightarrow \underline{\Delta}}{\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}' \Rightarrow \underline{\Delta}}\ \mathsf{LQ}$	$\dfrac{x \le z, y \le z, w \le x, w \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{w \le x, w \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{dir}$
GD	$\dfrac{\Gamma_1^x,\Gamma_1^y,\Gamma_2^y,\underline{\Gamma}' \Rightarrow \underline{\Delta}',\Delta_1^x,\Delta_2^x,\Delta_2^y \quad \Gamma_1^x,\Gamma_2^x,\Gamma_2^y,\underline{\Gamma}' \Rightarrow \underline{\Delta}',\Delta_1^x,\Delta_1^y,\Delta_2^y}{\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}' \Rightarrow \underline{\Delta}',\Delta_1^x,\Delta_2^y}\ \mathsf{Com_m}$	$\dfrac{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta} \quad y \le x, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{lin}$
Cl	$\dfrac{\Gamma_1^z,\Gamma_2^z,\underline{\Gamma}' \Rightarrow \underline{\Delta}',\Delta_1^z,\Delta_2^z}{\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}' \Rightarrow \underline{\Delta}',\Delta_1^x,\Delta_2^y}\ \mathsf{S}$	$\dfrac{x \le y, y \le x, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{sym}$

FIGURE 1.2. A comparison of labelling and relational rules.

By using the persistence property as a means for abbreviating the antecedent of the simply labelled sequent and making the relationships between labels explicit in (2), we obtain the following:

$$x \le y, y \le z; A^y, B^x, C^x \Rightarrow D^x, D^y, A^z \tag{3}$$

which is a simple relational sequent. One problem with using shared formulae between labels to determine relations is that there is not a one-to-one correspondence between simply labelled sequents and relational sequents. The sequent in (2) may also be equivalent to

$$x \le y; A^y, A^z, B^x, B^z, B^z, C^x, C^z \Rightarrow D^x, D^y, A^z \tag{4}$$

$$y \le z; A^y, B^x, B^y, C^x, C^y \Rightarrow D^x, D^y, A^z \tag{5}$$

However, all of the relational sequent variants of (2) in some sense belong to the same equivalence class, since they all are abbreviated forms of the same simply labelled sequent. (We consider a notion of a normal form of relational sequents in Chapter 8.)

The notation for relational sequents can be further simplified, by absorbing the relations into the labels, for example, as

$$A^{23}, B^{123}, B^{123}, C^{123} \Rightarrow D^{12}, A^{123} \tag{6}$$

which is a prefix sequent. An advantage of prefix sequents over labelled sequents with relational terms is that the relationship between labels is determined simply by examining the labels, rather than referring to a separate set of relations (and possibly inferring a relation using transitivity, where it is present in a system)—in the above example, $1 \leq 12$ and $12 \leq 123$, which corresponds to $x \leq y$ and $y \leq z$ if we assume $x = 1$, $y = 12$ and $z = 123$. However, a disadvantage of prefix sequent notation is that the relationships between labels that can expressed is limited. (We discuss these disadvantages in more detail later in Chapter 5.)

Another variation on the notation of relational sequents is an indexed sequent, which combines the notation of hypersequents with prefix sequents, for example,

$$B, C \Rightarrow^{12} \mid A \Rightarrow^{23} D \mid B, C \Rightarrow^{123} A \tag{7}$$

In an indexed sequent, each component is labelled, but the labels indicate relationships between components.

As noted above, our conjecture is that we can view labelled calculi as an *alternative notation* for a given formalism, such as hypersequents, where labels denote the locations of formulae within a data structure, and relational formulae can be used to denote relationships between locations of a data structure. This allows information about the data structure can be incorporated in the object language of the calculus. By transforming labelled systems, they can be used as intermediaries for translations between other formalisms.

Thus we look at proof calculi as systems which operate on abstract *data structures* that contain formulae and other information pertaining to proof construction, rather than as systems which only operate on formulae or sets of formulae. In some cases this extra information may be implicit, such as the location of a formula in the structure. Or it may be explicit, such as the label. Likewise, the distinction between logical and structural rules can be viewed as the difference between whether (mostly) formulae or the data structure are manipulated. (Though for some logics, rules may do both.)

So a proof in a Hilbert-style system can be thought of as having the simplest structure: trees whose vertices contain a single formula. Sequents are more complex: trees with pairs of lists of formulae at the vertices. And hypersequents are trees with lists of sequents at the vertices.

No claims are being made in this thesis that labelled calculi are more amenable to efficient automation than hypersequents. Indeed, it might be that hypersequent-like implementations that keep formulae in distinct areas of memory may be more suited for parallel proof search. Having translations between formalisms motivates further research into transforming proof systems into variants suitable for parallelisation.

1.2.2. Outline of this Thesis. In the remainder of this chapter, we discuss related work, and note translations between specific calculi, as well as more general work on labelled calculi and hypersequents.

In Chapter 2, we define the general notation and terminology used in this thesis. We also discuss terminology and criteria for evaluating a proof system in Section 2.3.

In Chapter 3, we discuss several intermediate logics, as well as their corresponding Kripke semantics. We also identify a class of intermediate logics with geometric Kripke semantics (**Int***/`Geo`), to which most of the logics we discuss belong, and to which the techniques used in this thesis are applicable. (We will also briefly discuss Beth semantics, as it is relevant to some of the calculi discussed later in this thesis.) Finally, we introduce a novel cut-free framework of calculi **G3c**/**PSF*** for reasoning about the subset of first-order classical formulae (called partially shielded formulae, or `PSF`) that is adequate for expressing the truth conditions of formulae in **Int***/`Geo` in their corresponding geometric Kripke models. **G3c**/**PSF*** is used later in this thesis to show the semantic the relationship between formalisms, as well as to show the soundness of various techniques for manipulating the formalisms.

In Chapter 4, we introduce the notation and terminology of hypersequent calculi and their general properties, including issues with rule permutation and cut admissibility. We discuss the technique of creating hyperextensions of sequent calculi. We follow with a survey of several hypersequent calculi for intuitionistic logic and extensions for some intermediate logics, including a novel framework for intermediate logics, **HG3ipm*** based on a multisuccedent hyperextension of a calculus for **Int**. (**HG3ipm*** is used as a basis for further translations in later chapters.) We give a general method for obtaining hypersequent rules from geometric rules.

In Chapter 5, we introduce notation and terminology for various kinds of labelled sequent calculi, focusing on simply labelled and relational calculi. (Prefix calculi will be discussed briefly.) We provide examples of these calculi from the literature, emphasising calculi for **Int**, or adapting calculi for the modal logic **S4** to calculi for **Int** using the Gödel translation (discussed in Chapter 3). We also introduce the notion of equivalence modulo permutation of labels.

In Chapter 6, we show a *semantic* correspondence between hypersequent and simply labelled sequent calculi, using techniques from correspondence theory to provide a translation of sequents in intermediate logics into PSF (as introduced in Chapter 3).

In Chapter 7, we provide methods for translating between hypersequent and simply labelled sequent calculi. We use these methods to introduce a novel hypersequent calculus for **Int** with invertible rules, based on Maslov's labelled calculus **O** [Mas67] (given in Chapter 5), and we translate the hypersequent framework $\mathbf{HG3ipm}^*$ for intermediate logics introduced in Chapter 4 into the simply labelled framework $\mathbf{LG3ipm}^*$.

In Chapter 8, we revisit the semantics of the formalisms and show that there is a correspondence between simply labelled sequents and simple relational sequents that preserves *linear* models, which corresponds to logics at least as strong as **GD**.

In Chapter 9, we discuss how to extend a simply labelled sequent calculus into a relational calculus, and apply this technique to obtain the relational calculus $\mathbf{RG3ipm}'$ for **Int**. We improve the technique so that it preserves cut admissibility, and we translate the simply labelled framework from Chapter 7 into a relational framework $\mathbf{RG3ipm}^*$. We also show that this framework is equivalent to an existing framework $\mathbf{G3I}^*$ [DN10] for intermediate logics. Finally, we introduce a technique for translating proofs in $\mathbf{RG3ipm}^*$ into proofs in the simply labelled framework $\mathbf{LG3ipm}^* + \mathsf{Com}_{\mathsf{m}}$. (We examine issues related to extending the translation to weaker logics in Appendix H.)

In Chapter 9, we also investigate a conjecture that geometric relational rules can be translated into simply labelled and hypersequent rules that preserve cut admissibility.

In the Conclusion (Chapter 10), we sum up the work of this thesis, and discuss open problems and areas for future work.

1.2.3. Discussion. The translation from hypersequents to simply labelled sequents is fairly straightforward. The only complications are using the notion of equivalence between labelled sequents modulo renaming of labels, and hypersequents with empty components, e.g. $\Gamma \Rightarrow \Delta \mid \ \Rightarrow$ (discussed in Appendix E).

The translation from simply labelled sequents to relational sequents requires more care. One issue is that the general semantics of hypersequents and simply labelled sequents are disjunctive—that is, they are true in an interpretation $\mathfrak{I}$ iff at least one component or label "slice" is true in $\mathfrak{I}$, whereas relational sequents have a similar general semantics to Gentzen-style sequents—they are true in $\mathfrak{I}$ iff the truth (all of) the antecedent formulae in $\mathfrak{I}$ implies the truth of (some of) the succedent formula in $\mathfrak{I}$. This is addressed in Chapters 6 and 8 by translating both kinds of sequents into the corresponding first-order formulae for intuitionistic (and intermediate) Kripke Models, and showing that the translations are equivalent using **G3c**/**PSF***.

Another issue is the addition of relational formulae to the sequent. The folding rules,

$$\frac{\Sigma; A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\leq' \qquad \frac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ \mathsf{R}\leq'$$

are introduced semantically in Chapter 8 and as logical rules in Chapter 9. They are based on the persistence property of intuitionistic Kripke models, and are necessary for the translated calculi to be complete with respect to an extension of the corresponding logic with relational formulae. (This requires we extend the notion of a "logic" to include relational formulae, discussed in Chapter 5.)

We also show how to translate proofs in a simple relational calculus into proofs in a simply labelled calculus. This is done by a technique called **transitive unfolding**, which unfolds the relations so as to preserve transitive relationships between labels. Unfortunately, the resulting hypersequents and simple labelled sequents from transitively unfolding do not always share the same models and countermodels as the original relational sequents. In certain conditions, the relational sequents may be derivable in **Int**, whereas the simply labelled sequents or hypersequents resulting from the unfolding are derivable in the stronger logic **GD** (which corresponds to **Int** with linear models).

This is due to the difference in languages: the relationship between labels is explicit for relational sequents. Note the relational sequent calculus derivation fragment below:

$$\frac{x \leq y, \Sigma; A^x \Rightarrow \underline{\Delta} \quad x \leq y, \Sigma; B^x \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; (A \vee B)^x \Rightarrow \underline{\Delta}}\ \mathsf{L}\vee$$

In the premisses, A^y and B^y occur implicitly due to the persistence property. On the other hand, in the following derivation fragment for a hypersequent calculus:

$$\dfrac{\dfrac{A\Rightarrow\Delta \mid A\Rightarrow\Delta' \quad A\Rightarrow\Delta \mid B\Rightarrow\Delta'}{A\Rightarrow\Delta \mid A\vee B\Rightarrow\Delta'}\,\mathsf{L}\vee \qquad \dfrac{B\Rightarrow\Delta \mid A\Rightarrow\Delta' \quad B\Rightarrow\Delta \mid B\Rightarrow\Delta'}{B\Rightarrow\Delta \mid A\vee B\Rightarrow\Delta'}\,\mathsf{L}\vee}{A\vee B\Rightarrow\Delta \mid A\vee B\Rightarrow\Delta'}\,\mathsf{L}\vee$$

the persistence property is not preserved in the two middle premisses. If we use subset relations between components to determine their corresponding relational formulae in a relational calculus, then those two middle premisses function as a counter-example to the translation, unless the hypersequent calculus admits some form of the communication rule (Com' in Figure 1.2 on page 10).

Another aspect of the implicit relationship between components in hypersequents is that the exact relationship cannot always be determined. For example, the hypersequent $A => A \mid A => A \mid A =>$ can be translated into $x \le y, y \le z; A^x \Rightarrow A^y$ or $x \le y, x \le z; A^x \Rightarrow A^y$. This suggests the need for canonical forms of hypersequents and relational sequents that have a one-to-one correspondence. We briefly investigate this question in Chapter 8, but do not find a solution. Such a notion appears to be connected to the problem of translating proofs to canonical proofs and showing proof equivalence, which is a larger problem that is beyond the scope of this thesis.

1.3. Related Work

We are not aware of any prior work on formal translations *between* hypersequent and labelled calculi *in general*, or for related families of calculi for particular families of logics. There is an informal discussion of the relationship between these formalisms in [Brü09a, Brü09b], where it is noted that labelled sequents are much more expressive (in terms of specifying explicit relationships) than other formalisms, such as hypersequents.

There is some work on formal translations between specific hypersequent, display and labelled calculi for specific systems, and (of more interest), some general work (discussed in the next subsection) on obtaining labelled systems from arbitrary logics, and on translating Hilbert-style axioms into structural sequent and hypersequent rules.

1.3.1. Specific Translations. Avron [Avr96] notes that a tableau system for the modal logic **S5** given by Mints [Min92] is essentially a hypersequent calculus with different notation:

$$\frac{\mathcal{H} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{L}\Box \qquad \frac{\mathcal{H} \mid \Gamma, A \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma', \Box A \Rightarrow \Delta'}\ \mathsf{L}\Box' \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Box A, \Delta}\ \mathsf{R}\Box$$

is roughly equivalent to the simply labelled system for **S5** given by Kanger [Kan57]:

$$\frac{\Gamma, A^y, \Box A^x \Rightarrow \Delta}{\Gamma, \Box A^x \Rightarrow \Delta}\ \mathsf{L}\Box_\dagger \qquad \frac{\Gamma \Rightarrow A^y, \Delta}{\Gamma \Rightarrow \Box A^x, \Delta}\ \mathsf{R}\Box_\dagger$$

(where y is fresh in $\mathsf{R}\Box_\dagger$), where the labels correspond to components in a hypersequent. The hypersequent rule $\mathsf{L}\Box$ corresponds to the labelled rule $\mathsf{L}\Box_\dagger$ where $y = x$, and the hypersequent $\mathsf{L}\Box'$ corresponds to the labelled rule $\mathsf{L}\Box_\dagger$ by the use of weakening and contraction. (Avron also notes that Mints' system is equivalent to the system that he introduces in [Avr96].) No formal translation is given between the calculi in [Avr96], nor is the relationship between hypersequents and labelled calculi generalised to other calculi.

Wansing [Wan98b] gives formal translations of hypersequents into display sequents for the logics **Ł₃**, **GD** and **S5**, based on hypersequent calculi introduced by Avron in [Avr91b, Avr91a, Avr96]. Although the translations are specific to each logic, it is noteworthy that the hypersequent "|" is translated using the same structural connective in display logic.

Restall in [Res06, Res07] presents a display calculus for normal modal logics, building on ideas from [Wan94, Wan98a]. A simple relational calculus is also given, as well as a procedure for translating display sequents and proofs into relational sequents and proofs, using the Kripke semantics of modal logics as a basis. Restall then replaces the labelled sequent with a hypersequent annotated with arrows between components in lieu of relational formulae to further simplicity the notation.In the case of **S5**, which has a universal accessibility relation, the arrows can be removed as redundant, leaving a hypersequent system.

P. Girard [Gir05] notes that Hacking's "starred calculi" for modal logics [Hac63] (which use the strict implication arrow "⥽" from [HC68]), where sequents are written as $\Gamma, \Theta^* \Rightarrow \Delta, \Lambda^*$, can be thought of as a hypersequents with at most two components,

$\Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda$. Girard adapts Hacking's notation into labelled hypersequent (Mints' indexed sequents [Min97]) rules, e.g.

$$\frac{\Gamma \Rightarrow \Delta \mid A \Rightarrow B}{\Gamma \Rightarrow \Delta, A \dashv\!\!3\, B}\ \mathsf{D}$$

Girard notes that Hacking's rule for **S5**,

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma^{*} \Rightarrow \Delta^{*}}\ {*2}$$

(where no formula in the premiss is labelled with $*$), becomes superfluous in the hypersequent calculus, and the elimination of the $*2$ rule simplifies cut-elimination. Girard then applied this technique to Nelson's calculus of strong negation **N3** [Nel49].

Metcalfe et al [MOG05] introduce hypersequent calculi for Abelian (**A**) and Łukasiewicz logics (**Ł**) where some of the rules lead to an exponential growth in the number of components (discussed later in Section 4.3.3 on page 76), e.g.

$$\frac{\mathcal{H} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \supset B \Rightarrow \Delta}\ \mathsf{L}{\supset}_{\text{Ł}_\iota}$$

As an alternative, they introduce simply labelled calculi that incorporate the complexity growth into the labels, e.g.

$$\frac{\underline{\Gamma}, B^{x}, B^{y} \Rightarrow A^{x}, A^{y}, \underline{\Delta}}{\underline{\Gamma}, (A \supset B)^{x} \Rightarrow \underline{\Delta}}\ \mathsf{L}{\supset}_{\text{Ł}_\iota \dagger}$$

(where y is fresh for the premisses). The actual notation appends atomic labels onto the existing label, but the systems only require one of the atomic labels to be significant, so they are equivalent to simply labelled calculi. A straightforward translation is provided from hypersequents into simply labelled sequents.

1.3.2. General Translations. Ciabattoni, Galatos and Terui [CGT08] introduce an algorithm for translating Hilbert-style systems into (hyper)sequent calculi that admit cut. This allows one to use the axiomatisations of various substructural logics (including some intermediate logics) to construct cut-free systems (with a single-succedent Full Lambek **FL** calculus as a base), by deriving a set of structural rules for each axiom. Of note is that they identify a *hierarchy of substructural logics* which is determined by the structure of the axioms. Certain levels of the hierarchy require the use of hypersequent calculi rather than sequent calculi. They also provide a technique called **completion** for transforming hypersequent rules into rules which admit cut, by semantic proof. (Some of their work on justifying that hypersequent rules admit cut is used to justify a conjecture about translating rules in Chapter 9.)

Gabbay [Gab96] presents a general theory of relational proof systems, called labelled deductive systems (or LDS, for short). Of note, he introduces a method of deriving a relational semantics for an arbitrary (non-labelled) calculus (a topic of interest in its own right) and then using that semantics as a basis for a relational calculus. We will omit a description of how to obtain a relational semantics from an arbitrary calculus, as that requires much technical detail that is irrelevant to this thesis (as the semantics of the logics studied are already known). An outline of the procedure for obtaining relational rules from semantic properties is as follows: the rules of the calculus are translated using this semantics, and the propositional variables in the rules are universally quantified over in second-order classical logic. The formulae are negated, and an algorithm (called "SCAN") [GO92, Gab96, GSS08] is applied to remove the second order quantifiers. The remaining clauses are resolved and "unskolemised", returning a negated form of the first-order formula that corresponds to the rule.

EXAMPLE 1.2. *From* [GO92, GSS08]*, we translate the modal axiom* 4*,* $\Box P \supset \Box\Box P$*, to its corresponding first-order frame axiom. First the formula is translated to its corresponding first-order formula using the standard translation of modal operators (see* [BdY01]*), and the predicates corresponding to propositional variables are quantified over as well:*

$$\forall P.\forall xy.(Rxy \supset Py) \supset \forall y.(Rxy \supset \forall z.Ryz \supset Pz)$$

The formula is then negated:

$$\exists P.\exists x.(\forall y.Rxy \supset Py) \wedge \exists y.(Rxy \wedge \forall z.(Ryz \wedge \neg Pz))$$

The SCAN algorithm is applied, resulting in the clauses (in normal form):

$$\neg Ray \vee Py, Rab, Rbc, \neg Pc$$

P is removed by resolution, returning $\neg Rac, Rab, Rbc$, which is unskolemised to

$$\exists abc.(\neg Rac \wedge Rab \wedge Rbc)$$

the negation of which is transitivity (which is the frame condition that 4 *corresponds to):*

$$\forall abc.(Rab \wedge Rbc) \supset Rac$$

The form of labelling rules that correspond to first-order axioms depends on the calculus, and is not explicitly covered.

The work in this thesis differs from Gabbay's by examining different labelled formalisms used to represent a related family of logics. In essence, we are approaching labelled calculi as an alternative notation about the relative *locations* of formulae within a data structure (such as a hypersequent), rather than as a notation that comes about by analysis of the consequence relations or connectives in a logic. We are considering the relationship between formalisms in general (although for a family of logics with common properties), rather than the logics themselves. We are also interested in using this relationship to translate labelled systems into non-labelled systems, such as hypersequent calculi.

1.4. Contributions of this Thesis

We list a summary of the contributions of this thesis below:

We identify a fragment of first-order logic PSF for expressing the truth conditions of Kripke models that correspond to logics in **Int***/Geo(the class of intermediate logics with geometric frame axioms), and we introduce a framework of sequent calculi **G3c/PSF*** for reasoning about these Kripke models. We use **G3c/PSF*** to demonstrate properties of these models by proof-theoretic means. We also provide translations of hypersequents, simply labelled sequents and relational sequents to and from sequents of formulae in PSF.

We show that hyperextensions of sequent calculi are conservative, and we introduce a novel multisuccedent hypersequent framework **HG3ipm*** for some intermediate logics in **Int***/Geo.

We give translations between hypersequents and simply labelled sequents that preserve models. We extend the translations to rules of hypersequent and simply labelled calculi, and use them to obtain a novel simply labelled framework **LG3ipm*** from the framework **HG3ipm***, and to translate an existing simply labelled calculus for **Int** called **O** [Mas67, Mas69] into a novel hypersequent calculus.

We introduce the notion of equivalence modulo permutation of labels for labelled sequents, which is akin to α-equivalence for unbound variables.

We show that, despite the different general semantics, simply labelled sequents can be considered as relational sequents with empty relational contexts (using the translation to PSF to show that the semantics are equivalent), and that there are sound folding rules that correspond to the persistence property of logics in **Int***. These folding rules are used to derive equivalent relational sequents. (However, the relationship is not one-to-one.)

We introduce techniques for translating simply labelled calculi in **Int***/Geo into relational calculi, and apply the technique to obtain the calculi **RG3ipm′** and the framework of calculi **RG3ipm***—the latter translation being mechanisable and preserving cut-admissibility.

We show that the relational calculi **G3I** and the intuitionistic fragment of **L** [PU09] are equivalent, and later show that the framework **G3I*** is equivalent to the framework **RG3ipm***.

We introduce a technique called transitive unfolding for transforming arbitrary relational sequents into simply labelled sequents that share the same linear Kripke models in **Int***. We apply this technique for translating relational proofs in **RG3ipm*** to simply labelled proofs in **LG3ip***+$\mathsf{Com_m}$.

CHAPTER 2

General Notation and Terminology

2.1. Notation and Terminology for Formulae and Proofs

The notation and terminology from Troelstra and Schwichtenberg [TS00] will be used for sequent calculi and derivations, and the terminology from Wansing [Wan94] for describing properties of the rules of logical systems will be used. Where appropriate, terminology will be extended in a natural manner to describe analogous features of hypersequent and labelled sequent calculi.

Some definitions and remarks will be identified as "Notation" or "Terminology". Defined terms will be highlighted in **boldface**. Some lemmas will be identified as "Propositions". The distinction between propositions, lemmas and theorems is somewhat arbitrary. Propositions generally indicate results proven elsewhere, or minor results, where the proofs are considered straightforward. Lemmas are considered noteworthy and are often used by other lemmas or theorems. Theorems are noteworthy or important results.

Notation 2.1 (Metavariables). Following the conventions used in [TS00], Roman letters P,Q,R will be used to denote atomic formulae, and the Roman letters A,B,C,D,E,F will be used to denote arbitrary formulae.

The uppercase Greek letters Γ,Δ will be used to denote lists or multisets of 0 or more side formulae. The comma will be used as shorthand for list append or multiset union (depending on the context). In a multiset context, Γ,A is equivalent to $\Gamma\cup\{A\}$. The uppercase Roman letter S (with primes or subscripts) may be used as shorthand for sequents, components of hypersequents or slices of labelled sequents.

Metavariables may be differentiated by adding one or more prime markers (e.g. A') or a subscript (e.g. A_1). Metavariables for formulae or multisets of formulae with numeric exponents (e.g. A^n) denotes a multiset of n occurrences of that formula or multiset. However, where this can be easily confused with labels, (e.g. A^x), $n\cdot A$ will be used to denote a multiset of n occurrences of A. Metavariables with an asterisk exponent (e.g. A^*) denote multisets of 0 or more occurrences of its value. Metavariables with a plus exponent (e.g. A^+) denote 1 or more occurrences.

Notation 2.2 (Types and Functions). We will use a notation for types as a convenience, rather than use formal type theory, e.g. we will write $A \in \mathtt{Prop}$ rather than write out a phrase such as "A is a propositional formula".

Types will generally be denoted using monospaced font, with the first letter of the name capitalised, e.g. $\mathtt{Var}$, a denumerable set of all propositional variables.

The type can also be thought of as denoting a set of **expressions** in a **language** where the type is defined inductively, e.g. $\mathtt{Prop}$ (Definition 2.6 on the next page).

Cartesian products will be denoted by $\mathtt{T}_1 \times \mathtt{T}_2$. Elements of $\mathtt{T}_1 \times \mathtt{T}_2$ may also be denoted by tuples, e.g. $\langle x, y\rangle$, where $x \in \mathtt{T}_1$ and $y \in \mathtt{T}_2$.

Disjoint unions are denoted by $\mathtt{T}_1 + \mathtt{T}_2$, which indicates that an element of that set is either in $\mathtt{T}_1$ or $\mathtt{T}_2$.

Parametric types that range over a class of sets may be represented with calligraphic letters as subscripts, e.g. $\mathtt{Seq}_{\mathcal{L}}$ as the type of sequents of formulae in $\mathcal{L}$, where $\mathcal{L}$ denotes an unspecified formula type. (Where parameters is irrelevant to or understood from the context, the subscript is omitted, e.g. we will write $\mathtt{Seq}$ instead of $\mathtt{Seq}_{\mathcal{L}}$.)

An asterisk as a superscript denotes a set of lists parametrised by a type, e.g. $\mathtt{Prop}^*$ denotes the set of lists of propositional formulae. (Whether the list is interpreted as an actual list or multiset depends on the context.)

A plus as a superscript denotes a set of non-empty lists parametrised by a type.

Numeric exponents are shorthand for tuples, e.g. $\mathtt{Var}^2 = \mathtt{Var} \times \mathtt{Var}$.

Named functions will be represented using (generally lowercase) monospaced font, e.g. $\mathtt{lab}$. (Some well-known functions such as min will be in Roman lettering, or in all caps, such as $\mathtt{FV}$.)

Notation 2.3 (Currying). Let f be a function of type $\mathtt{T}_1 \times \mathtt{T}_2 \to \mathtt{T}_3$ (which is equivalent to $\mathtt{T}_1 \to \mathtt{T}_2 \to \mathtt{T}_3$), and $x \in \mathtt{T}_1$. We will use $(f\ x)$ to denote a particular function of type $\mathtt{T}_2 \to \mathtt{T}_3$.

Definition 2.4 (Projection Functions). Let $\mathtt{T}_1 \times \mathtt{T}_2$ be a type. We define the projection functions $\pi_1 : \mathtt{T}_1 \times \mathtt{T}_2 \to \mathtt{T}_1$ and $\pi_2 : \mathtt{T}_1 \times \mathtt{T}_2 \to \mathtt{T}_2$ by

$$\pi_1 \langle x, y \rangle =_{def} x \tag{8}$$

$$\pi_2 \langle x, y \rangle =_{def} y \tag{9}$$

Definition 2.5 (First-Order Terms). Let $\mathtt{Term}_0$ be a denumerable **set of (atomic) first-order terms** (also called **individual variables**), and for each $k \geq 0$ let $\mathtt{Func}_k$ be a finite set of function symbols of arity k. We define the set $\mathtt{Term}$ of **terms** inductively:

(1) $x \in \mathtt{Term}$ if $x \in \mathtt{Term}_0$;
(2) $f(x_1, \ldots, x_k) \in \mathtt{Term}$ if $f \in \mathtt{Func}_k$ and $\{x_1, \ldots, x_k\} \subset \mathtt{Term}$.

Definition 2.6 (Second-Order Logical Formulae). For each $n \geq 0$ let $\mathtt{Pred}_n$ be a denumerable set of first-order predicate symbols of arity n. We define the set $\mathtt{Form}_2$ of **(second-order) logical formulae** inductively:

(1) $Px_1, \ldots, x_k \in \mathtt{Form}_2$ (called an **atomic formula**, or an **atom**, with arity k) if $P \in \mathtt{Pred}_k$ and $\{x_1, \ldots, x_k\} \subset \mathtt{Term}$ for $k \geq 0$;
(2) $\bot \in \mathtt{Form}_2$;
(3) $A \wedge B \in \mathtt{Form}_2$ if $A, B \in \mathtt{Form}_2$;
(4) $A \vee B \in \mathtt{Form}_2$ if $A, B \in \mathtt{Form}_2$;
(5) $A \supset B \in \mathtt{Form}_2$ if $A, B \in \mathtt{Form}_2$;
(6) $\forall x.A \in \mathtt{Form}_2$ if $x \in \mathtt{Term}_0$ and $A \in \mathtt{Form}_2$;
(7) $\exists x.A \in \mathtt{Form}_2$ if $x \in \mathtt{Term}_0$ and $A \in \mathtt{Form}_2$;
(8) $\forall P.A \in \mathtt{Form}_2$ if $P \in \mathtt{Pred}_k$ and $A \in \mathtt{Form}_2$;
(9) $\exists P.A \in \mathtt{Form}_2$ if $P \in \mathtt{Pred}_k$ and $A \in \mathtt{Form}_2$.

A formula is in the **first-order fragment** $\mathtt{Form}_1$ of $\mathtt{Form}_2$ iff there are no quantifiers on predicates (cases 8 and 9). A formula is in the **propositional fragment** $\mathtt{Prop}$ of $\mathtt{Form}_1$ iff it contains no quantifiers (cases 6 through 9), and all predicate symbols have arity 0.

Notation 2.7. Second-order quantifiers may be written as $\forall P^k.A$ or $\exists P^k.A$ to make the arity of P explicit.

Remark 2.8. We opt to use the term "formula" rather than the common term "well-formed formula", because all expressions matching the grammar defined above are, by definition, formulae, and other expressions containing propositional variables and connectives that do not match the grammar are not formulae.

NOTATION 2.9 (Substitution). The expression $[t/s]E$ denotes the result of substituting every occurrence of s with t in expression E. (It is assumed that both $E \in \mathtt{T}$ and $[t/s]E \in \mathtt{T}$, for some type $\mathtt{T}$.)

NOTATION 2.10 (Vector). We use $\bar{x}$ to denote a **vector** of variables $x_1, \ldots, x_n$. In a (multi)set context, $\bar{x}$ denotes a set of variables $\{x_1, \ldots, x_n\}$ in $\mathtt{Term}_0$.

In a slight abuse of notation, $\bar{x}\#X$ is used as shorthand for $\bar{x} \cap X = \emptyset$ and $\bar{x} \subseteq X$ for $\bar{x} \cap X = \bar{x}$.

We use $[\bar{y}/\bar{x}]A$ to denote the substitutions $[y_1/x_1], \ldots, [y_n/x_n]A$, where $\bar{y}$ and $\bar{x}$ are disjoint (and thus the order of substitution is irrelevant).

NOTATION 2.11. We combine like quantifier symbols and write $\forall x_1, \ldots, x_k.A$ as shorthand for $\forall x_1, \forall x_2, \ldots, \forall x_k$ (and similar for existential quantifiers). When the number of variables is indeterminate, we write $\forall \bar{x}.A$ (or $\exists \bar{x}.A$).

DEFINITION 2.12 (Free Variables). Let $A \in \mathtt{Form}_2$. We use $\mathtt{FV}(A)$ to denote the **set of free (individual) variables** in A. It is defined formally below:

(1) $\mathtt{FV}(Px_1, \ldots, x_k) =_{def} \bigcup_{i=1}^{k} \mathtt{FV}'(x_i)$, where $P \in \mathtt{Pred}_k$ and

$$\mathtt{FV}'(x) =_{def} \begin{cases} \{\ x\ \} & \text{iff } x \in \mathtt{Term}_0; \\ \bigcup_{i=1}^{k} \mathtt{FV}'(x'_i) & \text{if } x = f(x'_1, \ldots, x'_k) \text{ where } f \in \mathtt{Func}_k. \end{cases}$$

(2) $\mathtt{FV}(\bot) =_{def} \emptyset$;

(3) $\mathtt{FV}(A \wedge B) =_{def} \mathtt{FV}(A) \cup \mathtt{FV}(B)$;

(4) $\mathtt{FV}(A \vee B) =_{def} \mathtt{FV}(A) \cup \mathtt{FV}(B)$;

(5) $\mathtt{FV}(A \supset B) =_{def} \mathtt{FV}(A) \cup \mathtt{FV}(B)$;

(6) $\mathtt{FV}(\forall x.A) =_{def} \mathtt{FV}(A) \setminus \{x\}$, where $x \in \mathtt{Term}_0$; (x is called **bound** in A);

(7) $\mathtt{FV}(\exists x.A) =_{def} \mathtt{FV}(A) \setminus \{x\}$, where $x \in \mathtt{Term}_0$; (x is called **bound** in A);

(8) $\mathtt{FV}(\forall P.A) =_{def} \mathtt{FV}(A)$, where $P \in \mathtt{Pred}_k$;

(9) $\mathtt{FV}(\exists P.A) =_{def} \mathtt{FV}(A)$, where $P \in \mathtt{Pred}_k$.

REMARK 2.13. The codomain of $\mathtt{FV}$ is $\mathtt{Term}_0$ (the set of individual variables). A similar function $\mathtt{FV}_2$ could be defined with a codomain of $\mathtt{Var}$ (the set of propositional variables), but this is not needed here.

REMARK 2.14. We will adopt the usual conventions about free and bound occurrences of variables in quantified formulae.

DEFINITION 2.15 (Defined Connectives). Unless otherwise noted, the following connectives are defined in terms of other connectives:

$$\neg A =_{def} A \supset \bot$$

$$\top =_{def} \bot \supset \bot$$

$$A \equiv B =_{def} (A \supset B) \wedge (B \supset A)$$

where $A \in \mathtt{Form}_2$. Likewise, rules for these connectives in a calculus will be assumed derivable in the usual way.

REMARK 2.16. This work will examine translations of propositional calculi (or the propositional fragments of calculi) between formalisms. Some of the work will use first- or second-order logic to provide a semantic basis for the translation.

DEFINITION 2.17 (Implicational Fragment). The **implicational fragment** of formulae in $\mathtt{Form}_1$ (resp. $\mathtt{Prop}$), denoted by $\mathtt{Form}_1/\supset$ (resp. $\mathtt{Prop}/\supset$), is the set of all formulae in $\mathtt{Form}_1$ (resp. $\mathtt{Prop}$) that do not have $A \wedge B$ or $A \vee B$ as subformulae.

DEFINITION 2.18 (Formula Size). Let $A \in \mathtt{Form}_2$. The **size** of A, written as $|A|$, is defined inductively:

(1) $|Px_1, \ldots, x_k| =_{def} 0$ if $P \in \mathtt{Pred}_k$;
(2) $|\bot| =_{def} 1$;
(3) $|A \wedge B| =_{def} |A| + |B| + 1$;
(4) $|A \vee B| =_{def} |A| + |B| + 1$;
(5) $|A \supset B| =_{def} |A| + |B| + 1$;
(6) $|\forall\alpha.A =_{def} |A| + 1$, where $\alpha \in \mathtt{Term}_0 + \mathtt{Pred}_k$;
(7) $|\exists\alpha.A =_{def} |A| + 1$ where $\alpha \in \mathtt{Term}_0 + \mathtt{Pred}_k$;

Informally, the formula size is the number of connectives.

DEFINITION 2.19 (Prime Formulae). A formula $A \in \mathtt{Form}_2$ is **prime** iff A is atomic or $A = \bot$.

REMARK 2.20. The distinction between a prime formula and an atomic formula is somewhat arbitrary here.

DEFINITION 2.21 (Sequents). Let $\mathcal{L}$ denote a type of formulae, e.g. $\mathtt{Prop}$ or $\mathtt{Form}_1$. The set $\mathtt{Seq}_{\mathcal{L}}$ of **$\mathcal{L}$-sequents** of $\mathcal{L}$-formulae is a pair in $\mathcal{L}^* \times \mathcal{L}^*$, written (schematically) as $\Gamma \Rightarrow \Delta$, where $\Gamma, \Delta \in \mathcal{L}^*$. "$\Rightarrow$" is called the **sequent arrow**. When $\mathcal{L}$ is obvious from or irrelevant to the context, we shall simply refer to L-sequents as "sequents".

NOTATION 2.22. Empty antecedents or succedents in sequents are generally written as a blank space, rather than using the empty set symbol, e.g. "$\Rightarrow \Delta$" rather than "$\emptyset \Rightarrow \Delta$". (In some cases, the empty set symbol will be used for clarity.)

REMARK 2.23 (Order of Representation). We note that while antecedents, succedents and hypersequents are multisets, their *representations* are in a particular order, as a list. For the purposes of the functions given in this thesis that have multisets in their domains or codomains, we will assume that there is a particular order of those multisets that is determined by their representations, and thus without loss of generality we will treat them as if they as if they are indexed multisets, where the index of an element corresponds to the order of its occurrence in a representation.

DEFINITION 2.24 (Polarity). A formula has a **polarity** that is either **positive** or **negative**, depending on where it occurs in the structure of a compound formula or a sequent:

(1) if $A \wedge B$ has a positive (resp. negative) polarity, then both A and B have a positive (resp. negative) polarity;
(2) if $A \vee B$ has a positive (resp. negative) polarity, then both A and B have a positive (resp. negative) polarity;
(3) if $\forall \bar{x}.A$ has a positive (resp. negative) polarity, then A has a positive (resp. negative) polarity;
(4) if $\exists \bar{x}.A$ has a positive (resp. negative) polarity, then A has a positive (resp. negative) polarity;
(5) if $A \supset B$ has a positive (resp. negative) polarity, then A has a negative (resp. positive) polarity and B has a negative (resp. positive) polarity.

(6) if A occurs in the antecedent (resp. succedent) of a sequent, then A has a negative (resp. positive) polarity.

REMARK 2.25. The location of a component in a hypersequent has no bearing on its polarity.

DEFINITION 2.26 ($\bigwedge\!\!\!\bigwedge$ and $\bigvee\!\!\!\bigvee$ operators). The symbols $\bigwedge\!\!\!\bigwedge$ and $\bigvee\!\!\!\bigvee$ are normally used as alternative symbols to denote the iterated conjunction and disjunction of formulae in a multiset, i.e.

$$\bigwedge\!\!\!\bigwedge\{A_1,\ldots,A_n\} =_{def} \bigwedge_{i=1}^{n} A_i \qquad \bigvee\!\!\!\bigvee\{A_1,\ldots,A_n\} =_{def} \bigvee_{i=1}^{n} A_i$$

(Unless otherwise noted, $\bigwedge\!\!\!\bigwedge\emptyset =_{def} \top$ and $\bigvee\!\!\!\bigvee\emptyset =_{def} \bot$.) Here they will be used to denote *functions* from multisets of formulae into formulae of specific forms. This allows the use of $\bigwedge\!\!\!\bigwedge^{-1}$ and $\bigvee\!\!\!\bigvee^{-1}$ as notation to denote their respective *inverse functions* from formulae of those forms into multisets of formulae. That is,

$$\bigwedge\!\!\!\bigwedge\{A_1,\ldots,A_n\} = A_1 \wedge \ldots \wedge A_n \iff \bigwedge\!\!\!\bigwedge^{-1} A_1 \wedge \ldots \wedge A_n = \{A_1,\ldots,A_n\}$$

and similarly for $\bigvee\!\!\!\bigvee$ and $\bigvee\!\!\!\bigvee^{-1}$.

REMARK 2.27. We note that the order of formulae for the $\bigwedge\!\!\!\bigwedge$ and $\bigvee\!\!\!\bigvee$ operators is irrelevant with respect to the semantics of the logics we will be examining in this thesis (discussed in Chapter 3). However, conjunction and disjunction are commutative and associative in these logics, so we therefore allow ourselves to choose an arbitrary representative of their iterated operations. It is routine to check that all formulae obtained this way are equivalent.

NOTATION 2.28 (Precedence). For clarity, we may omit brackets around subformulae. The precedence of logical connectives not delineated by brackets should be read as follows:

(1) $\supset$ is right associative, so $A \supset B \supset C$ is read as $A \supset (B \supset C)$;
(2) $\wedge$ and $\vee$ are left associative, so $A \wedge B \wedge C$ is read is $(A \wedge B) \wedge C$;
(3) $\neg$ applies to the formula immediately to the right of it, so that $\neg A \supset B$ is read as $(\neg A) \supset B$;

(4) $\wedge$ and $\vee$ have equal precedence over $\supset$, so that $A \wedge B \supset C \vee D$ is read as $(A \wedge B) \supset (C \vee D)$.

NOTATION 2.29. When hypersequents or labelled sequents are translated into formulae in SPSF, numeric subscripts will sometimes be added to logical symbols $\wedge$ and $\vee$ (and their iterated forms $\bigwedge$ and $\bigvee$) to differentiate explicit connectives from implicit connectives, e.g. to distinguish the translations of formulae $A \vee (B \vee C)$ from $A, B \vee C$ in the succedent of a sequent, we might use $A \vee_0 (B \vee_0 C)$ and $A \vee_1 (B \vee_0 C)$. The subscripts are useful for pattern matching the arguments of the inverse translation functions. Otherwise the subscripts have no meaning.

TERMINOLOGY 2.30 (Formalism). We will use the term **formalism** to denote the general type of data structure, e.g. hypersequent or simply labelled sequent. (In the context of calculi or proofs, formalism will denote the calculi of that data structure, or proofs in such calculi.)

TERMINOLOGY 2.31 (Framework). We will use the term **framework** to denote a set of calculi of a particular formalism, where the calculi share a set of base rules.

DEFINITION 2.32 (Metaformula). A **metaformula** is a either a metavariable, or a formula with metavariables as subformulae.

NOTATION 2.33 (Rules). The names of rules for calculi will be denoted using a sans-serif font, e.g. $\mathsf{L}\wedge$.

NOTATION 2.34 (Derivations). Trees constructed according to the rules of a calculus are called **derivation fragments**. A **derivation** or **proof** has axioms as leaves. (Where the difference is obvious or irrelevant, we will use "derivation" to refer to both derivation fragments and derivations.)

In a derivation, rule names given in parenthesis, e.g.

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta}\ (\mathsf{LW})$$

denote an application of a non-primitive (admissible or derived) rule. An exponent (a number, asterisk or plus) on a rule name in a derivation signifies multiple instances of a rule in much the same way an exponent does for multiple occurrences of a formula or multiset. An exponent of -1 indicates the inverted form of a rule (see below).

For rules with multiple premisses that follow a pattern, we will use angle brackets. For example, if a rule has n premisses of the form $\Gamma_i \Rightarrow A_i$, then it may be shown schematically (or even in a derivation) as

$$\frac{\langle \Gamma_i \Rightarrow A_i \rangle_{i=1}^n}{\Gamma_0 \Rightarrow \Delta_0}$$

An **end sequent** (**end hypersequent**) is the root node of a derivation tree. An **initial sequent** (**initial hypersequent**) is a leaf node of a derivation tree, i.e., an axiom.

We use the notation **GS** $\vdash \Gamma \Rightarrow \Delta$ to indicate that sequent $\Gamma \Rightarrow \Delta$ is derivable in calculus **GS** for logic **S**. Where the calculus is obvious from the context, we say simply $\vdash \Gamma \Rightarrow \Delta$. This is extended to hypersequents and label sequents.

In derivation fragments, the notation

$$\frac{\vdots\ \delta}{\Gamma \Rightarrow \Delta}\ \rho$$

denotes a derivation of $\Gamma \Rightarrow \Delta$ from δ (often represented as an equation number or lemma).

Notation 2.35 (Rule Instantiation). Let ρ be a schematic rule in a calculus. A (partial) instantiation σ is a substitution of metavariables such that $\sigma\rho$ is a form of ρ with the substitution applied to it's metavariables. $\sigma\rho$ may be another schematic rule (for example, substituting an active formula with one of a specific form), or it may be an **inference** (an instance of ρ with no metavariables).

Terminology 2.36 (nth Premiss). For rules with more than two premisses, the nth premiss will refer to the nth premiss from the left.

Definition 2.37 (Invertibility). A rule in a calculus **GS**

$$\frac{\Gamma_1' \Rightarrow \Delta_1' \quad \ldots \quad \Gamma_n' \Rightarrow \Delta_n'}{\Gamma \Rightarrow \Delta}\ \rho$$

is **invertible** iff the rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma_i' \Rightarrow \Delta_i'}\ \rho_i^{-1}$$

for all $1 \leq i \leq n$, are admissible in **GS**.

Remark 2.38. This definition is applicable irrespective of the formalism of **GS**, e.g. sequent, hypersequent, labelled sequent, etc.

Notation 2.39. Inverted rules are denoted with the exponent -1, and a subscript for the ith premiss (when there are more than one premiss), e.g. ρ_{i}^{-1}

Definition 2.40 (Derivation depth). The expression $\mathsf{h}(\delta)$ denotes the **depth** (or **height**) of the derivation, which is defined as following:

(1) Axioms (initial sequents or hypersequents, etc.) have a derivation depth of 0.
(2) The derivation depth of a rule instance is equal to 1 plus the maximum derivation depths of the premisses.

Remark 2.41. The term “depth” is used instead of “height”, since rules of lower depth are closer to the top of the derivation.

Definition 2.42 (Depth-Preserving Admissible). A rule is **depth-preserving admissible**

$$\frac{\Gamma'_1 \Rightarrow \Delta'_1 \quad \ldots \quad \Gamma'_n \Rightarrow \Delta'_n}{\Gamma \Rightarrow \Delta}\ \rho$$

in a calculus **GS** iff for all instantiations σ of ρ, whenever the maximum derivation depth of the premisses of $\sigma\rho$ is h, then the conclusion of $\sigma\rho$ can be derived at a derivation depth no greater than h.

Remark 2.43. This definition is applicable irrespective of the formalism of **GS**, e.g. sequent, hypersequent, labelled sequent, etc.

Definition 2.44 (Depth-Preserving Invertible). A rule ρ is **depth-preserving invertible** in a calculus **GS** iff ρ is invertible, and the inverted rule(s) are depth-preserving admissible in **GS**.

Notation 2.45 (Freshness). $t\#E$ indicates that expression t is not a subexpression of E, read as t is **fresh** for E.

The expression $t \in E$ may be used to indicate that t is a subexpression of E, when it is clear from the context type t belongs to relative to E.

2.2. Other Notation

Notation 2.46. The operator $\circledast$ is used as an alternative for the `map` function:

$$f \circledast \Gamma =_{def} \mathtt{map}\ f\ \Gamma$$

where f is a function and Γ is a collection (such as a list, multiset or set). In other words, if f has type $A \to B$, and Γ has type A^*, then $f \circledast \Gamma$ has type B^*.

Proposition 2.47 (Identity). $id \circledast \Gamma = \Gamma$.

Proof. Trivial. □

Proposition 2.48 (Composition). $f \circledast (g \circledast \Gamma) = (f \circ g) \circledast \Gamma$.

Proof. Straightforward. □

Proposition 2.49 (Distribution). $(f \circledast \Gamma) \cup (f \circledast \Delta) = f \circledast (\Gamma \cup \Delta)$*, where* Γ, Δ *are multisets.*

Proof. Straightforward. □

Remark 2.50. A general treatment of the $\circledast$ operator can be found in [MP08] (from which this notation was adapted) and [Wad89].

2.3. On the Criteria for Evaluating a Proof System

Much of the literature on extensions to Gentzen-style calculi (e.g. [Wan94] and [Avr96]) includes discussions about what makes a "good" proof system. Although the criteria in the literature have philosophical and even computational rationale, we find it to be inadequate for evaluating proof formalisms with regards to their actual usage in the literature. However, the vocabulary from the literature that is used for describing the properties of proof systems is useful for the discussion of proof systems later on in this work, so we will provide an overview of some of these properties, which we (informally) introduce below.

2.3.1. General Syntactic Considerations. According to Avron [Avr96], a formalism should have a "simple and uniform syntax", and the structures of a framework should be built on the formulae of the logic. Clearly, sequents contain syntactic elements which are not part of the logic, such as the comma or sequent arrow. But these new connectives (in most sequent calculi) can be translated into connectives in the corresponding logic. So we extend the notion to say that a formalism should be **natural**—that is, there should be a function that translates arbitrary structures in the formalism to formulae in the corresponding logic.

REMARK 2.51. These syntactic considerations are given in [Avr96] as a criticism of display calculi [Bel82], which have structural connectives that allow for the reorganisation of underlying data structure.

The relational sequent calculi examined in this thesis also lack a naturalness property, as the relational formulae do not have a corresponding formula in the standard language of the logics that the calculi correspond to (such as modal logics).

Another corollary to this is **syntactic purity** [Avr96, Pog08b]: The semantics of the logic should not be explicit in the formalism.

The desire for syntactic purity is often given as a critique of labelled calculi which incorporate the semantics of the logic into the calculus. However, the case against including semantics in the calculus is weaker when appealing to notions of syntactic purity than to naturalness. The semantics of a logic is *encoded* in the syntax of logical rules, e.g. the disjunction property of constructive logics in the single-succedent restriction, or the components of hypersequents correspond to points in a Kripke frame. A formalism that uses brackets rather than labels and relations to denote complex semantic structures such as trees, as with the tree hypersequents in [Pog08b], is still explicitly embedding the semantics into the syntax.

A semantic structure such as a relational frame or algebra can be considered an alternative form of a logic. In some contexts, it is more intuitive to refer to the semantics of a logic with respect to some operators. For example, the notion of an accessible "world" provides meaning to the $\Box$ and $\Diamond$ operators of a modal logic that a "syntactically pure" notation does not.

2.3.2. Inferential Semantics. Gentzen [Gen35] specified that the meaning of a logical connective $\odot$ is given syntactically by the rules that introduce $\odot$ in their principal formulae. (This is consistent with the BHK semantics for **Int** discussed later in Chapter 3.) Wansing [Wan94] extended this with the notion of explicit meaning: a rule ρ for a connective $\odot$ is called **weakly explicit** iff the connective $\odot$ occurs only in the conclusion of ρ as a principal formula, and not in any premiss of ρ as an active formula. ρ is called **explicit** iff it is weakly explicit and there is only one principal formula with $\odot$ in the conclusion. Thus, a rule such as

$$\frac{\Gamma, \odot A, A \Rightarrow \Delta}{\Gamma, \odot A \Rightarrow \Delta}\ \mathsf{L}\odot$$

is not (weakly) explicit, because the presence of $\odot$ in the premiss makes it less clear what the $\odot$ connective means.

A related notion is that of **separation** [Wan94]: the meaning of a logical connective $\odot$ should not be defined in terms of other logical connectives. That is, the rules for $\odot$ should not contain another connective $\odot'$ in the active or principal formulae. (Note that we are referring to schematic formulae here, so that connectives in the subformulae of the principal formula are irrelevant.)

Wansing [Wan94] also gives the notion of symmetry: for sequent calculi, if a rule ρ for $\odot$ is either an antecedent rule or succedent rule, then ρ is called **weakly symmetric**. If all of the rules for $\odot$ are weakly symmetric, and there exist both antecedent and succedent rules for $\odot$, then that set of rules for $\odot$ is called **symmetric**.

We note that symmetry is (roughly) a weaker notion than **harmony** [Pra65], which requires a relationship in natural deduction systems between introduction and elimination rules (that roughly correspond to succedent and antecedent rules of sequent calculi) for a connective.

Clearly the notion of symmetry is specific to sequent calculi, although it can be extended to hypersequent and labelled calculi easily. (What a corresponding notion to symmetry is for an arbitrary data structure is a topic for future work.)

The **subformula property** of a rule holds when premisses of a rule for a connective $\odot$ should only contain the subformulae of the principal formulae as active formulae. The **weak subformula property** only requires that the active formulae in the premisses be built up from subformulae of the principal formulae, or nullary connectives. For example, the $\mathsf{L}\vee$ rule from **G3c** (Figure A.1 on page 233):

$$\frac{A,\Gamma\Rightarrow\Delta \quad B,\Gamma\Rightarrow\Delta}{A\vee B,\Gamma\Rightarrow\Delta}\ \mathsf{L}\vee$$

has the subformula property, whereas the rule $\mathsf{L}\vee\supset$ from the calculus **G4ip** for **Int** from [Dyc92],

$$\frac{C\supset B, D\supset B,\Gamma\Rightarrow\Delta}{(C\vee D)\supset B,\Gamma\Rightarrow\Delta}\ \mathsf{L}\vee\supset$$

has the weak subformula property. The subformula property is noted as desirable in many works, e.g. [CT07, Pog08b]. Clearly a rule which has the weak subformula property but not the (strong) subformula property does not have either the separation or explicitness properties. However, there are calculi such as **G4ip** or various calculi for **GD** (discussed in Section 3.2.2 on page 39) [AF96, AFM99a, Dyc99, Avr00, GLWS07] which have only the weak subformula property, but which are also contraction free, and allow for terminating proof search.

Wansing [Wan94] notes that systems where the rules are separated, symmetric and weakly explicit and admit cut imply the subformula property, if they do not contain what he calls "silly structural rules" that allow for the elimination of arbitrary formulae, e.g.

$$\frac{\Gamma,\Gamma'\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}$$

Avron [Avr96] also suggests that a calculus should be **concise**, that is, it should have a minimal number of rules, and the rules should have a small and fixed number of premisses. (This is a criticism of calculi with rules such as [AFM99a, §3] for **GD** which has a form of $\mathsf{R}\supset$ that has n premisses—one premiss for each implication or negation. In the case of [AFM99a], the calculus is cut- and contraction-free, making it more suited for automated deduction.

2.3.3. The Došen Principle. The Došen Principle [Doš85b, Wan94, Avr96] states that a **framework** (a family of similar calculi with shared rules) should be defined so that:

(a) logical connectives have an inferential semantics (as defined above), where
(b) the meaning of the conjunction and disjunction connectives should be close to the standard meaning of those connectives in classical logic; and

(c) calculi in the family shall differ only by the structural rules—that is, they share a base calculus of logical rules. (This last property is called **modularity**.)

Avron [Avr96] adds that a "good framework" should give a better understanding of the corresponding logics for each calculus and their differences, and even suggest new logics (e.g. restricting the structural rules of sequent calculi suggested linear logic [Gir87]).

Avron [Avr96] suggests that all members of a good framework should have **cut freedom**—that is, each calculus in the framework is complete without use of some form of the cut rule.

CHAPTER 3

Intermediate Logics

3.1. Overview

3.1.1. Preliminaries. In this chapter, we will discuss intermediate logics and their semantics. The chapter is organised as follows: in Section 3.2, we give a brief introduction to intuitionistic and some well-known intermediate propositional logics, and discuss some of their properties and their relationships to each other. We also give Gödel's interpretation of intuitionistic logic (and some of the other intermediate logics that we discuss) as extensions of the modal logic **S4**. (This will be useful later for translating calculi for modal logics into calculi for corresponding intermediate logics.)

In Section 3.3, we discuss the Kripke semantics of intuitionistic logic, and give the Kripke semantics for the intermediate logics discussed in the previous section. We also recall a subset of first-order formulae called **geometric implications** from [Pal02], and use it to define a subclass of intermediate logics called $\mathbf{Int}^*/\mathtt{Geo}$ determined by Kripke frames axiomatised by such formulae.[1] Note that most of the intermediate logics we discuss here are in $\mathbf{Int}^*/\mathtt{Geo}$.

In Section 3.5, we introduce the set of partially shielded formulae ($\mathtt{PSF}$) and an interpretation of these formulae in Kripke models. We then present a cut-free sequent calculus $\mathbf{G3c}/\mathbf{PSF}$ (Figure 3.2 on page 57) for $\mathtt{PSF}$ based on **G3c** (Appendix A). We extend the calculus with geometric rules based on the frame axioms of the logics discussed in the previous section, so that sequents proven in the extended framework of calculi $\mathbf{G3c}/\mathbf{PSF}^*$ correspond to sequents derivable in various intermediate logics. (As we note later, there exists prior work showing that geometric rules can be added to calculi based on **G3c** without affecting the admissibility of cut.) We also prove "persistence lemmas" which show the admissibility of rules that have corresponding rules in relational sequents for

[1] The class of logics $\mathbf{Int}^*/\mathtt{Geo}$ was implicitly introduced for the framework $\mathbf{G3I}^*$ in [Neg07, DN10].

Int*/Geo. These rules will be useful later in the chapters on semantics (Chapters 6 and 8), and used as a basis for translating between simply labelled and relational proofs.

3.2. Intermediate Logics

Recall that a propositional logic **S** can be considered to be a set of formulae $\mathbf{S} \subset \mathtt{Prop}$ that is closed under modus ponens and substitution of propositional variables by arbitrary formulae.

3.2.1. Intuitionistic Logic.

Intuitionistic logic (**Int**) was philosophically motivated by the work of Brouwer [Bro07, Bro75], who considered the law of excluded middle (LEM), $A \vee \neg A$, to be unacceptable for mathematical reasoning. Briefly, Brouwer considered logic to be an abstraction of mathematical reasoning, and a proof to be a **construction** of mathematical objects. A proof of a disjunction requires a proof of one of the disjuncts.

Definition 3.1 (Brouwer-Heyting-Kolmogorov Interpretation). The Brouwer-Heyting-Kolmogorov (BHK) interpretation [Kol32, Hey56] (cf. [GLT89, CZ97]) of propositional intuitionistic logic is as follows: Let S be a set of unanalysable (atomic) facts describing a situation. Then

(1) P is proven if $P \in \mathsf{S}$;
(2) $\perp$ is not proven (that is, $\perp \notin \mathsf{S}$);
(3) $A \wedge B$ is proven if both A and B are proven;
(4) $A \vee B$ is proven if either A or B is proven;
(5) $A \supset B$ is proven if, given a proof of A, one can construct a proof of B.

A Hilbert-style axiomatisation for **Int** was first given by Heyting [Hey30], which is identical to the axiomatisation for classical logic (**Cl**) without the axiom of excluded middle or double negation, $\neg\neg A \supset A$. (Multiple Hilbert-style axiomatisations of **Int** can be found in [van02].)

Various formal semantics for **Int** have been introduced. We will mainly be concerned with Kripke semantics here—see Section 3.3 on page 44 below. (We will give a brief overview of Beth semantics in Section 3.3.4 on page 50, which are similar to Kripke semantics, and relevant to some of the calculi discussed later.) For a survey of other kinds of semantics for **Int**, see [van02].

3.2.2. Intermediate Logics. Intermediate logics (also called **superintuitionistic logics**) are logics between **Int** and **Cl**, inclusive. We use $\mathbf{Int}^*$ to denote the class of all intermediate logics.

There are uncountably many logics in $\mathbf{Int}^*$ [Göd33a, Jan68b]. The set of logics in $\mathbf{Int}^*$ form a distributive lattice where join corresponds to set inclusion [CZ97].

DEFINITION 3.2 (Disjunction Property). A logic **S** has the **disjunction property** when $\mathbf{S} \vdash A \vee B$ iff $\mathbf{S} \vdash A$ or $\mathbf{S} \vdash B$. An intermediate logic is called **constructive** if it has the disjunction property.

Not all intermediate logics are constructive. (Clearly **Cl** is an example.)

Following are some of the well-known intermediate propositional logics which will be discussed later in this thesis. Intermediate predicate logics will not be discussed. The applications for these logics will not be discussed explicitly, though we cite sources where the applications are discussed.

Classical Logic: has the following alternative axiomatisations:

$$\mathbf{Cl} =_{def} \mathbf{Int} + A \vee (A \supset B) \tag{10}$$

$$=_{def} \mathbf{Int} + \neg\neg A \supset A \tag{11}$$

$$\mathbf{Cl}/\supset =_{def} \mathbf{Int} + ((A \supset B) \supset A) \supset A \tag{12}$$

where $\mathbf{Cl}/\supset$ is the implicational fragment of **Cl**. (Clearly, LEM, $A \vee \neg A$ is included in $A \vee (A \supset B)$.)

Jankov-De Morgan Logic: was introduced by Jankov [Jan68a]. It has the following alternative axiomatisations:

$$\mathbf{Jan} =_{def} \mathbf{Int} + \neg\neg A \vee \neg A \tag{13}$$

$$=_{def} \mathbf{Int} + \neg(A \wedge B) \supset (\neg A \vee \neg B) \tag{14}$$

Jan is also known as the logic of "weak excluded middle" (after the corresponding formula in the first axiomatisation) or as **LQ** e.g. [Avr91a] or **KC** e.g. [CZ97].

Gödel-Dummett Logic: Dummett [Dum59] generalised the logics $\mathbf{G_k}$ (below) to infinite-valued logics. It has the following alternative axiomatisations:

$$\mathbf{GD} =_{def} \mathbf{Int} + (A \supset B) \vee (B \supset A) \tag{15}$$

$$=_{def} \mathbf{Int} + (A \supset (B \vee C)) \supset ((A \supset B) \vee (A \supset C)) \tag{16}$$

$$\mathbf{GD}/\supset =_{def} \mathbf{Int}/\supset + ((A \supset B \supset C) \supset (((B \supset A) \supset C) \supset C) \tag{17}$$

where $\mathbf{GD}/\supset$ is the implicational fragment [Bac68, Bul62, Dum59]. **GD** has also been known in the literature as **G** e.g. [BCF03a] $\mathbf{G}_\omega$ e.g. [Got01], or **LC** (for the logic of "linear chains", referring to the Kripke semantics) e.g. [Avr91a].

Gödel Logics: Gödel [Göd33a] introduced a class of (finite) many-valued logics $\mathbf{G_k}$. The class is defined inductively [BCF03a]:

$$\mathbf{G_1} =_{def} \mathbf{GD} \tag{18}$$

$$\mathbf{G_{k+1}} =_{def} \mathbf{GD} + A_1 \vee (A_1 \supset A_2) \vee \ldots \vee ((A_1 \wedge \ldots \wedge A_k) \supset A_{k+1}) \tag{19}$$

where $k \geq 1$. For example,

$$\mathbf{G_2} = \mathbf{GD} + A \vee (A \supset B) \tag{20}$$

$$\mathbf{G_3} = \mathbf{GD} + A \vee (A \supset B) \vee ((A \wedge B) \supset C) \tag{21}$$

Clearly $\mathbf{G_2} = \mathbf{Cl}$, and $\mathbf{G_3} = \mathbf{Sm}$ (given below, with an alternative axiomatisation).

A survey of applications of these logics can be found in [BCF03a].

Logics of Bounded Depth: is a class of logics with Kripke frames of bounded depth. Each logic $\mathbf{BD_k} =_{def} \mathbf{Int} + \mathtt{BD}_k$, where the the characteristic axiom $\mathtt{BD}_k$ is defined inductively [CZ97]:

$$\mathtt{BD}_1 = A_1 \vee \neg A_1 \tag{22}$$

$$\mathtt{BD}_{k+1} = A_{k+1} \vee (A_{k+1} \supset \mathtt{BD}_k) \tag{23}$$

For example,

$$\mathbf{BD_1} = \mathbf{Int} + A \vee \neg A \tag{24}$$

$$\mathbf{BD_2} = \mathbf{Int} + A \vee (A \supset (B \vee \neg B)) \tag{25}$$

Clearly $\mathbf{BD_1} = \mathbf{Cl}$. An alternative axiomatisation uses generalisations of Peirce's Law of the characteristic axiom [van02]:

$$\mathtt{BD}'_1 = ((A_1 \supset A_0) \supset A_1) \supset A_1 \tag{26}$$

$$\mathtt{BD}'_{k+1} = ((A_{k+1} \supset \mathtt{BD}'_k) \supset A_{k+1}) \supset A_{k+1} \tag{27}$$

which can be used to define the implicational fragments $\mathbf{BD_k}/\supset$. $\mathbf{BD_k}$ has also been known in the literature as $\mathbf{BH_k}$ (for "bounded height") and $\mathbf{LP_k}$ (in reference to the generalisation of Peirce's Law, e.g. [van02]).

Logics of Bounded Width: is a class of logics with Kripke frames of bounded width. The axiomatisation of $\mathbf{BW_k}$ [CZ97] is by:

$$\mathbf{BW_k} =_{def} \mathbf{Int} + \bigvee_{i=0}^{k} (A_i \supset \bigvee_{j \neq i} A_j) \tag{28}$$

For example,

$$\mathbf{BW_1} = \mathbf{Int} + (A \supset B) \vee (B \supset A) \tag{29}$$

$$\mathbf{BW_2} = \mathbf{Int} + (A \supset B \vee C) \vee (B \supset A \vee C) \vee (C \supset A \vee B) \tag{30}$$

Clearly $\mathbf{BW_1} = \mathbf{GD}$. $\mathbf{BW_k}$ is also known in the literate as $\mathbf{BA_k}$ (for "bounded anti-chains") [van02].

Logics of Bounded Top Width: is a class of logics with Kripke frames of bounded width at the top of their trees. The axiomatisation of $\mathbf{BTW_k}$ is [CZ97, van02]:

$$\mathbf{BTW_k} =_{def} \mathbf{Int} + \left(\bigwedge_{0 \leq i < j \leq k} \neg(\neg A_i \wedge \neg A_j) \right) \supset \bigvee_{i=0}^{k} \left(\neg A_i \supset \bigvee_{j \neq i} \neg A_j \right) \tag{31}$$

In [FM93], the axiomatisation is given as

$$\mathbf{BTW_k} = \mathbf{Int} + \neg A_1 \vee \bigvee_{i=2}^{k-1} \big(\bigwedge_{j=1}^{i} \neg A_j \supset \neg A_{i+1} \big) \vee \big(\bigwedge_{i=1}^{k-1} \neg A_i \supset \neg\neg A_k \big) \tag{32}$$

From the semantics given below, we will see that $\mathbf{BTW_1} = \mathbf{Jan}$.

Logics of Bounded Cardinality: is a class of logics with Kripke frames of bounded size. The axiomatisation of a logic $\mathbf{BC_k}$ is defined as $\mathbf{BC_k} =_{def} \mathbf{Int} + \mathtt{BC_k}$, where the characteristic axiom $\mathtt{BC_k}$ is defined inductively [CZ97]:

$$\mathtt{BC}_0 =_{def} A_0 \tag{33}$$

$$\mathtt{BC}_{k+1} =_{def} \mathtt{BC}_k \vee (A_0 \wedge \ldots \wedge A_k) \supset A_{k+1} \tag{34}$$

for $k \geq 1$. Alternatively it can be defined is [van02]:

$$\mathtt{BC}_k =_{def} \bigvee_{0 \leq i < j \leq k} A_i \equiv A_j \tag{35}$$

For example,

$$\mathbf{BC_1} = \mathbf{Int} + A \vee (A \supset B) \tag{36}$$

$$\mathbf{BC_2} = \mathbf{Int} + A \vee (A \supset B) \vee ((A \wedge B) \supset C) \tag{37}$$

Clearly the logics $\mathbf{BC_1} = \mathbf{Cl}$, and $\mathbf{G_k} = \mathbf{GD} + \mathbf{BC_k}$.

Greatest Semiconstructive Logic: A **semiconstructive logic** is a logic where a proof of a disjunction $A \vee B$ requires a proof of A or a proof of B, but the proof of A or B is not necessarily constructive. The greatest (or maximal) semiconstructive logic **GSc** [FM93] is axiomatised by:

$$\mathbf{GSc} =_{def} \mathbf{Int} + \mathtt{BD}_2 + \neg A \vee (\neg A \supset \neg B) \vee (\neg A \supset \neg\neg B) \tag{38}$$

GSc is also known as **GS** or $\mathbf{BD_2F_2}$ in [AFM99b]. Axiomatisations for several families of semiconstructive logics are given in [FM93]. For brevity, we will not discuss them here.

Smetanich Logic: **Sm** is the logic obtained by combining **GD** and $\mathbf{BD_2}$, or equivalently [DN10]:

$$\mathbf{Sm} =_{def} \mathbf{Int} + (\neg B \supset A) \supset (((A \supset B) \supset A \supset A) \tag{39}$$

It is equivalent to $\mathbf{G_3}$, and is also known in the literature as $\mathbf{LC_2}$ (cf. [DN10]) or **HT**, for the "logic of here and there" [Pea99, LPV01].

Kreisel-Putnam Logic: was introduced by Kreisel and Putnam [KP57] as a counterexample to a conjecture by Łukasiewicz that **Int** was the maximally consistent logic with the disjunction property [Łuk52]. It is axiomatised by

$$\mathbf{KP} =_{def} \mathbf{Int} + (\neg A \supset B \vee C) \supset (\neg A \supset B) \vee (\neg A \supset C) \tag{40}$$

Exactly seven intermediate logics are interpolable [Mak79], all of them are noted above. They form the lattice in Figure 3.1, based on set inclusion [DN10].

$$\begin{tikzcd}
 & \mathbf{BD_2} \arrow[r] & \mathbf{GSc} \arrow[dr] & & \\
\mathbf{Int} \arrow[ur] \arrow[dr] & & & \mathbf{Sm} \arrow[r] & \mathbf{Cl} \\
 & \mathbf{Jan} \arrow[r] & \mathbf{GD} \arrow[ur] & &
\end{tikzcd}$$

FIGURE 3.1. The lattice of the seven interpolable intermediate logics.

Semantics for these logics will be given in Section 3.3 below. There are several other well-known intermediate propositional logics that we will not cover here, largely because the do not have Kripke models which are axiomatised by first-order formula, and so are not relevant to the methods discussed in this thesis.

A more detailed introduction to the philosophical motivations and history of intuitionistic and some intermediate logics can be found in [van02]. A detailed exposition of the semantics of intermediate logics can be found in [CZ97].

3.2.3. Modal Interpretation. An interpretation of **Int** in terms of the modal logic **S4** was given by Gödel [Göd33b]. We note the translation here, as it is used later as a basis for adapting calculi for model logics into calculi for **Int**.

Definition 3.3 (Gödel Translation). The Gödel translation of formulae [Göd33b] of formulae in **Int** to formulae in **S4** is the function given below:

$$
\begin{aligned}
P^{\Box} &= \Box P && \text{where } P \text{ is atomic}\\
(A \wedge B)^{\Box} &= A^{\Box} \wedge B^{\Box}\\
(A \vee B)^{\Box} &= A^{\Box} \vee B^{\Box}\\
(A \supset B)^{\Box} &= \Box(A^{\Box} \supset B^{\Box})
\end{aligned}
$$

Lemma 3.4. **Int** $\vdash A$ *iff* **S4** $\vdash A^{\Box}$.

Proof. Cf. [MT48, Art01] □

This can be extended to various intermediate logics, so that **Jan** corresponds to **S4.2**, **GD** corresponds to **S4.3** and **Cl** corresponds to **S5** [DN10].

This interpretation will be used later to adapt some calculi for **S4** into calculi for **Int**.

3.3. Kripke Semantics

3.3.1. Preliminaries. Kripke semantics are a kind of relational semantics that were introduced for modal logics in [Kri59a, Kri59b], and for **Int** were in [Kri65]. We give the notation and terminology of Kripke semantics that is relevant to this thesis below.

Definition 3.5 (Kripke Frame and Model). A **Kripke frame** M is a structure $\langle W, R\rangle$, where W is a non-empty set of **points**, sometimes called "nodes", "individuals", "worlds" or "states", $R \subseteq W^2$ is a binary relation between points. We may abbreviate $(x, y) \in R$ as Rxy. A **Kripke model** $\mathfrak{M} = \langle W, R, v\rangle$ is a model with a Kripke frame $\langle W, R\rangle$ and an **interpretation function** v. of type $W \times \mathtt{Var} \to \mathtt{Bool}$, where $\mathtt{Var}$ is the set of propositional variables, and $\mathtt{Bool}$ is the set of Boolean values $\{0, 1\}$.

A Kripke model belonging to a particular class $\mathcal{K}$ (e.g. intuitionistic Kripke models), will also have an **extended interpretation function** v' of type $W \times \mathtt{Prop} \to \mathtt{Bool}$ that extends v to cover all propositional formulae. (Because the definition of v' will depend on the class of Kripke models that $\mathfrak{M}$ belongs to, and not the specific model $\mathfrak{M}$, so it is omitted from the signature.) Using v', we define the **forcing relation**:

$$\mathfrak{M}, x \Vdash A =_{def} v'(x, A) = 1 \tag{41}$$

which we read as stating that formula A is true in state x. (We abbreviate $\neg(\mathfrak{M}, x \Vdash A)$ as $\mathfrak{M}, x \nVdash A$.) When the specific model is clear from the context, we simply write $x \Vdash A$ (or $x \nVdash A$).

We call a formula A **true** in a Kripke model $\mathfrak{M}$, written as $\mathfrak{M} \Vdash A$ (or as $\Vdash A$ when $\mathcal{K}$ is clear from the context), iff for all $w \in W.w \Vdash A$.

We call a formula A **valid** in a class $\mathcal{K}$ of Kripke models, written as $\mathcal{K} \Vdash A$ (or simply as $\Vdash A$ when $\mathcal{K}$ is obvious from the context), to denote that the formula A is true in all Kripke models from $\mathcal{K}$.

Remark 3.6. An alternative definition of a model is to use a domain function D in place of v such that $P \in D(w)$ iff $v(w, P) = 1$. This is why we distinguish between the interpretation function v and the extended interpretation function v'.

Definition 3.7 (Completeness). Let **S** be a propositional logic, and let $\mathcal{K}$ be a class of Kripke models. **S** is **complete** for $\mathcal{K}$ when for all $A \in \texttt{Prop}$, $A \in \mathbf{S}$ iff $\mathcal{K} \Vdash A$.

Definition 3.8 (Finite Kripke Frame). Let $M = \langle W, R \rangle$ be a Kripke frame. M is **finite** iff W is finite.

Definition 3.9 (Finite Model Property). A logic **S** has the **finite model property** (FMP) iff it is complete for a class of finite Kripke frames $\mathcal{K}$. That is, a if a formula $A \notin \mathbf{S}$, then there exists a finite Kripke model $\mathfrak{M} \in \mathcal{K}$ such that $\mathfrak{M} \nvDash A$.

3.3.2. Intuitionistic Kripke Frames and Models.

We recall the definition of intuitionistic Kripke frames and models below (cf. [CZ97, Min00, van02]).

Definition 3.10 (Intuitionistic Kripke Models). The class $\mathcal{K}_{\mathbf{Int}}$ of **intuitionistic Kripke models** is the class of all Kripke models $\mathfrak{M} = \langle W, R, v \rangle$ with the properties:

(1) R is a **pre-ordering** on W—i.e. R is **reflexive** and **transitive**.
(2) $\mathfrak{M}$ is **persistent**[2], that is, for all $x, y \in W$ and $P \in \texttt{Var}$, if $(x, y) \in R$ and $v(x, P) = 1$, then $v(y, P) = 1$;

The extended interpretation function v' for intuitionistic Kripke models is defined for all points $x, y \in W$ and formulae $A, B \in \texttt{Prop}$ as:

(1) $v'(x, A) = v(x, A)$ iff A is a propositional variable;

[2] Persistence is also called "monotonicity" or "heredity".

(2) $v'(x, \bot) = 0$;

(3) $v'(x, A \wedge B) = 1$ iff $v'(x, A) = 1$ and $v'(x, B) = 1$;

(4) $v'(x, A \vee B) = 1$ iff either $v'(x, A) = 1$ or $v'(x, B) = 1$;

(5) $v'(x, A \supset B) = 1$ iff whenever $(x, y) \in R$, if $v'(y, A) = 1$ then $v'(y, B) = 1$.

Proposition 3.11 (General Persistence). *In the class of Intuitionistic Kripke frames* $\mathcal{K}_{\mathbf{Int}}$*, for all states* $x, y \in W$*, if* Rxy *and* $v'(x, A) = 1$ *then* $v'(y, A) = 1$*, for all compound formulae* A*.*

Proof. By induction on the structure of A. See [Min00]. □

Lemma 3.12. **Int** *has the FMP.*

Proof. From completeness with respect to finite intuitionistic Kripke frames, e.g. [Min00]. □

Definition 3.13 (Rooted frame and model). A **rooted (Kripke) frame** (also called a pointed frame) M is a Kripke frame $\langle W, R \rangle$, where there exists $x \in W$ such that, for all $y \in W$, $(x, y) \in R$. x is called the **root** of a Kripke frame (also called the **distinguished point**).

A **rooted (Kripke) model** is a Kripke model based on a rooted Kripke frame.

Lemma 3.14. *A rooted intuitionistic Kripke frame is partially ordered—that is, it is reflexive, transitive and antisymmetric.*

Proof. [Min00, §7.2] or [CZ97, §2.3]. □

Theorem 3.15. *A formula* A *is valid in all intuitionistic Kripke models iff it is valid in all rooted intuitionistic Kripke models.*

Proof. [Min00, §7.2] or [CZ97, §2.3]. □

The existence of a distinguished point in an intuitionistic Kripke frame is generally used for completeness proofs for calculi, e.g. [Min00]. Later we will use Theorem 3.15 to justify the soundness of a root rule for relational sequent calculi. (For example, see Lemma 9.30 on page 205.)

3.3.3. Semantics of Intermediate Logics. We present the extensions for various intermediate logics belows, but first define useful notation and terminology with respect to the properties of some of these logics.

Because extensions to intuitionistic Kripke frames can be considered rooted, and thus partially ordered, we can apply the terminology of trees and directed graphs (e.g. from [Die05]) to them:

DEFINITION 3.16 (Subframe). Let $M = \langle W, R\rangle$ be an intuitionistic Kripke frame. We call $M' = \langle W', R'\rangle$ a **subframe** of M if $W' \subseteq W$, $R' \subseteq R$ and R' is a partial order.

DEFINITION 3.17 (Chain). Let $M = \langle W, R\rangle$ be an intuitionistic Kripke frame. A set of points $W' \subseteq W$ forms a **chain** in M iff for all $x, y \in W'$, either $(x, y) \in R$ or $(y, x) \in R$.

DEFINITION 3.18 (Depth of a Kripke Frame). The **depth** of a Kripke Frame $M = \langle W, R\rangle$ is the size of the largest chain in M.

DEFINITION 3.19 (Antichain). Let $M = \langle W, R\rangle$ be an intuitionistic Kripke frame. A set of points $W' \subseteq W$ forms an **antichain** in M iff for all $x, y \in W'$, both $(x, y) \notin R$ and $(y, x) \notin R$.

DEFINITION 3.20 (Width of a Kripke Frame). The **width** of a Kripke Frame $M = \langle W, R\rangle$ is the size of the largest antichain in M.

DEFINITION 3.21 (Cardinality of a Kripke Frame). The **cardinality** of a Kripke Frame $\langle W, R\rangle$ is $|W|$ (the size of W).

We now give the semantic properties of the intermediate logics listed in Section 3.2.2 above, defined as extensions to an intuitionistic Kripke frame $\mathfrak{M} = \langle W, R\rangle$. We assume that quantified variables below belong to the set of points W, and that relations belong to the set of relations R, but generally omit specific reference to them below for brevity.

The frame properties given below come from [CZ97] unless otherwise noted. All of these logics have the finite model property (FMP) [CZ97, van02].

Classical Logic: Cl corresponds to the class of **symmetric** Kripke frames:

$$\forall xy.Rxy \supset Ryx \tag{42}$$

Because we can assume that the Kripke frames are rooted, from antisymmetry, we can conclude that $\forall xy.x = y$, and thus $|W| = 1$.

Jankov-De Morgan Logic: Jan corresponds to the class of **directed** (or converging) Kripke frames:

$$\forall wxy.(Rwx \wedge Rwy) \supset \exists z.(Rxz \wedge Ryz) \tag{43}$$

Because frames are rooted, we can simplify this to $\exists z.(Rxz \wedge Ryz)$.

Gödel-Dummett Logic: GD corresponds to the class of **linear** Kripke frames:

$$\forall xy.Rxy \vee Ryx \tag{44}$$

Logics of Bounded Depth: $\mathbf{BD_k}$ corresponds to the class of Kripke frames with bounded depth k:

$$\forall x_0, \ldots, x_k. \bigwedge_{i=0}^{k-1} Rx_ix_{i+1} \supset \bigvee_{i \neq j} x_i = z_j \tag{45}$$

Using antisymmetry, this can be simplified to

$$\forall x_0, \ldots, x_k. \bigwedge_{i=0}^{k-1} Rx_ix_{i+1} \supset \bigvee_{i=0}^{k-1} Rx_{i+1}x_i \tag{46}$$

For example, the characteristic frame condition of $\mathbf{BD_2}$ is:

$$\forall xyz.((Rxy \wedge Ryz) \supset (Ryx \vee Rzy)) \tag{47}$$

Gödel Logics: $\mathbf{G_k}$ corresponds to the class of linear Kripke frames (44) of bounded depth $k-1$ (46). For example, $\mathbf{G_3}$ (also called **Sm**) is the union of **GD** and $\mathbf{BD_2}$.

Logics of Bounded Width: $\mathbf{BW_k}$ corresponds to the class of Kripke frames where every rooted subframe is of a width bounded by k:

$$\forall x, y_0, \ldots, y_k. \bigwedge_{i=0}^{k} Rxy_i \supset \bigvee_{j \neq i} Ry_iy_j \tag{48}$$

Because we can assume that the frames are rooted, this can be simplified to

$$\forall x, y_0, \ldots, y_k. \bigvee_{j \neq i} Ry_iy_j \tag{49}$$

Logics of Bounded Top Width: $\mathbf{BTW_k}$ corresponds to the class of Kripke frames where there are at most k points at the top of the frame:

$$\forall x, y_0, \ldots, y_k. \bigwedge_{i=0}^{k} Rxy_i \supset \exists z. \bigvee_{j \neq i} (Ry_i z \wedge Ry_j z) \tag{50}$$

Because we can assume that the frames are rooted, we can simplify this to

$$\forall y_0, \ldots, y_k \exists z. \bigvee_{i=0, j \neq i}^{k} (Ry_i z \wedge Ry_j z) \tag{51}$$

Logics of Bounded Cardinality: $\mathbf{BC_k}$ corresponds to the class of Kripke frames where $|W| = k$, which can be expressed as the frame condition

$$\forall x_0, x_1, \ldots, x_k. \bigwedge_{i=1}^{k} Rx_0 x_i \supset \bigvee_{j \neq i} x_j = x_i \tag{52}$$

Using antisymmetry, this can be simplified to

$$\forall x_0, x_1, \ldots, x_k. \bigwedge_{i=1}^{k} Rx_0 x_i \supset \bigvee_{i=1}^{k} Rx_i x_0 \tag{53}$$

Greatest Semiconstructive Logic: GSc corresponds to the class of Kripke frames that have a depth of at most 2 (47) and a bounded top width (51) of 2. That is for any three distinct points, there is a point accessible from two of them . This corresponds to (47) and (51) for $k = 2$:

$$\forall wxy. \exists z. ((Rwz \wedge Rxz) \vee (Rxz \wedge Ryz) \vee (Rwz \wedge Ryz)) \tag{54}$$

Kreisel-Putnam Logic: KP corresponds to the class of Kripke frames with the following condition:

$$\forall xyz. (Rxy \wedge Rxz \wedge \neg Ryz \wedge \neg Rzy \supset \exists u. (Rxu \wedge Ruy \wedge Ruz \wedge \mathtt{fin}_2(u, y, z))) \tag{55}$$

where we adapt (51) to be the function

$$\mathtt{fin}_k(x, y_0, \ldots, y_k) =_{def} \forall w. Rxw \supset \exists z. Rwz \wedge \bigvee_{i=0, j \neq i}^{k} (Ry_i z \wedge Ry_j z)$$

REMARK 3.22. We have not discussed other well-known intermediate logics, such as the logic of bounded branching ($\mathbf{BB_k}$) [GD74], Scott logic (**SL**) or Anti-Scott logic (**ASL**) [KP57] because they do not have Kripke frames that are axiomatised by first-order formulae. (See [FM93] for further discussion.)

3.3.4. Beth Semantics. We give a brief outline of Beth semantics for **Int** here, as it is an alternative relational semantics (that predates Kripke semantics), and inspired the hypersequent calculus by Beth from [Bet59] (shown in Figure 4.1 on page 83), and possibly other calculi such as Maslov's **O** [Mas67, Mas69] (Figure 5.1 on page 112).

The description of Beth semantics given here is based on [van02]. The reader is also referred to [Tv88, Ch. 13], and to [Tv99] for the historical context.

DEFINITION 3.23 (Beth Frames and Models). We define Beth Frames and Models similarly to Kripke Frames and Models (Definition 3.5 on page 44). A **Beth frame** M is a tree (or **spread**) $\langle W, R\rangle$, where W is a non-empty set of **points**, $R \subseteq W^2$ is a binary relation between points We may abbreviate $(x,y) \in R$ as Rxy. M is rooted, that is, there exists a distinguished point $x \in W$ such that for all $y \in W$, $(x,y) \in R$.

Let $M = \langle W, R\rangle$ be a Beth frame. A **path** $P \subseteq W$ through a point $x \in W$ is a maximal, linearly ordered set (on R). A **bar** for $x \in W$ is a subset $B \subseteq W$ such that if P is a path through x (that is, $x \in P$), then there exists $y \in B$ such that $y \in P$. Informally, all paths through x pass through the bar B.

The definitions relating to Kripke models are extended naturally to Beth models.

DEFINITION 3.24 (Intuitionistic Beth Models). The class $\mathcal{B}_{\mathbf{Int}}$ of **intuitionistic Beth models** is the class of all Beth models $\mathfrak{M} = \langle W, R, v\rangle$ with the properties:

(1) R is a **pre-ordering** on W—i.e. R is **reflexive** and **transitive**.
(2) $\mathfrak{M}$ is **persistent**, that is, for all $x \in W$ and $P \in \mathtt{Var}$, if $v(x,P) = 1$, then there exists a bar $B \subseteq W$ for x such that for all $y \in B$, $v(y,P) = 1$;;
(3) For all $x \in W$, $v(x,\perp) = 0$;
(4) For all $x \in W$, $v(x, A \wedge B) = 1$ iff $v(x,A) = 1$ and $v(x,B) = 1$;
(5) $v(x, A \vee B) = 1$ iff there exists a bar $B \subseteq W$ for x such that for all $y \in B$, either $v(y,A) = 1$ or $v(y,B) = 1$;
(6) $v(x, A \supset B) = 1$ iff whenever $(x,y) \in R$, if $v(y,A) = 1$ then $v(y,B) = 1$.

Remark 3.25. For simplicity, we do not describe the class of intuitionistic Beth models in terms of an extended interpretation function as we do for intuitionistic Kripke models.

We note that intuitionistic Kripke models can be considered intuitionistic Beth models where the bars in cases 2 and 5 are restricted to singletons. (In [van02], both models are given using the same notation with this very distinction.)

A method for translating Kripke models into Beth models is given in [Tv88, p. 680]. However, according to [Tv88], arbitrary Beth models cannot be translated into Kripke models, because there are Beth models $\mathfrak{M}$ such that $\mathfrak{M} \nvDash A \vee B$ but $\mathfrak{M} \vDash A$ and $\mathfrak{M} \vDash B$.

Note that unlike Kripke models, Beth models which refute non-intuitionistic formulae (formulae that are not in **Int**) may have infinite branches. Beth models which refute disjunctions have bars which are pushed infinitely upwards [van02, p. 30].

3.4. Geometric Formulae

In [Nv98] it was shown that axioms of certain forms can be added as rules to the systems based on the calculi **G3**[**cim**] (see Appendix A on page 233) without losing the admissibility of weakening, contraction and cut. In [Neg03], this result was extended to **geometric rules**—that is, rules based on formulae of the form $\forall\bar{x}.(A \supset B)$, where A, B do not contain implications or universal quantifiers. This work was used for the development of labelled sequent frameworks for various non-classical logics in [Neg05, Neg07, DN10], including **G3I*** (Figure 5.4 on page 122), which will be examined later in this thesis.

Definition 3.26 (Geometric Formulae). We define the set of **geometric formulae** inductively:

(1) $P\bar{x}$ is geometric, if $P \in \mathtt{Pred}_k$ for some $k \geq 0$;
(2) $\perp$ is geometric;
(3) $A \wedge B$ is geometric, if A and B are geometric;
(4) $A \vee B$ is geometric, if A and B are geometric;
(5) $\exists\bar{x}.A$ is geometric, if A is geometric.

The set of **positive geometric formulae** is the set of geometric formulae excluding $\perp$ as a subformula.

Definition 3.27 (Geometric Implications). A **geometric implication** is a formula of the form $\forall\bar{x}.(A \supset B)$, where A and B are geometric formulae.

Definition 3.28. Let G_R be the set of positive geometric formulae with atomic formulae restricted to instances of a single binary relation R. Let H_R denote the set of geometric implications constructed from geometric formulae in G_R. We define $\mathsf{Geo} =_{def} G_R + H_R$.

Remark 3.29. In [Pal02], $\top$ is included as a primitive atomic formulae, and is thus a geometric formula, even though as defined $\top$ is an implication. However, $\forall\bar{x}.(\top \supset A)$ is intuitionistically equivalent to $\forall\bar{x}.A$, so we define Geo as the union of both sets.

Int*/Geo denotes the class of intermediate logics determined by a class of intuitionistic Kripke models with frame axioms that are in Geo. (We call these **geometric Kripke models**.) We estimate that **Int***/Geo contains uncountably many logics by assuming that it corresponds with the power set of the denumerable set Geo, although this does not account for the fact that the logics form a lattice, where combining axioms corresponds to join operations (i.e, some axioms subsume others, e.g. linearity + symmetry is the same as symmetry.) So determining the actual cardinality of **Int***/Geo is an open problem.

Determining what subset of logics in **Int***/Geo have the finite model property is also an open problem. Clearly, some of them do, as some logics which have that property are in **Int***/Geo:

Proposition 3.30. *The logics* **Int**, $\mathbf{BD_k}$, $\mathbf{BW_k}$, $\mathbf{BTW_k}$ *and* $\mathbf{BC_k}$ *are in* **Int***/Geo.

Proof. Their characteristic frame axioms are geometric implications. □

Corollary 3.31. *The logics* **Cl**, **Jan**, **GD**, $\mathbf{G_k}$ *and* **GSc** *are in* **Int***/Geo.

Proof. As special cases of the logics in Proposition 3.30. □

We note that the logic **KP** does not appear to be in **Int***/Geo, as it contains universally quantified formulae and implications as subformulae. We are not aware of an equivalent form of (55) that is in Geo. (Whether **KP** is in **Int***/Geo is an open problem.)

3.5. Partially Shielded Formulae

We introduce the fragment of first-order formulae (called here **partially shielded formulae**, or PSF for short) that is adequate for expressing the translation of formulae in

Int*/$\mathtt{Geo}$ into corresponding predicate formulae for intermediate Kripke models. We also introduce a calculus for **G3c/PSF** (Figure 3.2 on page 57) for manipulating sequents of formulae in $\mathtt{PSF}$, and extend it with rules that preserve the admissibility of structural rules and cut but allow it to be used for deriving the translations of formulae in intermediate logics.

A discussion of embedding modal formulae into classical predicate logic, such that the resulting formulae express the Kripke semantics of the original modal logics, can be found in [BdY01]. A general discussion of embedding non-classical formulae into first-order classical logic can be found in [ONdG01].

Notation 3.32. The expression $A\{\bar{x}\}$ denotes a formula A such that $\mathtt{FV}(A) \subseteq \bar{x}$. The expression $\Gamma\{\bar{x}\}$ denotes a multiset of formulae such that for all $A \in \Gamma$, $\mathtt{FV}(A) \subseteq \bar{x}$.

The expression $A\{\bar{x}\bar{y}\}$ denotes a formula A such that $\mathtt{FV}(A) \subseteq \bar{x} \cup \bar{y}$, where $\bar{x}$ and $\bar{y}$ are not necessarily disjoint, since the variables are named. We make no assumption about the order of variables within the vector.

For simplicity, $A\bar{x}$ will be used when it is clear from the context that it denotes a formula A such that $\mathtt{FV}(A) \subseteq \bar{x}$ rather than a formula of the form $A\bar{x}$ such that $\mathtt{FV}(A\bar{x}) = \bar{x}$. (For example, in the presentation of the calculus **G3c/PSF** in Figure 3.2 on page 57.)

Definition 3.33 (Strict Partially Shielded Formula). We define the set $\mathtt{SPSF}$ of **strict partially shielded formulae** inductively:

(1) $Px \in \mathtt{SPSF}$ if $P \in \mathtt{Pred}_1$;
(2) $\bot \in \mathtt{SPSF}$;
(3) $A \wedge B \in \mathtt{SPSF}$ if $A, B \in \mathtt{SPSF}$;
(4) $A \vee B \in \mathtt{SPSF}$ if $A, B \in \mathtt{SPSF}$;
(5) $\forall y.(\mathcal{R}xy \wedge A\{y\}) \supset B\{y\} \in \mathtt{SPSF}$ if $A, B \in \mathtt{SPSF}$.

Note that $\mathcal{R}$ is a fixed binary predicate symbol in $\mathtt{Pred}_2$, with $\mathcal{R}xy \in \mathtt{RF}$ (also known as $\mathcal{R}$-formulae) called the **shield** of the formula in case 5. Although $\mathcal{R}xy$ is a subformula in that case, $\mathcal{R}xy \notin \mathtt{SPSF}$.

Lemma 3.34. *Let* $A \in \mathtt{SPSF}$. *Then* $|FV(A)| \leq 1$.

Proof. By induction on the structure of A. □

Formulae in SPSF can be used to adequately express the translation of intuitionistic formulae into their corresponding first-order formulae for intuitionistic Kripke models, for example, $A \supset (B \supset A)$ corresponds to $\forall y_1.((\mathcal{R}xy_1 \wedge \hat{A}y_1) \supset (\forall y_2.(\mathcal{R}y_1y_2 \wedge \hat{B}y_2) \supset \hat{A}y_2))$, using the translation function from formulae to SPSF is given later in Chapter 6. However, the frame axioms for this class of Kripke models—reflexivity and transitivity—and for classes that correspond to stronger logics in $\mathbf{Int}^*/\mathtt{Geo}$, cannot be expressed in SPSF. So we extend the definition:

Definition 3.35 (Partially Shielded Formulae). We define the set PSF of **partially shielded formulae** inductively:

(1) $A \in$ PSF, if $A \in$ SPSF;
(2) $\mathcal{R}xy \in$ PSF, where $\mathcal{R}xy \in$ RF;
(3) $\top \in$ PSF (see the remark below);
(4) $(A \wedge B) \in$ PSF if $A, B \in$ PSF;
(5) $(A \vee B) \in$ PSF if $A, B \in$ PSF;
(6) $\forall \bar{y}.(P\{\bar{x}\bar{y}\} \wedge A_1\{\bar{x}\bar{y}\} \wedge \ldots \wedge A_n\{\bar{x}\bar{y}\}) \supset B\{\bar{x}\bar{y}\} \in$ PSF where $n \geq 0$, if $P, A_1, \ldots, A_n, B \in$ PSF, P is atomic and $\bar{y} \cap FV(P) \neq \emptyset$;
(7) $\forall \bar{y}.A \in$ PSF if $A \in$ PSF;
(8) $\exists \bar{y}.A \in$ PSF if $A \in$ PSF;

In case 6, P is called the **shield** of the formula. Note that case 5 of the definition for SPSFis also a special case of case 6.

Remark 3.36. $\top$ is explicitly included in PSF as a primitive symbol because $\bot \supset \bot$ (as $\top$ is normally defined) is not in PSF.

Definitions 3.33 and 3.35 are *syntactic*: although $A \vee B$ is classically equivalent to $\neg A \supset B$, the latter is not in SPSF or PSF.

Proposition 3.37. SPSF $\subset$ PSF $\subset \mathtt{Form}_1$.

Proof. From Definitions 3.33 and 3.35. □

Terminology 3.38. A **multiset is in** SPSF, written as SPSF*, iff all of the formulae in that multiset are in SPSF. We extend this terminology to sequents and to PSF naturally.

NOTATION 3.39. A sequent $(\Gamma \Rightarrow \Delta) \in \mathtt{SPSF}^{*}{}_{\leq}$, iff $\Gamma \in (\mathtt{SPSF} + \mathtt{RF})^{*}$ and $\Delta \in \mathtt{SPSF}^{*}$. This corresponds to a basic relational sequent, as will be shown later in Chapter 8. (Note that this is a slice abuse of notation, as $\mathcal{R}$ formulae are not in SPSF.)

REMARK 3.40. The sets GF and LGF of **guarded formulae** and **loosely guarded formulae** [AvN96], have similarities to the set PSF. Recall that formulae in LGF are of the form $\forall \bar{y}.(P_1\{\bar{x}\bar{y}\} \wedge \ldots \wedge P_k\{\bar{x}\bar{y}\}) \supset A\{\bar{x}\bar{y}\}$, where $P_i\{\bar{x}\bar{y}\}$ is an atom, and that formulae in GF are special cases of LGF where $k = 1$. The formula to the left of the implication is called the **guard**.

Because PSF includes non-atomic formulae in the scope of what would be called the guard, the term "shielded" is used instead. Unlike GF and LGF, the formula to the right of the implication must also be in PSF.

Note also that GF and LGF allow predicates in $\mathtt{Pred}_3$ or higher arity as guards. The motivation for defining the set PSF rather than using GF or LGF, was that all subformulae of a formula in PSF are in PSF, and that geometric implications are in PSF.

We note that implications in SPSF (case 5) are intuitionistically equivalent to the guarded formula $\forall y.\mathcal{R}xy \supset (A\{y\} \supset B\{y\})$.

3.5.1. A Kripke Semantics for Partially Shielded Formulae.

In later chapters of this thesis, we use translations of hypersequents and labelled sequents into sequents in $\mathtt{PSF}^{*}$ to show the relationships between these formalisms, or to justify the soundness of these rules. This requires a definition of the interpretation of formulae in PSF with respect to Kripke models.

DEFINITION 3.41 (Satisfaction of a Formulae in PSF by a Kripke Model). Let $\mathfrak{M} = \langle W, R, v\rangle$ be a Kripke model, and let $D(\hat{x})$ be an surjective function from $\mathtt{Term}_0$ to W. We define the **satisfiability** of a formula $A \in \mathtt{PSF}$ by a Kripke model inductively:

(1) $\mathfrak{M} \models \mathcal{R}\hat{x}\hat{y}$ iff $(x, y) \in R$, where $D(\hat{x}) = x$ and $D(\hat{y}) = y$;
(2) $\mathfrak{M} \models \hat{P}\hat{x}$ iff $v(x, P) = 1$, where $D(\hat{x}) = x$;
(3) $\mathfrak{M} \not\models \bot$;
(4) $\mathfrak{M} \models \top$;
(5) $\mathfrak{M} \models A \wedge B$ iff $\mathfrak{M} \models A$ and $\mathfrak{M} \models B$;
(6) $\mathfrak{M} \models A \vee B$ iff either $\mathfrak{M} \models A$ or $\mathfrak{M} \models B$;

(7) $\mathfrak{M} \vDash \forall\bar{y}.(P\{\bar{x}\bar{y}\} \wedge A_1\{\bar{x}\bar{y}\} \wedge \ldots \wedge A_n\{\bar{x}\bar{y}\}) \supset B\{\bar{x}\bar{y}\}$ iff $\mathfrak{M} \vDash P\{\bar{x}\bar{y}\} \wedge A_1\{\bar{x}\bar{y}\} \wedge \ldots \wedge A_n\{\bar{x}\bar{y}\}$ implies $\mathfrak{M} \vDash B\{\bar{x}\bar{y}\}$;

(8) $\mathfrak{M} \vDash \forall\hat{x}.A$ iff for all $w \in W$, $\mathfrak{M} \vDash [\hat{w}/\hat{x}]A$, where $D(\hat{w}) = w$;

(9) $\mathfrak{M} \vDash \exists\hat{x}.A$ iff there exists $w \in W$ such that $\mathfrak{M} \vDash [\hat{w}/\hat{x}]A$, where $D(\hat{w}) = w$.

where P is the propositional variable that corresponds to the unary predicate $\hat{P}$.

We extend this notion naturally for sequents in $\mathsf{SPSF}^{*2}_{\leq}$ and PSF^2 so that $\mathfrak{M} \vDash \Gamma \Rightarrow \Delta$ iff $\mathfrak{M} \vDash \bigwedge\Gamma \supset \bigvee\Delta$.

Definition 3.42 (Validity of a Formulae in PSF in a Class of Kripke Models). A formula $A \in \mathsf{PSF}$ is **valid** for a class of Kripke models $\mathcal{K}$ iff for every Kripke model $\mathfrak{M} \in \mathcal{K}$ there exists a mapping D from $\mathtt{Term}_0$ to W such that $\mathfrak{M} \vDash A$.

3.5.2. The calculus G3c/PSF. We introduce the calculus **G3c/PSF** in Figure 3.2 on the facing page.

Terminology 3.43. In the $\mathsf{L}\forall\supset$ (viz. $\mathsf{R}\forall\supset$) rule of **G3c/PSF** (Figure 3.2 on the next page), the active and principal formula $\mathcal{R}\bar{x}\bar{z}$ is called the **shield**, the variables $\bar{z}$ of $\mathcal{R}$ are called **bindable**, and the variables $\bar{x}$ are called **unbindable**.

Remark 3.44. We (again) note that the $\mathsf{L}\forall$ and $\mathsf{R}\forall$ rules of **G3c/PSF** cannot be applied to shielded implications (or specifically, to all bound variables in shielded implications) because the formulae in the premisses would not be partially shielded.

Below we discuss the properties of that calculus, including an embedding in **G3c** and cut elimination.

Remark 3.45. Ideas for **G3c/PSF**—in particular for the $\mathsf{L}\forall\supset$ rule that uses the shield formula as a "key" to unlock the shielded formula in **G3c/PSF** were influenced by a general calculus for guarded logic that was developed in [DS06].

We do not present complexity or decidability results about **G3c/PSF** here, as that is beyond what is needed for our purposes in this work. It is an area for future investigation.

Clearly some of the formulae $A_i\{\bar{x}\bar{y}\}$ in the principal formulae of $\mathsf{L}\forall\supset$ and $\mathsf{R}\forall\supset$ can be formulae in RF. Where corresponding $A_i\{\bar{x}\bar{z}\}$ formulae occur as side formulae in the antecedent, the ith premiss is an axiom and can be ignored. For example, we suppose

$$\frac{}{\Gamma, P \Rightarrow P, \Delta}\ \mathsf{Ax} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \mathsf{L}\bot \qquad \frac{}{\Gamma \Rightarrow \top, \Delta}\ \mathsf{R}\top$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ \mathsf{R}\wedge$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ \mathsf{R}\vee$$

$$\frac{\langle\ \Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.\ldots \Rightarrow A_i\bar{x}\bar{z}, \Delta\ \rangle_{i=1}^{n} \quad \Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.\ldots, B\bar{x}\bar{z} \Rightarrow \Delta}{\Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}) \supset B\bar{x}\bar{y} \Rightarrow \Delta}\ \mathsf{L}\forall\supset$$

$$\frac{\Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z} \Rightarrow B\bar{x}\bar{z}, \Delta}{\Gamma \Rightarrow \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}) \supset B\bar{x}\bar{y}, \Delta}\ \mathsf{R}\forall\supset$$

$$\frac{\Gamma, \forall\bar{x}.A, [\bar{y}/\bar{x}]A \Rightarrow \Delta}{\Gamma, \forall\bar{x}.A \Rightarrow \Delta}\ \mathsf{L}\forall \qquad \frac{\Gamma \Rightarrow [\bar{z}/\bar{x}]A, \Delta}{\Gamma \Rightarrow \forall\bar{x}.A, \Delta}\ \mathsf{R}\forall$$

$$\frac{\Gamma, [\bar{z}/\bar{x}]A \Rightarrow \Delta}{\Gamma, \exists\bar{x}.A \Rightarrow \Delta}\ \mathsf{L}\exists \qquad \frac{\Gamma \Rightarrow [\bar{y}/\bar{x}]A, \exists\bar{x}.A, \Delta}{\Gamma \Rightarrow \exists\bar{x}.A, \Delta}\ \mathsf{R}\exists$$

We omit the curly brackets for brevity, e.g. using $A\bar{x}$ instead of $A\{\bar{x}\}$. All formulae $A, A_1, \ldots A_n, B, P \in \mathsf{PSF}$, with $\mathcal{R}$ being atomic, and $\bar{z}$ is *fresh* for the conclusion of the $\mathsf{R}\forall$, $\mathsf{L}\exists$ and $\mathsf{R}\forall\supset$ rules, and $\forall\bar{y}\ldots$ is an abbreviation for $\forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}) \supset B\bar{x}\bar{y}$.

FIGURE 3.2. The calculus **G3c/PSF** for sequents of partially shielded formulae.

formulae $A_1\bar{x}\bar{y}, \ldots, A_m\bar{x}\bar{y}$ (for $m \leq n$) to be formulae in RF:

$$\frac{\langle\ \Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{y}, \ldots, A_m\bar{x}\bar{y}, \forall\bar{y}.\ldots \Rightarrow A_i\bar{x}\bar{z}, \Delta\ \rangle_{i=1}^{n} \quad \Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.\ldots, B\bar{x}\bar{z} \Rightarrow \Delta}{\Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{y}, \ldots, A_m\bar{x}\bar{y}, \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}) \supset B\bar{x}\bar{y} \Rightarrow \Delta}\ \mathsf{L}\forall\supset$$

This allows us to give a simpler form of the $\mathsf{L}\forall\supset$ rule below:

PROPOSITION 3.46. *The* $\mathsf{L}\forall\supset'$ *rule*

$$\frac{\Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z}, \forall\bar{y}\ldots, B\bar{x}\bar{z} \Rightarrow \Delta}{\Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z}, \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge \bigwedge A_i\bar{x}\bar{y}) \supset B\bar{x}\bar{y} \Rightarrow \Delta}\ (\mathsf{L}\forall\supset')$$

is derivable in **G3c/PSF**.

Proof. Straightforward. (See Proposition B.1 on page 237.) □

REMARK 3.47. The $\mathsf{L}\forall\supset'$ rule is useful for cases where some of the $A_i\bar{x}\bar{y}$ that are in negative positions of the principal formula are relational formulae. In these cases, the premisses that contain a relational formula in the succedent are axioms, and can be ignored. This is important for translations between partially shielded sequents and languages for relational sequents (such as the one used in this thesis) that do not allow relational formulae in the succedent.

We note that proofs in **G3c**/**PSF** can be embedded in the system **G3c**:

LEMMA 3.48 (Embedding). *If* **G3c**/**PSF** $\vdash \Gamma \Rightarrow \Delta$, *then* **G3c** $\vdash \Gamma \Rightarrow \Delta$.

Proof. Straightforward. (The proof is written out in Lemma B.2 on page 237.) □

COROLLARY 3.49 (Soundness). **G3c**/**PSF** *is sound.*

Proof. From Lemma 3.48 and the soundness of **G3c** [TS00]. □

DEFINITION 3.50. A first-order term t is **free for** x **in** A iff t does not contain a free variable y that would become bound by replacing x with t in A. (In the case of formulae in PSF, which does not allow functions in terms, this means that t is not equal to a bound variable y in A such that replacing x with t would change a free variable into a bound variable.)

This notion is extended to multisets and sequents naturally.

LEMMA 3.51 (Substitution). *Variable substitution*

$$\frac{\Gamma \Rightarrow \Delta}{[t/x]\Gamma \Rightarrow [t/x]\Delta}\ \mathsf{[t/x]}$$

where t is free for x in Γ, Δ, *is depth-preserving admissible in* **G3c**/**PSF**.

Proof. By induction on the derivation depth. □

NOTATION 3.52. For readability, we omit parentheses from instances of the substitution rule.

LEMMA 3.53 (Weakening). *The weakening rules*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathsf{(LW)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\ \mathsf{(RW)}$$

are depth-preserving admissible in **G3c**/**PSF**.

Proof. By induction on the derivation depth. □

LEMMA 3.54 (Generalised axioms). *Sequents of the form $A, \Gamma \Rightarrow \Delta, A$, where A is an arbitrary formula in* PSF*, are derivable in* $\mathbf{G3c/PSF}$.

Proof. By induction on the size of A. □

LEMMA 3.55 (Invertibility). *The rules of* $\mathbf{G3c/PSF}$ *are depth-preserving invertible.*

Proof. Straightforward, For $\mathsf{L}\forall$ and $\mathsf{L}\forall\supset$, LW is used. For all other rules, by simultaneous proof, using induction on the derivation depths. □

LEMMA 3.56 (Nullary Connective Deletion). *The following constant deletion rules*

$$\frac{\top, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathsf{L}\top) \qquad \frac{\Gamma \Rightarrow \Delta, \bot}{\Gamma \Rightarrow \Delta}\ (\mathsf{R}\bot)$$

are depth-preserving admissible in $\mathbf{G3c/PSF}$.

Proof. By induction on the derivation depth. (See Lemma B.4 on page 238.) □

LEMMA 3.57 (Contraction). *The contraction rules*

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (\mathsf{LC}) \qquad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}\ (\mathsf{RC})$$

are depth-preserving admissible in $\mathbf{G3c/PSF}$.

Proof. Straightforward simultaneous induction on the derivation depth. (The proof is written out in Lemma B.5 on page 239.) □

THEOREM 3.58 (Cut). *The context-splitting cut rule*

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathsf{cut})$$

is admissible in $\mathbf{G3c/PSF}$.

Proof. By induction on the cut rank (a lexically-ordered pair consisting of the size of the cut formula and sum of the depths of the premisses). (The proof is given in Theorem B.6 on page 240.) □

COROLLARY 3.59 (Cut). *The context-sharing cut rule*

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ (\mathsf{cut}')$$

is admissible in **G3c/PSF**.

Proof. From Theorem 3.58, using LC and RC. □

3.5.3. Geometric Extension of G3c/PSF. In later chapters we will introduce translations of hypersequents, simply labelled sequents and relational sequents into PSF, such that a (hyper)sequent is valid in a class of Kripke models iff its translation into PSF is derivable in **G3c/PSF**. But in order to use **G3c/PSF** to validate sequents that are translations of formulae in an intermediate logic, the axioms corresponding to the properties of the corresponding Kripke models must be included. For example,the sequent $\mathcal{R}xy, \mathcal{R}yz, Ax \Rightarrow Az$ is only derivable in **G3c/PSF** when some form of the transitivity and persistence axioms are included.

In [Min00], a translation ϕ is given for a formula A that is true with respect to a set of axioms that is *parametrised* by the formulae A. Adapted for the notation used here:

$$\kappa_A =_{def} \forall x.\mathcal{R}xx, \forall xyz.\mathcal{R}xy \wedge \mathcal{R}yz \supset \mathcal{R}xz, \bigcup_{i=1}^{n} \{ \ \forall xy.Rxy \supset \hat{P}_i x \supset \hat{P}_i y \ \}$$

corresponds to the reflexivity, transitivity and persistence axioms of intuitionistic Kripke models, such that for all formulae $A \in \mathtt{Prop}$, **LJpm** $\vdash \kappa_A \supset \phi A$ iff **LJpm** $\vdash A$, where each $\hat{P}_i \in \mathtt{Pred}_1$ corresponds to a propositional variable $P_i \in \mathtt{Var}_1$, for all atomic propositional variables that are subformulae of A, and **LJpm** is a sound and complete sequent calculus for **Int** (see Figure A.4 on page 236).

This could be adapted to extend the proof theory of **G3c/PSF**, where of sequents of the form $\kappa_X, \Gamma \Rightarrow \Delta$, where $\kappa_X, \Gamma, \Delta \subset \mathtt{PSF}$, and $X \subset \mathtt{Var}$. The parametrisation could even be eliminated by extend the language of PSF and the rules of **G3c/PSF** for second-order quantification and use a second-order definition of the persistence axiom:

$$\kappa =_{def} \forall x.\mathcal{R}xx, \forall xyz.\mathcal{R}xy \wedge \mathcal{R}yz \supset \mathcal{R}xz, \forall P^1.\forall xy.(\mathcal{R}xy \wedge Px) \supset Py$$

where $P \in \mathtt{Pred}_1$. Work on a second-order extension of **G3c/PSF** is an area for future investigation. Another alternative is to define the the $\mathcal{R}$ formula as an abbreviation for

$$\mathcal{R}xy =_{def} \forall P^1.(\mathcal{R}xy \wedge Px) \supset Py$$

It appears that reflexivity and transitivity follow from this defined relation, but it is unclear whether a calculus using this definition correspond with **Int** or a stronger logic. This too is an area for later research.

Because we are using **G3c/PSF** to logics in **Int**$^*/\mathtt{Geo}$, a simpler alternative is to extend **G3c/PSF** with **geometric rules** (Definition 3.60 below) that correspond to the axioms in κ. Recall the work cited in Section 3.4 on page 51 that adding such rules to a calculus based on **G3c** does not affected the admissibility of cut, weakening or contraction.

Definition 3.60 (Geometric Rule). A **geometric rule** [Neg03, Neg07, DN10] is a rule of the form

$$\frac{[\bar{z}/\bar{y}]\bar{A}_1, \bar{A}_0, \Gamma \Rightarrow \Delta \quad \ldots \quad [\bar{z}/\bar{y}]\bar{A}_n, \bar{A}_0, \Gamma \Rightarrow \Delta}{\bar{A}_0, \Gamma \Rightarrow \Delta} \tag{56}$$

where the variables $\bar{z}$ do not occur free in the conclusion, and each $\bar{A}_i$ (in an abuse of notation) is a multiset $P_{i_1}, \ldots, P_{i_{k(i)}}$.

Here we show that geometric rules are derivable from geometric implications in the antecedent in **G3c/PSF**. This we can analyse the frame axioms of **Int**$^*/\mathtt{Geo}$ and obtain geometric rules that can be added to **G3c/PSF** to obtain a calculus for the Kripke models that validate the corresponding logics.

Proposition 3.61. *If A is a geometric formulae, then $A \in \mathtt{PSF}$.*

Proof. By induction on the structure of A. □

Lemma 3.62. *Let $A' \in \mathtt{Geo}$. Then there is a geometric rule that corresponds to analysing $A', \Gamma \Rightarrow \Delta$ in* **G3c/PSF**.

Proof. Note that a formula in $\mathtt{Geo}$ is either G_R or H_R (Definition 3.28 on page 52). If $A' \in G_R$, then treat it as $\forall \bar{x}.(\top \supset A')$ in the procedure below.

Not all geometric implications are in $\mathtt{PSF}$, e.g. $\forall \bar{x}.((C \vee D) \supset B)$. However, in [Pal02], it was shown that any set of geometric implications is *intuitionistically equivalent* to a set consisting of formulae of the form $\forall\bar{x}(A_0 \supset \exists\bar{y}.(A_1 \vee \ldots \vee A_n))$, where each A_i is a conjunction of atomic formulae, which *is* in PSF.

We then transform that formula into a rule by analysing it:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\bar{z}/\bar{y}]\bar{A}_1, \exists\bar{y}\ldots, \forall\bar{x}\ldots, \bar{A}_0, \Gamma \Rightarrow \Delta}{[\bar{z}/\bar{y}]A_1, \exists\bar{y}\ldots, \forall\bar{x}\ldots, \bar{A}_0, \Gamma \Rightarrow \Delta}\,\mathsf{L}\wedge^* \quad \ldots \quad \dfrac{[\bar{z}/\bar{y}]\bar{A}_n, \exists\bar{y}\ldots, \forall\bar{x}\ldots, \bar{A}_0, \Gamma \Rightarrow \Delta}{[\bar{z}/\bar{y}]A_n, \exists\bar{y}\ldots, \forall\bar{x}\ldots, \bar{A}_0, \Gamma \Rightarrow \Delta}\,\mathsf{L}\wedge^*}{[\bar{z}/\bar{y}](A_1 \vee \ldots \vee A_n), \exists\bar{y}\ldots, \forall\bar{x}\ldots, \bar{A}_0, \Gamma \Rightarrow \Delta}\,\mathsf{R}\vee^*}{\exists\bar{y}.(A_1 \vee \ldots \vee A_n), \exists\bar{y}\ldots, \forall\bar{x}\ldots, \bar{A}_0, \Gamma \Rightarrow \Delta}\,\mathsf{L}\exists}{\exists\bar{y}.(A_1 \vee \ldots \vee A_n), \forall\bar{x}\ldots, \bar{A}_0, \Gamma \Rightarrow \Delta}\,(\mathsf{LC})}{\forall\bar{x}.(A_0 \supset \exists\bar{y}.(A_1 \vee \ldots \vee A_n)), \bar{A}_0, \Gamma \Rightarrow \Delta}\,(\mathsf{L}\forall\supset')$$

This clearly matches the geometric rule schema. □

REMARK 3.63. An alternative to defining the language of PSF is to combine the language of SPSF (Definition 3.33 on page 53) with that of geometric formulae (Definition 3.26). However, such a calculus would require two kinds of $\mathsf{L}\forall\supset$ and $\mathsf{R}\forall\supset$ rules.

DEFINITION 3.64. Let **G3c/PSF*** be the system obtained by adding to the rules of **G3c/PSF** the rules from Figures 3.3 and 3.5 on the facing page.

$$\dfrac{\mathcal{R}xx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathsf{refl} \qquad \dfrac{\mathcal{R}xz, \mathcal{R}xy, \mathcal{R}yz, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, \mathcal{R}yz, \Gamma \Rightarrow \Delta}\,\mathsf{trans}$$

$$\dfrac{\mathcal{R}xy, Px, Py, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, Px, \Gamma \Rightarrow \Delta}\,\mathsf{LF}_0 \qquad \dfrac{\mathcal{R}zx_1, \ldots, \mathcal{R}zx_n, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathsf{root}$$

where Px and Py are atomic in LF_0 and z is not free in Γ, Δ in root.

FIGURE 3.3. Extension rules of **G3c/PSF*** for **Int**.

REMARK 3.65. The root rule (Figure 3.3) is considered an extension rule for **Int** by Theorem 3.15 on page 46.

The root rule can be shown eliminable for **G3c/PSF*** with the extension rules for **Int** but not in the presence of dir rule. Hence, the rule is given as primitive.

While the root rule is not necessary for proving formulae in logics in **Int***/Geo, it is useful for proving properties about those logics. Hence the inclusion here.

We introduce rules corresponding to the semantics of the more general intermediate logics introduced earlier in Figure 3.4 on the next page, and give rules for some special cases of them in Figure 3.5 on the facing page.

$$\frac{\langle \mathcal{R}x_{i+1}x_i, \mathcal{R}x_1x_2, \ldots, \mathcal{R}x_{k-1}x_k, \Gamma \Rightarrow \Delta \rangle_{i=1}^{k-1}}{\mathcal{R}x_1x_2, \ldots, \mathcal{R}x_{k-1}x_k, \Gamma \Rightarrow \Delta} \ \mathsf{BD_k}$$

$$\frac{\langle \mathcal{R}x_ix_1, \ldots, \mathcal{R}x_ix_k, \Gamma \Rightarrow \Delta \rangle_{i=1, i\neq k}^{k}}{\Gamma \Rightarrow \Delta} \ \mathsf{BW_k}$$

$$\frac{\langle \mathcal{R}x_1z_i, \ldots, \mathcal{R}x_kz_i, \Gamma \Rightarrow \Delta \rangle_{i=1}^{k}}{\Gamma \Rightarrow \Delta} \ \mathsf{BTW_k}$$

where $z_1, \ldots, z_k$ do not occur free in Γ, Δ for $\mathsf{BTW_k}$.

$$\frac{\langle \mathcal{R}x_ix_o, \mathcal{R}x_0x_1, \ldots, \mathcal{R}x_0x_k, \Gamma \Rightarrow \Delta \rangle_{i=1}^{k}}{\mathcal{R}x_0x_1, \ldots, \mathcal{R}x_0x_k, \Gamma \Rightarrow \Delta} \ \mathsf{BC_k}$$

FIGURE 3.4. Extension rules of **G3c**/**PSF*** for some logics in **Int***/Geo.

$$\frac{\mathcal{R}wx, \mathcal{R}xz, \mathcal{R}wy, \mathcal{R}yz, \Gamma \Rightarrow \Delta}{\mathcal{R}wx, \mathcal{R}wy, \Gamma \Rightarrow \Delta} \ \mathsf{dir}$$

where z is not free in Γ, Δ.

$$\frac{\mathcal{R}xy, \Gamma \Rightarrow \Delta \quad \mathcal{R}yx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ \mathsf{lin} \qquad \frac{\mathcal{R}xy, \mathcal{R}yx, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, \Gamma \Rightarrow \Delta} \ \mathsf{sym}$$

FIGURE 3.5. Special cases of extension rules of **G3c**/**PSF*** for logics in **Int***/Geo.

PROPOSITION 3.66 (Weakly directed rule). *With* root*, the* ***weakly directed*** *rule*

$$\frac{\mathcal{R}xz, \mathcal{R}yz, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ \mathsf{wk\ dir}$$

where z is not free in Γ, Δ, and dir *are interderivable in* **G3c**/**PSF***.

Proof.

$$\cfrac{\cfrac{\cfrac{\mathcal{R}xz, \mathcal{R}yz, \Gamma \Rightarrow \Delta}{\mathcal{R}wx, \mathcal{R}xz, \mathcal{R}wy, \mathcal{R}yz, \Gamma \Rightarrow \Delta} \ (\mathsf{LW})^+}{\mathcal{R}wx, \mathcal{R}wy, \Gamma \Rightarrow \Delta} \ \mathsf{dir}}{\Gamma \Rightarrow \Delta} \ \mathsf{root}$$

The dir rule can also be considered a special case of the wk dir rule. □

REMARK 3.67. The wk dir rule corresponds with the LQ rule for simply labelled sequents and hypersequents, where rootedness is not explicit.

with

3.5.4. Persistence Lemmas. Below the persistence property is proven and folding rules are shown. These allow for redundant formulae to be eliminated from the sequent, and are crucial for extending the simple correspondence.

For the proofs below, it is assumed that the sequents are in $\mathsf{SPSF}^{*2}_{\leq}$, that is, there are no $\mathcal{R}$-formulae in the succedents.

LEMMA 3.68. $\mathbf{G3c/PSF}^* \vdash \mathcal{R}xy, A, \Gamma \Rightarrow \Delta, [y/x]A$, *for all* $A \in \mathsf{SPSF}$, *where* $y\#A$.

Proof. By induction on the size of the formulae A. (The proof is written out in Lemma B.7 on page 242.) □

REMARK 3.69. Lemma 3.68 corresponds to the axiom, $x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y$, from $\mathbf{G3I}^*$.

PROPOSITION 3.70. *The* ***atomic right-folding*** *rule*

$$\frac{\mathcal{R}xy, \Gamma \Rightarrow \Delta, Px, Py}{\mathcal{R}xy, \Gamma \Rightarrow \Delta, Py}\ \mathsf{RF}_0$$

where Px *and* Py *are atomic, is admissible in* $\mathbf{G3c/PSF}^*$.

Proof. By induction on the derivation depth. (The proof is written out in Proposition B.8 on page 243.) □

LEMMA 3.71 (Folding). *The* ***general folding rules***

$$\frac{\mathcal{R}xy, A, [y/x]A, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, A, \Gamma \Rightarrow \Delta}\ (\mathsf{LF}) \qquad \frac{\mathcal{R}xy, \Gamma \Rightarrow A, [y/x]A, \Delta}{\mathcal{R}xy, \Gamma \Rightarrow [y/x]A, \Delta}\ (\mathsf{RF})$$

where $A \in \mathsf{SPSF}$, *are admissible in* $\mathbf{G3c/PSF}^*$.

Proof. Proof by simultaneous induction on the rank determined by the formula size and derivation depth. (The proof is written out in Lemma B.9 on page 244.) □

3.6. Conclusion

This chapter is largely an extension to the previous chapter on notation, with a focus on intermediate logics and Kripke semantics. What is novel is the identification of a subclass of intermediate logics, $\mathbf{Int}^*/\mathsf{Geo}$, or intermediate logics with geometric Kripke models. However, we do not analyse the properties of this class. (Much of that is left to future investigation.) What is noteworthy about geometric Kripke models is that cut free labelled calculi can be introduced with rules based on their frame axioms.

Another novel contribution of this chapter is a new framework of sequent calculi, called **G3c/PSF***, for a subset of first-order logic, called partially shielded formulae (PSF), that can be used to apply proof-theoretic techniques to the model theory of intermediate logics. (Later in Chapters 6 and 8 we use this calculus to prove useful properties about the relationships between hypersequents and labelled sequents.)

The the next two chapters, we will provide definitions and results about hypersequent and labelled sequent calculi, along with example calculi from the literature.

CHAPTER 4

Hypersequent Calculi

4.1. Overview

4.1.1. Preliminaries. As noted in the Introduction, hypersequents are non-empty multisets of sequents (called **components**), $\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$, where the pipe operator "|" denotes a meta-level disjunction. This is a natural extension of standard sequent calculi, where the calculus has disjunctive as well as conjunctive branching. This allows alternative choices in proof search to be incorporated into the system, as part of the object language of the calculus, e.g.

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \mid \Gamma \Rightarrow \Delta, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee'$$

Such rules allow for more powerful calculi than standard sequent calculi, but can also increase the difficulty of proving properties about a calculus, such as cut-admissibility.

In this chapter, we will followup with a brief survey of hypersequent calculi in Section 4.2. In Section 4.3 we will provide formal definitions and some general lemmas pertaining to hypersequent calculi, including hyperextensions of sequent calculi. In Section 4.4 we will survey several hypersequent calculi for and frameworks of calculi for intuitionistic and intermediate logics, and introduce a multisuccedent hypersequent framework for logics in **Int***/Geo.

4.2. A Brief Survey of Hypersequent Calculi

Hypersequent calculi were apparently first introduced by Beth [Bet59] in the 1950s in a multisuccedent calculus for **Int** (shown in Figure 4.1 on page 83) which incorporated *disjunctive* as well as conjunctive branching into the rules. While disjunctive branching can be considered to be natural extension of Gentzen-style sequent calculi that corresponded with Beth's relational semantics for **Int** (Section 3.3.4 on page 50), it also adds more complexity to the proof system. Beth's extension to Gentzen-style calculi was not presented as a general formalism.

Hypersequent-like formalisms have been used for the proof theory of various non-classical logics, for example, single-sided systems for finitely many-valued logics, e.g. [Rou67, Tak70] (where components corresponded to possible valuations), and a calculus for Gödel-Löb provability logic in [SV82]. Pottinger introduced a hypersequent framework for the modal logics **T**, **S4** and **S5** [Pot83] (although hypersequents were only needed for **S5**), and Mints gave a single-sided hypersequent calculus (as a tableau calculus) for **S5** [Min92] (noted by Avron in [Avr96]).

The literature generally credits the invention of hypersequents to Avron [Avr87] in work on the relevant logic **RM** (with Pottinger noted sometimes as having independently originating the formalism). Avron is responsible for the name **hypersequent**, and for much of the terminology and general theory about hypersequent calculi, for example [Avr91a, Avr96].

Hypersequents have been used to develop the proof theory of relevance, paraconsistent and three-valued logics, e.g. [Avr87, Avr90, Avr91c, Avr91b]. In the past decade, hypersequent calculi have been used to develop the proof theory for many intermediate logics, e.g. [Avr00, CF00, CF01b, Fer03] and fuzzy logics, e.g. [Avr99, Avr00, BZ00, CF01a, CM03, MOG05, CFM04, MM07]. Surveys can be found in [BCF03a, GMO04]. More recent work examines the use of hypersequents with respect to work in substructural logics, e.g. [CGT08].

A survey of hypersequent calculi for intermediate logics will be given later in this chapter. However, where useful in illustrating properties of hypersequent calculi, examples will be given for hypersequent calculi for other kinds of logics, such as the model logics **S4** and **S5**, or Łukasiewicz Logic (**Ł**).

4.3. Formal Definitions for Hypersequent Calculi

4.3.1. Definitions. Although much of the terminology given below was introduced by Avron in [Avr91a, Avr96], we will deviate from it or introduce new terminology suitable to the work in this thesis.

DEFINITION 4.1 (Hypersequent). Let $\mathcal{L}$ be a set of logical formulae, and $\mathtt{Seq}_{\mathcal{L}}$ be the set of sequents of formulae in $\mathcal{L}$ (Definition 2.21 on page 26).A **hypersequent** in $\mathtt{Seq}^{+}_{\mathcal{L}}$ is a non-empty, finite multiset of sequents in $\mathtt{Seq}_{\mathcal{L}}$, connected with the pipe operator, "|".

$\mathtt{Seq}_{\mathcal{L}}$ is referred to in a general way without respect to the logic or the structure of sequents, including whether sequents contain lists or multisets of formulae. (Although sequents have been defined as pairs of arbitrary lists of formulae in $\mathcal{L}$, this definition could be extended to cover single-sided sequents.)

Reference to a particular set of formulae $\mathcal{L}$ will be omitted when the language (or class of languages) is obvious from the context.

NOTATION 4.2. Calligraphic Roman letters $\mathcal{G},\mathcal{H}$ will be used to denote multisets of 0 or more components of a hypersequent. When the context assumes derivability, e.g. "$\vdash \mathcal{H}\ldots$", they denote hypersequents, which by definition are not empty.

NOTATION 4.3. The notation for hypersequents in $\mathtt{Seq}^{+}_{\mathcal{L}}$ is defined inductively:

- $\Gamma \Rightarrow \Delta \in \mathtt{Seq}^{+}_{\mathcal{L}}$ if $\Gamma \Rightarrow \Delta \in \mathtt{Seq}_{\mathcal{L}}$, where $\Gamma \Rightarrow \Delta$ is called a **single-component hypersequent**;
- $\mathcal{H} \mid \Gamma \Rightarrow \Delta \in \mathtt{Seq}^{+}_{\mathcal{L}}$ iff $\mathcal{H} \in \mathtt{Seq}^{+}_{\mathcal{L}}$ and $\Gamma \Rightarrow \Delta \in \mathtt{Seq}_{\mathcal{L}}$, where $\Gamma \Rightarrow \Delta$ is called a **component**

NOTATION 4.4. We extend the standard (multi)set operators to hypersequents, i.e.,

(1) $\Gamma \Rightarrow \Delta \in \mathcal{H}$ means that $\Gamma \Rightarrow \Delta$ is a component of $\mathcal{H}$;
(2) $\mathcal{H} \setminus \Gamma \Rightarrow \Delta$ denotes $\mathcal{H}$ minus one instance of $\Gamma \Rightarrow \Delta$;
(3) $\mathcal{H} \cup \mathcal{H}'$ denotes $\mathcal{H} \mid \mathcal{H}'$.

Set intersection ("$\cap$") is not used for hypersequents.

NOTATION 4.5. For most of the calculi examined in this work, hypersequents will be multisets of components. For cases where hypersequents are lists of components, e.g. in [Avr96], they will be represented using double-bars ("||"), e.g.

$$\Gamma_1 \Rightarrow \Delta_1 \parallel \Gamma_2 \Rightarrow \Delta_2 \parallel \ldots \parallel \Gamma_n \Rightarrow \Delta_n$$

In this context, the metavariables $\mathcal{G},\mathcal{H}$, etc. will denote lists of components. Only the membership operator from Note 4.4 is extended to lists of components.

Remark 4.6. Clearly, the “|” operator is associative and commutative. Likewise for the “||” operator where permutation of components (see Remark 4.31 on page 74) is permitted.

Notation 4.7. The following alternative notation for sequents and components of hypersequents may be used:

$$S_i =_{def} \Gamma_i \Rightarrow \Delta_i \tag{57}$$

$$S_i, S_j =_{def} \Gamma_i, \Gamma_j \Rightarrow \Delta_i, \Delta_j \tag{58}$$

$$\mathcal{H} \mid S_i =_{def} \mathcal{H} \mid \Gamma_i \Rightarrow \Delta_i \tag{59}$$

Notation 4.8. Let $S_1 \mid \ldots \mid S_n$ be a hypersequent. Then the hypersequent may also be represented as $|_{i=1}^{n} S_i$.

Definition 4.9 (Solid hypersequents). The **empty sequent** is $\emptyset \Rightarrow \emptyset$, more commonly written as “ $\Rightarrow$ ”. When the empty sequent is a component of a hypersequent, it is called an **empty component** of that hypersequent. A **solid hypersequent** is a hypersequent with no empty components. The set of solid hypersequents is denoted by $\mathtt{Seq}^{+} \setminus (\Rightarrow)$.

Definition 4.10 (Standard Semantics of Hypersequents). A hypersequent $\mathcal{H}$ is **true** in an interpretation $\Im$, written $\Im \models \mathcal{H}$, iff there is a component $S \in \mathcal{H}$ such that $\Im \models S$. S is called a **significant component** for $\Im$ in $\mathcal{H}$. (There may be more than one significant component for $\Im$ in $\mathcal{H}$.)

Likewise, a hypersequent $\mathcal{H}$ is **false** in $\Im$, written as $\Im \not\models \mathcal{H}$, iff for all components $S \in \mathcal{H}$, $\Im \not\models S$.

Remark 4.11. Note that the semantics given in Definition 4.10 is *generic*, and not dependent on the structure of the components or the logic, or the semantics of the constituent components (sequents).

For logics in **Int***/$\mathtt{Geo}$, there is a standard translation from arbitrary hypersequents for calculi in those logics into formulae in $\mathtt{Prop}$:

Definition 4.12 (Standard Translation of Hypersequents). Recall that the standard translation of sequents into formulae is

$$\mathtt{form}\,(\Gamma \Rightarrow \Delta) =_{def} \bigwedge \Gamma \supset \bigvee \Delta \tag{60}$$

where $\bigwedge\!\!\!\bigwedge\emptyset =_{def} \top$ and $\bigvee\!\!\!\bigvee\emptyset =_{def} \bot$. This is extended to hypersequents by using the general semantics in Definition 4.10 on the preceding page:

$$\texttt{form}\,(\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n) =_{def} \bigvee_{i=1}^{n} \texttt{form}\,(\Gamma_1 \Rightarrow \Delta_i) \tag{61}$$

REMARK 4.13. Not all hypersequent calculi use hypersequents with standard translations. For example, arbitrary components in the hypersequent systems for **A** and **Ł** in [MOG05] have no known translation into $\texttt{Prop}$, because the commas in the antecedents and succedents of components are not known to correspond to connectives in those logics.

LEMMA 4.14. *Let $\mathcal{H}$ be a hypersequent and $\mathfrak{M}$ an intuitionistic Kripke model. Then $\mathfrak{M} \vDash \mathcal{H}$ iff $\mathfrak{M} \vDash \texttt{form}(\mathcal{H})$.*

Proof. From left-to-right, there exists $\Gamma \Rightarrow \Delta \in \mathcal{H}$ such that $\mathfrak{M} \vDash \Gamma \Rightarrow \Delta$, i.e. $\mathfrak{M} \vDash \bigwedge\!\!\!\bigwedge\Gamma \supset \bigvee\!\!\!\bigvee\Delta$. So $\mathfrak{M} \vDash \texttt{form}(\Gamma \Rightarrow \Delta)$. From the Kripke semantics for **Int**, $\mathfrak{M} \vDash \texttt{form}(\Gamma \Rightarrow \Delta) \vee \texttt{form}(\mathcal{H} \setminus \Gamma \Rightarrow \Delta)$, i.e. $\mathfrak{M} \vDash \texttt{form}(\mathcal{H})$. From right-to-left, using the reverse steps of left-to-right. □

DEFINITION 4.15 (Schematic Hypersequents). Let $\texttt{HyperseqVar}$ be the denumerable set of hypercontext variables, e.g. $\mathcal{H}$.

A **schematic hypersequent** is an expression of the form $\alpha_1 \mid \ldots \mid \alpha_n$ for $n \geq 1$ where each α_i is

(1) an expression in the language of schematic sequents $\texttt{Seq}^{\mu}_{\mathcal{L}}$, which denotes a single component;
(2) a **hypercontext variable** $\mathcal{G}$ or $\mathcal{H}$ (with optional primes or subscripts), which can denote a multiset of 0 or more components.

For example, $\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \wedge B$ is a schematic hypersequent.

We denote the set of schematic hypersequents as $(\texttt{Seq}^{\mu}_{\mathcal{L}} + \texttt{HyperseqVar})^{+}$, or simply as $(\texttt{Seq}^{\mu} + \texttt{HyperseqVar})^{+}$ when the set of formulae $\mathcal{L}$ is obvious from the context.

REMARK 4.16. We give a formal definition of schematic hypersequents (and later of schematic labelled sequents) so as to formally define translation functions that can be applied to rules of calculi, and not just to proofs containing instantiated hypersequents or labelled sequents in inferences.

DEFINITION 4.17 (Hypercontext). Let ρ be a (schematic) rule of a hypersequent calculus. A hypercontext variable which occurs in a premiss of ρ and in the conclusion of ρ, with no annotations to indicate that the variable has different values between the conclusion and premiss, is a **hypercontext** of ρ.

Let σ be a substitution of schematic variables such that $\sigma\rho$ is an inference (i.e., an instance σ of rule ρ). The components that are instantiated for the hypercontext variables are considered to be part of the hypercontext(s) of $\sigma\rho$. (They are also called the **side components** of $\sigma\rho$.)

If the hypercontext of the conclusion of ρ is the same as the hypercontext of every premiss of ρ, then ρ is called **hypercontext-sharing**. Otherwise ρ is called **hypercontext-splitting**.

EXAMPLE 4.18. *Hypercontext-sharing and splitting forms of the communication rule from* [Avr96] *are shown below:*

$$\frac{\mathcal{H} \mid \Gamma_1,\Gamma_2 \Rightarrow \Delta_1 \quad \mathcal{H} \mid \Gamma_1,\Gamma_2 \Rightarrow \Delta_2}{\mathcal{H} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ \mathsf{Com} \qquad \frac{\mathcal{H}_1 \mid \Gamma_1,\Gamma_2 \Rightarrow \Delta_1 \quad \mathcal{H}_2 \mid \Gamma_1,\Gamma_2 \Rightarrow \Delta_2}{\mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ \mathsf{Com}_{\mathsf{cs}}$$

The metavariables $\mathcal{H},\mathcal{H}_1$ and $\mathcal{H}_2$ are hypercontexts. The rule on the left is hypercontext-sharing, and the rule on the right is called hypercontext-splitting.

REMARK 4.19. In a schematic hypersequent, the hypercontext is not necessarily a hypersequent, since it may be denote an empty multiset of components. However, the schematic hypersequent $\mathcal{H}$, which consists of a single hypercontext variable, denotes a hypersequent, which by definition, is non-empty multiset of components.

REMARK 4.20. A sequent calculus is a hypersequent calculus with no hypercontexts in the rules [Avr96].

DEFINITION 4.21 (Active and principal components). The components in a premiss of rule ρ that are not part of the hypercontext are called **active components**. The components in the conclusion of a rule ρ that are not part of the hypercontext are called **principal components**.

The active and principal components of a rule ρ are said to *play a role* in rule ρ. (Likewise, the components of the hypercontext(s) of ρ *play no role* in ρ.)

EXAMPLE 4.22. *In Example 4.18 on the facing page (for both rules), the active components are* $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1$ *in the left premiss, and* $\Gamma_1, \Gamma_2 \Rightarrow \Delta_2$ *in the right premiss. The principal components are* $\Gamma_1 \Rightarrow \Delta_1$ *and* $\Gamma_2 \Rightarrow \Delta_2$.

DEFINITION 4.23 (Side Formulae). The **side formulae** of a rule ρ are the formulae that are neither active nor principal formulae. They may be in the the side components (hypercontext) or in the active or principal components. These are differentiated as **hypercontextual side formulae** for hypercontext formulae, **active side formulae** for side formulae in the active component, and **principal side formulae** for side formulae in the principal component.

EXAMPLE 4.24. *In the following rule,*

$$\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \wedge B, \Gamma \Rightarrow \Delta} \text{ R}\supset$$

The formulae in $\mathcal{H}, \Gamma$ *and* Δ *are side formulae. Specifically, the formulae in* $\mathcal{H}$ *are hypercontextual side formulae; the formulae in* Γ, Δ *in the premiss are active side formulae as well as principal side formulae (since the metavariables occur in both the active and principal components).*

DEFINITION 4.25 (Internal and External Rules). **Internal rules** are hypersequent rules which have exactly one active component per premiss and one principal component in the conclusion. **External rules** are hypersequent rules which are not internal rules.

EXAMPLE 4.26. *In Example 4.18 on the preceding page, the rules are external rules. In Example 4.24, the rule is an internal rule.*

REMARK 4.27. The literature on hypersequents generally uses this distinction only for structural rules. Here the distinction is also used for logical rules as well. Where necessary, the distinction will be made for internal (resp. external) structural rules and internal (resp. external) logical rules.

REMARK 4.28. Internal rules can be thought of as standard sequent rules with added hypercontexts.

DEFINITION 4.29 (Standard External Rules). The **external weakening** and **external contraction**, called the **standard external (structural) rules**, are given below:

$$\frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EW} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}\ \mathsf{EC}$$

REMARK 4.30. Clearly the $\mathcal{H}$ in the premiss of the EW rule is non-empty.

REMARK 4.31. In systems where hypersequents are sequences (lists) of components rather than multisets, e.g. [Avr96], the standard external rules include the **external permutation** rule:

$$\frac{\mathcal{H} \parallel \Gamma \Rightarrow \Delta \parallel \Gamma' \Rightarrow \Delta' \parallel \mathcal{H}'}{\mathcal{H} \parallel \Gamma' \Rightarrow \Delta' \parallel \Gamma \Rightarrow \Delta \parallel \mathcal{H}'}\ \mathsf{EP}$$

(The rule is also known as **external exchange**.)

LEMMA 4.32. *Let* **HS** *be a calculus with* LW, RW *and* EW *rules. Then the* ***general weakening*** *rule*

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid \mathcal{H}'}\ (\mathsf{GW})$$

is admissible in **HS**.

Proof. Straightforward. □

LEMMA 4.33. *Let* **HS** *be a calculus with* LC, RC *and* EC *rules. Then the* ***general contraction*** *rule*

$$\frac{\mathcal{H} \mid \Gamma, m_1 \cdot \Gamma' \Rightarrow \Delta, n_1 \cdot \Delta' \mid \ldots \mid \Gamma, m_k \cdot \Gamma' \Rightarrow \Delta, n_k \cdot \Delta'}{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathsf{GC})$$

is admissible in **HS**.

Proof. Straightforward. □

REMARK 4.34. The GW and GC rules are useful as shorthand notation that combines multiple instances of weakening and contraction rules, respectively.

DEFINITION 4.35. A **single-conclusioned hypersequent** (also called a **single succedent hypersequent**) is one where all of the components have at most one formula in the succedent. A **multiple-conclusioned hypersequent** (also called a **multisuccedent hypersequent**) is one that is not a single-conclusioned hypersequent—that is, at least one component has two or more formulae in the succedent.

4.3.2. Hyperextensions of Sequent Calculi. Here we discuss extending sequent calculi to hypersequent calculi.

Definition 4.36 (Hyperextension). Let **GS** be a standard sequent calculus for a logic **S**, and let **HGS** be the resulting hypersequent calculus by adding the standard external rules (Definition 4.29 on page 73) to **GS**, and augmenting the premisses and conclusions of the original rules of **GS** with hypercontext variables. Then **HGS** is called the **hyperextension** of **GS**.

Remark 4.37. The schematic axioms of a hyperextension need not contain hypercontext variables.

Example 4.38. *The calculus* **HG1ip** *in Figure 4.2 on page 84 is an example of a hyperextension of the calculus* **G1ip**.

Definition 4.39. A hyperextension where all of the premisses and the conclusion of the rules have the same hypercontexts is called a **(hyper)context-sharing hyperextension**. Otherwise it is called a **(hyper)context-splitting hyperextension**.

Remark 4.40. When discussing the relationship between a sequent calculus and its hyperextension, reference to the underlying logic will be omitted when it is not necessary.

Remark 4.41. A standard technique for introducing hypersequent calculi is to take the hyperextension of a base calculus for a logic such as **Int** or Linear Logic and add external rules to extend the logic, for example, [Avr91a, Avr96, CGO99]. This technique will be used to give a framework of calculi for some intermediate logics later in this chapter.

Definition 4.42 (Redundant Hyperstructure). Let **HGS** be the hyperextension of a sequent calculus **GS**. Then **HGS** has a **redundant hyperstructure** when for all $\mathcal{H}$, **HGS** $\vdash \mathcal{H}$ iff for some component $S \in \mathcal{H}$, **GS** $\vdash S$.

Lemma 4.43. *Let* **GS** *be a standard sequent calculus, and let* **HGS** *be the hyperextension of* **GS**. *Then* **HGS** *has a redundant hyperstructure.*

Proof. By induction on derivation depth in **HGS**. □

COROLLARY 4.44. *Let* **GS** *be a standard sequent calculus, and let* **HGS** *be the hyperextension of* **GS**, *such that the axioms of* **HGS** *have hypercontexts. Then the standard external rules (*EW *and* EC*) are eliminable.*

Proof. By induction on the derivation depth in **HGS**. □

THEOREM 4.45 (Conservativity). *Let* **GS** *be a sequent calculus, and let* **HGS** *be the corresponding hyperextension. Then* **HGS** *is conservative.*

Proof. Follows from Corollary 4.44 on the previous page. □

REMARK 4.46. An alternative form of the Conservativity Theorem is to show that for all hypersequents $\mathcal{H}$, **GS** $\vdash \texttt{form}(\mathcal{H})$ iff **HGS** $\vdash \texttt{form}(\mathcal{H})$. However, such a proof relies on the specific rules of the calculi **GS** and **HGS**. In the case of hyperextensions to calculi with non-invertible rules, such as **HG1ip** (Figure 4.2 on page 84), this can be difficult to prove.

Because hypersequents for logics in **Int***/`Geo` can be translated into formulae using the standard translation, there is in theory no concern about the completeness of a hyperextension with respect to the language of hypersequents for a logic **S**.

However, merely adding hypercontexts to the rules (but not axioms) of a sequent calculus will not necessarily enable the derivation of multi-component hypersequents, without the addition of the EW rule. And without adding the EC rule, one may not be able to show that cut is admissible. (Issues related to cut admissibility proofs for hypersequent calculi will be discussed later in this chapter.)

4.3.3. General Properties of Hypersequent Calculi.

LEMMA 4.47 (Hypercontext-Sharing and Splitting). *For calculi with the standard external rules the hypercontext-splitting and hypercontext-sharing rules are interderivable.*

Proof. Straightforward. (The proof is written out in Lemma C.1 on page 249.) □

DEFINITION 4.48 (Trivially-Invertible Form). Let ρ be a hypercontext-sharing rule

$$\frac{\mathcal{H} \mid \mathcal{G}_1 \quad \dots \quad \mathcal{H} \mid \mathcal{G}_n}{\mathcal{H} \mid \mathcal{G}_0}\ \rho$$

where $\mathcal{G}_1,\ldots,\mathcal{G}_n$ denotes the active components and $\mathcal{G}_0$ denotes the principal component(s). The **trivially invertible form** of ρ is

$$\frac{\mathcal{H} \mid \mathcal{G}_0 \mid \mathcal{G}_1 \quad \ldots \quad \mathcal{H} \mid \mathcal{G}_0 \mid \mathcal{G}_n}{\mathcal{H} \mid \mathcal{G}_0} \; \rho_\iota$$

REMARK 4.49. Trivially invertible rules are hypercontext-sharing, by definition.

LEMMA 4.50. *Let* **HGS** *be a calculus with the standard external rules and a hypercontext-sharing rule* ρ *of the form*

$$\frac{\mathcal{H} \mid \mathcal{G}_1 \quad \ldots \quad \mathcal{H} \mid \mathcal{G}_n}{\mathcal{H} \mid \mathcal{G}_0} \; \rho$$

Then the rule ρ *and the trivially-invertible form of* ρ *are interderivable.*

Proof. Trivial, using EW and EC. □

COROLLARY 4.51. *Let* **HGS** *be a calculus with the standard external rules and a hypercontext-splitting rule* ρ *of the form*

$$\frac{\mathcal{H}_1 \mid \mathcal{G}_1 \quad \ldots \quad \mathcal{H}_n \mid \mathcal{G}_n}{\mathcal{H}_1 \mid \ldots \mid \mathcal{H}_n \mid \mathcal{G}_0} \; \rho$$

Then the rule ρ *and the trivially-invertible form of* ρ *are interderivable.*

Proof. From Lemmas 4.47 on the facing page and 4.50. □

A useful feature of hypersequent calculi is that they allow one to combine multiple rules into a single rule:

LEMMA 4.52 (Rule composition). *Let* **HGS** *be a hypersequent calculus with the* EC *rule. Let* $\rho_1,\ldots,\rho_n$*, for* $n \geq 2$*, be hypercontext-sharing rules of* **HGS***, and let* $\sigma_1,\ldots,\sigma_n$ *be partial variable substitutions such that the rules* $\sigma_1\rho_1,\ldots,\sigma_n\rho_n$ *have the same conclusions. (Note that each* σ_i *is a replacement of variables by other variables, such that* $\sigma_i\rho_i$ *is a schematic rule, and not a rule instance.) Then the rules may be composed into a single rule,* $\sigma_2\rho_1 \circ \ldots \circ \sigma_n\rho_n$.

Proof. Straightforward, by induction on n. (The proof is written out in Lemma C.2 on page 249.) □

EXAMPLE 4.53. *The rules for* $\mathsf{R}\vee$ *in the single-succedent calculus* **HG1ip** *(Figure 4.2 on page 84)*

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow A \vee B}\ \mathsf{R}\vee_1 \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \vee B}\ \mathsf{R}\vee_2$$

can be combined into a "parallel" rule:

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \vee B}\ \mathsf{R}\vee'$$

Note that composing several non-invertible rules does not necessarily make an invertible rule, although the composed rule may reduce backtracking in root-first proof search. When the rules are composed with nullary substitutions, and the calculus allows for rule decomposition (shown below), then the rules may be replaced by the single composed rule.

While combining rules may be useful for proof-theoretic purposes, if the number of primitive rules can be reduced, or if the rules are invertible, it may be detrimental as a basis for root-first proof search, by increasing the complexity of the premisses of rules. In studies of the system **GŁ** for **Ł** from [CM03, GMO04, MOG05], we compared proof search using the $\mathsf{L}\supset$ rule with an invertible form of the rule, which can be seen as a composition of $\mathsf{L}\supset_{\text{Ł}}$ with LW:

$$\frac{\mathcal{H} \mid \Gamma, B \Rightarrow A, \Delta}{\mathcal{H} \mid \Gamma, A \supset B \Rightarrow \Delta}\ \mathsf{L}\supset_{\text{Ł}} \qquad \frac{\mathcal{H} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \supset B \Rightarrow \Delta}\ \mathsf{L}\supset_{\text{Ł}_\iota}$$

The extra component in $\mathsf{L}\supset_{\text{Ł}_\iota}$ leads to an exponential increase in the number of formulae that may need to be analysed. Take a single-component hypersequent with $n+k \geq 1$ formulae in the antecedent (where at least $n \geq 1$ formulae are implications) and $m \geq 1$ formulae in the succedent: $A_1 \supset B_1, \ldots, A_n \supset B_n, G_1, \ldots, G_k \Rightarrow D_1, \ldots, D_m$. This hypersequent can be analysed in proof search by successive applications of the $\mathsf{L}\supset_{\text{Ł}_\iota}$ rule into a hypersequent with 2^n components. Additional components also increase the work for the single-component $\mathsf{R}\supset_{\text{Ł}}$ rule:

$$\frac{\mathcal{H} \mid \Gamma, A \Rightarrow B, \Delta \quad \mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow A \supset B, \Delta}\ \mathsf{R}\supset_{\text{Ł}}$$

Consider the following derivation fragment (abbreviating formulae of the form $A \supset B$ with AB),

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma, E, C \Rightarrow AB, D, \Delta \mid \Gamma, E \Rightarrow D, \Delta \mid \Gamma, C \Rightarrow AB, \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, E, C \Rightarrow AB, D, \Delta \mid \Gamma, E \Rightarrow D, \Delta \mid \Gamma, (AB)C \Rightarrow \Delta}\ \mathsf{L}{\supset}_{Ł_\iota}}{\mathcal{H} \mid \Gamma, (AB)C, E \Rightarrow D, \Delta \mid \Gamma, (AB)C \Rightarrow \Delta}\ \mathsf{L}{\supset}_{Ł_\iota}}{\mathcal{H} \mid \Gamma, (AB)C, DE \Rightarrow \Delta}\ \mathsf{L}{\supset}_{Ł_\iota}$$

There are two components with the formula $A \supset B$ in the premiss. Analysing formulae of the form $A \supset B$ in the succedents would require *three* applications of the $\mathsf{R}{\supset}_{Ł}$ rule and *four* branches in the proof. (The larger derivation fragment is not shown owing to space constraints on the page.)

We note that rudimentary experiments in [Rot06b] with an partial implementation of **GŁ** in Prolog for axioms of Łukasiewicz Logic showed that the use of the composed rule required significantly more time to find proofs for the axioms.

Lemma 4.54 (Rule decomposition). *Let* **HGS** *be a hypersequent calculus with the* EW *rule, and an external rule*

$$\dfrac{\mathcal{H} \mid \mathcal{X}_1 \mid \mathcal{Y}_1 \quad \dots \quad \mathcal{H} \mid \mathcal{X}_n \mid \mathcal{Y}_n}{\mathcal{H} \mid \mathcal{G}}\ \rho$$

where all $\mathcal{X}_i$ and $\mathcal{Y}_i$ are (possibly empty) active components of ρ. Then the rule

$$\dfrac{\mathcal{H} \mid \mathcal{X}_1 \quad \dots \quad \mathcal{H} \mid \mathcal{X}_n}{\mathcal{H} \mid \mathcal{G}}\ \rho'$$

is admissible in **HGS**.

Proof. By derivation using EW. □

Definition 4.55 (Merge and Split Rules). We give the merge (M) and split (S) rules below:

$$\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{M} \qquad \dfrac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}\ \mathsf{S}$$

Lemma 4.56 (Merge). *Let* **HGS** *be a hypersequent calculus with the* LW*,* RW *(internal weakening) and* EC *rules. Then the merge rule is admissible in* **HGS**.

Proof. By derivation. (See Lemma C.3 on page 250.) □

DEFINITION 4.57. The empty component elimination ($\mathsf{E}\emptyset$) and $\sqsupset$ rules are

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\ (\mathsf{E}\emptyset) \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Delta' \mid \Gamma \Rightarrow \Delta'}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Delta'}\ \sqsupset$$

COROLLARY 4.58 (Elimination of Empty Components). *Let* **HGS** *be a hypersequent calculus with the* M *rule (primitive or admissible). Then the* $\mathsf{E}\emptyset$ *rule is admissible in* **HGS**.

Proof. $\mathsf{E}\emptyset$ is a special case of the M rule. □

COROLLARY 4.59. *Let* **HGS** *be a hypersequent calculus with the* M *rule (primitive or admissible). Then the* $\sqsupset$ *rule is admissible in* **HGS**.

Proof. $\sqsupset$ is a special case of the M rule. □

REMARK 4.60. The $\sqsupset$ rule is named after Beth, who introduced the rule in [Bet59].

LEMMA 4.61. *Let* **HGS** *be a hypersequent calculus with the* EW *and* $\sqsupset$ *rules. Then the* $\sqsupset$ *rule is invertible, i.e.* $\sqsupset^{-1}$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Delta'}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Delta' \mid \Gamma \Rightarrow \Delta'}\ \sqsupset^{-1}$$

is admissible in **HGS**.

Proof. By derivation, using EW rule. □

LEMMA 4.62 (External Contraction). *Let* **HGS** *be a hypersequent calculus with the* LC*,* RC *(internal contraction) and* M *rules. Then the* EC *rule is admissible in* **HGS**.

Proof. By derivation. See Lemma C.4 on page 250. □

LEMMA 4.63 (External Weakening). *Let* **HGS** *be a hypersequent calculus with the* S *(split),* LW *and* RW *(internal weakening) rules. Then the* EW *(external weakening) rule is admissible in* **HGS**.

Proof. By derivation. See Lemma C.6 on page 250. □

4.3.4. Permutation of External Rules and Cut Admissibility. The permutation of external rules with multi-premiss rules can be problematic. For example,

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta_1\\ \Gamma,A\Rightarrow C,\Delta \mid \Gamma,B\Rightarrow C,\Delta\end{array}}{\Gamma,A\wedge B\Rightarrow C,\Delta}\,\mathsf{L}\wedge \quad \dfrac{\begin{array}{c}\vdots\ \delta_2\\ \Gamma,A\Rightarrow D,\Delta \mid \Gamma,B\Rightarrow D,\Delta\end{array}}{\Gamma,A\wedge B\Rightarrow D,\Delta}\,\mathsf{L}\wedge}{\Gamma,A\wedge B\Rightarrow C\wedge D,\Delta}\,\mathsf{R}\wedge$$

the rules $\mathsf{L}\wedge$ and $\mathsf{R}\wedge$ may not be permutable in a particular calculus.

$$\dfrac{\dfrac{\delta_1 \quad \begin{array}{c}\vdots\ ?\\ \Gamma,A\Rightarrow C,\Delta \mid \Gamma,B\Rightarrow D,\Delta\end{array} \quad \begin{array}{c}\vdots\ ?\\ \Gamma,A\Rightarrow D,\Delta \mid \Gamma,B\Rightarrow C,\Delta\end{array} \quad \delta_2}{\Gamma,A\Rightarrow C\wedge D,\Delta \mid \Gamma,B\Rightarrow C\wedge D,\Delta}\,\mathsf{R}\wedge^2}{\Gamma,A\wedge B\Rightarrow C\wedge D,\Delta}\,\mathsf{L}\wedge$$

In the above example, it's not clear that the two additional premisses are derivable from the local context. Additional proofs of the derivability of premisses where components are recombined are needed.

Similarly, difficulties of proving cut-admissibility arise owing to external rules such as EC. Take the following derivation fragment,

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma\Rightarrow\Delta,A \mid \Gamma\Rightarrow\Delta,A}{\mathcal{H} \mid \Gamma\Rightarrow\Delta,A}\,\mathsf{EC} \quad \mathcal{H}' \mid A,\Gamma'\Rightarrow\Delta'}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,\mathsf{cut}$$

If we try to permute the cut above the EC, then we still have one instance of cut at the same height as the original cut:

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma\Rightarrow\Delta,A \mid \Gamma\Rightarrow\Delta,A \quad \mathcal{H}' \mid A,\Gamma'\Rightarrow\Delta'}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma\Rightarrow\Delta,A \mid \Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,\mathsf{cut} \quad \mathcal{H}' \mid A,\Gamma'\Rightarrow\Delta'}{\mathcal{H} \mid \mathcal{H}' \mid \mathcal{H}' \mid \Gamma,\Gamma'\Rightarrow\Delta,\Delta' \mid \Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,\mathsf{cut}}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,\mathsf{EC}^+$$

Avron uses hypersequent forms of multicut (which we call $\mathsf{hypercut}$) where the cut formulae occur in multiple components, e.g. for the system **GS5** [Avr96], e.g.

$$\dfrac{\mathcal{H} \mid \Gamma_1\Rightarrow\Delta_1,A^+ \mid \ldots \mid \Gamma_m\Rightarrow\Delta_m,A^+ \quad \mathcal{H}' \mid A^+,\Gamma_1'\Rightarrow\Delta_1' \mid \ldots \mid A^+,\Gamma_n'\Rightarrow\Delta_n'}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1,\ldots,\Gamma_m,\Gamma_1',\ldots,\Gamma_n'\Rightarrow\Delta_1,\ldots,\Delta_m,\Delta_1',\ldots,\Delta_n'}$$

(As noted in the introduction, [CGT08] outlines properties of structural hypersequent rules that ensure the rule does not affect cut admissibility in certain contexts. The proof is semantic, and the applicability of those criteria to the calculi introduced here is unclear.)

In [Rot06a], a variant of **GŁ** [CM03] for **Ł** was given that absorbed the internal and external weakening rules into a single form of axiom. It was also conjectured that instances of the mix rule.

$$\dfrac{\mathcal{H} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{H}' \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ \mathsf{mix}$$

could be eliminated from proofs. A problematic example occurs with derivations that end with

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1, B \Rightarrow A, \Delta_1 \mid \Gamma_1 \Rightarrow \Delta_1}{\mathcal{H} \mid \Gamma_1, A \supset B \Rightarrow \Delta_1}\ \mathsf{L}{\supset_\iota} \quad \dfrac{\mathcal{H}' \mid \Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3}{\mathcal{H}' \mid \Gamma_2 \Rightarrow \Delta_2 \mid \Gamma_3 \Rightarrow \Delta_3}\ \mathsf{S}}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1, \Gamma_3, A \supset B \Rightarrow \Delta_1, \Delta_3 \mid \Gamma_2 \Rightarrow \Delta_2}\ \mathsf{mix}$$

The procedure for rewriting the derivations resulted in

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1, B \Rightarrow A, \Delta_1 \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{H}' \mid \Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1, \Gamma_2, \Gamma_3, B \Rightarrow A, \Delta_1, \Delta_2, \Delta_3 \mid \Gamma_1 \Rightarrow \Delta_1}\ \mathsf{mix} \quad \mathcal{H}' \mid \Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3}{\mathcal{H} \mid \mathcal{H}' \mid \mathcal{H}' \mid \Gamma_1, \Gamma_2, \Gamma_3, B \Rightarrow A, \Delta_1, \Delta_2, \Delta_3 \mid \Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3}\ \mathsf{mix}}{\mathcal{H} \mid \mathcal{H}' \mid \mathcal{H}' \mid \Gamma_1, \Gamma_2, \Gamma_3, A \supset B \Rightarrow \Delta_1, \Delta_2, \Delta_3}\ \mathsf{L}{\supset_\iota}}{\mathcal{H} \mid \mathcal{H}' \mid \mathcal{H}' \mid \Gamma_1, \Gamma_3, A \supset B \Rightarrow \Delta_1, \Delta_3 \mid \Gamma_2 \Rightarrow \Delta_2}\ \mathsf{S} \tag{62}$$

Clearly, if the derivation of (62) is a premiss of the following,

$$\dfrac{\dfrac{\mathcal{H}'' \mid \Gamma_4, D \Rightarrow C, \Delta_4 \mid \Gamma_4 \Rightarrow \Delta_4}{\mathcal{H}'' \mid \Gamma_4, C \supset D \Rightarrow \Delta_4}\ \mathsf{L}{\supset_\iota} \quad \dfrac{\vdots\ (62)}{\mathcal{H} \mid \mathcal{H}' \mid \mathcal{H}' \mid \Gamma_1, \Gamma_3, A \supset B \Rightarrow \Delta_1, \Delta_3 \mid \Gamma_2 \Rightarrow \Delta_2}\ \mathsf{S}}{\mathcal{H} \mid \mathcal{H}' \mid \mathcal{H}' \mid \mathcal{H}'' \mid \Gamma_1, \Gamma_3, \Gamma_4, A \supset B, C \supset D \Rightarrow \Delta_1, \Delta_3, \Delta_4 \mid \Gamma_2 \Rightarrow \Delta_2}\ \mathsf{mix}$$

then there is a question of whether the procedure for eliminating mix rules terminates. We have been unable to solve this issue. An alternative mix-free hypersequent calculus for **Ł** is given in [GS08].

REMARK 4.64. The rule is generally called "merge" in the literature. We call it "mix" here to avoid confusion with the inverse of the split rule that is called "merge" elsewhere in this thesis. This should not be confused with Gentzen's mix rule from [Gen35], which we call "multicut" here.

4.4. Hypersequent Calculi for Intermediate Logics

4.4.1. Beth's Calculus. Beth [Bet59, p. 449] apparently introduced the first hypersequent calculus, for **Int**. The propositional rules (adapted for modern notation, and treating $\neg$ as a defined connective) are given in Figure 4.1, which we call here **HGipm**$_{\sqsupset}$. (The calculus in [Bet59] is unnamed.)

$$\frac{}{\mathcal{H} \mid A,\Gamma \Rightarrow \Delta, A}\ \mathsf{Ax} \qquad \frac{}{\mathcal{H} \mid \bot,\Gamma \Rightarrow \Delta}\ \mathsf{L}\bot$$

$$\frac{\mathcal{H} \mid \Gamma, A, B \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \wedge B \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A, \Delta \quad \mathcal{H} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{H} \mid \Gamma \Rightarrow A \wedge B, \Delta}\ \mathsf{R}\wedge$$

$$\frac{\mathcal{H} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{H} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \vee B \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A, B, \Delta}{\mathcal{H} \mid \Gamma \Rightarrow A \vee B, \Delta}\ \mathsf{R}\vee$$

$$\frac{\mathcal{H} \mid A \supset B, \Gamma \Rightarrow \Delta, A \quad \mathcal{H} \mid B, A \supset B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \supset B, \Gamma \Rightarrow \Delta}\ \mathsf{L}{\supset_\iota} \qquad \frac{\mathcal{H} \mid A, \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \supset B}\ \mathsf{R}{\supset_\mathsf{I}}$$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \mid \Gamma \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}\ \sqsupset_1$$

FIGURE 4.1. Hypersequent rules for **HGipm**$_{\sqsupset}$ (Beth's Intuitionistic Calculus).

The system **HGipm**$_{\sqsupset}$ is noteworthy because it is a multisuccedent calculus where all of the rules are invertible. This is achieved by having a single-succedent $\mathsf{R}{\supset_\mathsf{I}}$ rule and the invertible structural rule $\sqsupset_1$, the combination of which can be used to derive the trivially invertible rule $\mathsf{R}{\supset_\iota}$:

$$\frac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \supset B \mid A, \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \supset B \mid \Gamma \Rightarrow A \supset B}\ \mathsf{R}{\supset_\mathsf{I}}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \supset B}\ \sqsupset_1 \tag{63}$$

We note also that $\sqsupset_1$ is the only primitive structural rule in **HGipm**$_{\sqsupset}$. However, the internal and external weakening and contraction rules are admissible:

PROPOSITION 4.65. *The standard internal and external structural rules are admissible in* **HGipm**$_{\sqsupset}$.

Proof. By induction on the derivation depth. □

4.4.2. HIL Framework. Avron [Avr91a] introduced the first hyperextension of a single-conclusioned sequent calculus for **Int**. It is called **HIL** in [BCF03a]. A variant called **HG1ip** with hypercontext-sharing rules and without primitive cut is given in Figure 4.2. A similar version with ¬ as a primitive connective and the hypersequent as a list rather than multiset is given in [CGO99].

$$\frac{}{\mathcal{H} \mid A \Rightarrow A}\,\mathsf{Ax} \qquad \frac{}{\mathcal{H} \mid \bot \Rightarrow}\,\mathsf{L\bot}$$

$$\frac{\mathcal{H} \mid \Gamma, A \Rightarrow C}{\mathcal{H} \mid \Gamma, A \wedge B \Rightarrow C}\,\mathsf{L\wedge_1} \qquad \frac{\mathcal{H} \mid \Gamma, B \Rightarrow C}{\mathcal{H} \mid \Gamma, A \wedge B \Rightarrow C}\,\mathsf{L\wedge_2} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A \quad \mathcal{H} \mid \Gamma' \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \wedge B}\,\mathsf{R\wedge}$$

$$\frac{\mathcal{H} \mid \Gamma, A \Rightarrow C \quad \mathcal{H} \mid \Gamma, B \Rightarrow C}{\mathcal{H} \mid \Gamma, A \vee B \Rightarrow C}\,\mathsf{L\vee} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow A \vee B}\,\mathsf{R\vee_1} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \vee B}\,\mathsf{R\vee_2}$$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow A \quad \mathcal{H} \mid \Gamma, B \Rightarrow C}{\mathcal{H} \mid \Gamma, A \supset B \Rightarrow C}\,\mathsf{L\supset} \qquad \frac{\mathcal{H} \mid \Gamma, A \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \supset B}\,\mathsf{R\supset}$$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow C}{\mathcal{H} \mid \Gamma, A \Rightarrow C}\,\mathsf{LW} \quad \frac{\mathcal{H} \mid \Gamma \Rightarrow}{\mathcal{H} \mid \Gamma \Rightarrow C}\,\mathsf{RW} \quad \frac{\mathcal{H} \mid \Gamma, A, A \Rightarrow C}{\mathcal{H} \mid \Gamma, A \Rightarrow C}\,\mathsf{LC}$$

$$\frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\,\mathsf{EW} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\,\mathsf{EC}$$

FIGURE 4.2. The hypersequent calculus **HG1ip**.

LEMMA 4.66. **HG1ip** *is a conservative hyperextension.*

Proof. From Theorem 4.45 on page 76. □

REMARK 4.67. The literature on hypersequents generally assumes that hyperextensions are conservative, e.g. [Avr91a]. We are unaware of formal proofs that hyperextensions in general are conservative.

PROPOSITION 4.68. *The composed rules*

$$\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta \mid B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \wedge B, \Gamma \Rightarrow \Delta}\,(\mathsf{L\wedge'}) \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \mid \Gamma \Rightarrow \Delta, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}\,(\mathsf{R\vee'})$$

are derivable in **HG1ip**.

Proof. By rule composition (Lemma 4.52 on page 77). □

REMARK 4.69. The rules from Proposition 4.68 on the facing page are not necessarily invertible, e.g., in the derivation of $B \vee C \Rightarrow (A \supset B) \vee C$, one cannot analyse the succedent first to obtain a proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B \mid B \Rightarrow C \qquad \overset{\vdots\, \delta}{C \Rightarrow B \mid B \Rightarrow C}}{B \vee C \Rightarrow B \mid B \Rightarrow C}\,\mathsf{L}\vee}{A, B \vee C \Rightarrow B \mid B \Rightarrow C}\,\mathsf{LW}}{B \vee C \Rightarrow (A \supset B) \mid B \Rightarrow C}\,\mathsf{R}\supset \qquad B \vee C \Rightarrow (A \supset B) \mid C \Rightarrow C}{B \vee C \Rightarrow (A \supset B) \mid B \vee C \Rightarrow C}\,\mathsf{L}\vee}{B \vee C \Rightarrow (A \supset B) \vee C}\,(\mathsf{R}\vee') \tag{64}$$

The second premiss identified by δ cannot be derived in **HG1ip**. (LC is of no use here.) However, the sequent can be derived in **HG1ip** by analysing the antecedent first:

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,\mathsf{LW}}{B \Rightarrow A \supset B}\,\mathsf{R}\supset}{B \Rightarrow (A \supset B) \vee C}\,\mathsf{R}\vee_1 \qquad \dfrac{C \Rightarrow C}{C \Rightarrow (A \supset B) \vee C}\,\mathsf{R}\vee_2}{B \vee C \Rightarrow (A \supset B) \vee C}\,\mathsf{L}\vee \tag{65}$$

Novel external structural rules were given by Avron in [Avr91a, Avr96] to extend the hyperextension of a calculus for **Int** into calculi for superintuitionistic logics, with the idea that they might correspond to concurrent computation. (As far as we are aware, that aim was never reached, although there is later work by Fermüller on parallel game semantics for hypersequents in [Fer03].)

Adding the intuitionistic splitting rule,

$$\dfrac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow A \mid \Gamma' \Rightarrow A}\,\mathsf{S_I}$$

to **HG1ip** gives a calculus for a logic called **LCW** in [Avr91a]), and cut-admissibility[1] for **HG1ip**+$\mathsf{S_I}$ is shown for the fragment of **HG1ip**+$\mathsf{S_I}$ without $\wedge$. The counter-example to cut-admissibility is a derivation of $A \Rightarrow B \mid B \Rightarrow A$ using $A \wedge B$ as the cut formula:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}\,\mathsf{LW} \qquad \dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,\mathsf{LW}}{A, B \Rightarrow A \wedge B}\,\mathsf{R}\wedge}{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,\mathsf{S_I} \qquad \dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A}\,\mathsf{L}\wedge_1}{A \Rightarrow A \wedge B \mid B \Rightarrow A}\,\mathsf{cut} \qquad \dfrac{B \Rightarrow B}{A \wedge B \Rightarrow B}\,\mathsf{L}\wedge_2}{A \Rightarrow B \mid B \Rightarrow A}\,\mathsf{cut} \tag{66}$$

[1] In [Avr91a], cut is a primitive rule, and cut-elimination is shown.

In [CGT08], the $\mathsf{S_I}$ rule is translated into the Com'' rule (discussed below) that allows cut-admissibility.

A variant of the $\mathsf{S_I}$ rule with empty succedents in the active and principal components is given in [CGO99, CF01b, BCF03a]:

$$\frac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow}{\mathcal{H} \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow}\ \mathsf{LQ}$$

which gives a calculus for **Jan**. Admissibility of cut is shown for **HG1ip**+LQ in [BCF03a]. LQ can be considered a special case of $\mathsf{S_I}$ where $A = \bot$. (Recall that

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \bot}{\mathcal{H} \mid \Gamma \Rightarrow}$$

is a sound inference for **Int**.)

Example. *An example derivation in* **HG1ip**+LQ *of the distinguishing axiom for* **Jan**. *(Recall that rules are ¬ are derived from the implication rules.)*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, \neg A \Rightarrow A}\ \mathsf{LW}}{A, \neg A \Rightarrow}\ \mathsf{L}\neg}{A \Rightarrow \ \mid \neg A \Rightarrow}\ \mathsf{LQ}}{A \Rightarrow \bot \mid \neg A \Rightarrow \bot}\ \mathsf{RW}^{+}}{\Rightarrow \neg A \mid \ \Rightarrow \neg\neg A}\ \mathsf{R}\neg^{+}}{\Rightarrow \neg A \vee \neg\neg A}\ (\mathsf{R}\vee')$$

HG1ip can similarly be extended to a single-succedent calculus for **Cl** by adding the classical splitting ($\mathsf{S_c}$) rule from [Avr98, CGO99]:

$$\frac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow A \mid \Gamma' \Rightarrow B}\ \mathsf{S_c}$$

(Some variants of the rule use an empty succedent in place of B in the conclusion, e.g. [CGO99] and Figure 1.2 on page 10) Note also that $\mathsf{S_I}$ and LQ (which are rules for weaker logics) can be considered special cases of $\mathsf{S_c}$.

EXAMPLE 4.70. *A classical proof of the law of excluded middle in* **HG1ip**+$\mathsf{S_c}$*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A, \bot}\ \mathsf{RW}}{\Rightarrow A \mid A \Rightarrow \bot}\ \mathsf{Cl}}{\Rightarrow A \mid \ \Rightarrow \neg A}\ \mathsf{R\neg}}{\Rightarrow A \vee \neg A}\ (\mathsf{R\vee'})$$

Note that while $\mathsf{S_c}$ can be used in the counterexample to cut-admissibility in (66), the same hypersequent can be derived without cut from $A, B \Rightarrow A$ simply by using $\mathsf{S_c}$. [CGO99] claims the proof of cut-admissibility in **HG1ip**+$\mathsf{S_c}$ is similar to the proof for **HG1ip**+LQ.

The symmetry property of classical Kripke frames (page 47) is better illustrated with the following variant of the rule

$$\dfrac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow A \mid \Gamma, \Gamma' \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \mid \Gamma' \Rightarrow B}\ \mathsf{S_c'}$$

which is interderivable with $\mathsf{S_c}$ using the standard external rules.

Avron [Avr98] notes that for **HG1ip**+$\mathsf{S_c}$, the standard translation of hypersequents is equivalent to the standard translation of sequents in the classical system **LK** [Gen35]. That is, **HG1ip** $+ \mathsf{S_c} \vdash \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n$, iff **LK** $\vdash \Gamma_1, \ldots, \Gamma_n \Rightarrow A_1, \ldots, A_n$.

Adding the communication rule (Com) to **HG1ip** gives a calculus for **GD**:

$$\dfrac{\mathcal{H} \mid \Gamma_1, \Gamma_1' \Rightarrow A_1 \quad \mathcal{H} \mid \Gamma_2, \Gamma_2' \Rightarrow A_2}{\mathcal{H} \mid \Gamma_1, \Gamma_2' \Rightarrow A_1 \mid \Gamma_2, \Gamma_1' \Rightarrow A_2}\ \mathsf{Com}$$

EXAMPLE 4.71. *A proof of the distinguishing axiom of* **GD** *in* **HG1ip**+Com*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{A, B \Rightarrow B}\ \mathsf{LW} \quad \dfrac{A \Rightarrow A}{A, B \Rightarrow A}\ \mathsf{LW}}{A, A \Rightarrow B \mid B, B \Rightarrow A}\ \mathsf{Com}}{A \Rightarrow B \mid B \Rightarrow A}\ \mathsf{LC^+}}{\Rightarrow A \supset B \mid \ \Rightarrow B \supset A}\ \mathsf{R\supset^+}}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (\mathsf{R\vee'})$$

HG1ip+Com is said by [Avr91a] to allow cut-admissibility. Proofs are given in [Avr91a, BCF03a] using a $\mathsf{hypercut}$ rule:

$$\dfrac{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n \quad \mathcal{H}' \mid \Gamma_1', k_1 \cdot A_1 \Rightarrow C_1 \mid \ldots \mid \Gamma_n', k_n \cdot A_n \Rightarrow C_n}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1, \Gamma_1' \Rightarrow C_1 \mid \ldots \mid \Gamma_n, \Gamma_n', k_n \Rightarrow C_n}\ \mathsf{hypercut}$$

of which the $\mathsf{multicut}$ rule

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow A \quad \mathcal{H}' \mid \Gamma', k \cdot A \Rightarrow C}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow C}\ \mathsf{multicut}$$

is a special case. In [Avr09], it was noted that there are problems with the disjunction case of the original cut-admissibility proof for **HG1ip**+Com.[2] Avron outlines a semantic proof of cut-admissibility in [Avr09].

In [Avr96], a simpler variant of the Com rule (suggested by Mints) is given:

$$\frac{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A_1 \quad \mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A_2}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2}\ \mathsf{Com}'$$

The use of this variant in the proof of linearity in Example 4.71 on the preceding page would not require contraction rules.

Avron [Avr96] notes that the Com and Com' rules are interderivable. (A proof is given in Lemma C.7 on page 251.)

In [Avr96, Avr00] the intuitionistic splitting rule and a variant of the communication rule are given in place of the single Com rule:

$$\frac{\mathcal{H}_1 \mid \Gamma_1 \Rightarrow A_1 \quad \mathcal{H}_2 \mid \Gamma_2 \Rightarrow A_2}{\mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1 \Rightarrow A_2 \mid \Gamma_2 \Rightarrow A_1}\ \mathsf{Com}''$$

Avron [Avr96] also notes that the $\mathsf{S_I}$ and Com'' rules are derivable from Com' (or Com) and vice versa [Avr96]. (A proof is given in Lemma C.8 on page 251.)

Remark 4.72. The hypercontext-splitting variants of the Com rule given in some papers on hyperextensions of **Int** with some form of the Com rule are inter-derivable with the hypercontext-sharing rules given above (Lemma 4.47).

Proposition 4.73. *The parallel* $\mathsf{R}\vee'$ *rule is invertible in* **HG1ip**+Com*, i.e.*

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \mid \Gamma \Rightarrow \Delta, B}\ (\mathsf{R}\vee'^{-1})$$

Proof. Noted in [Avr09]. □

Remark 4.74. Note that the combination of $\mathsf{S_I}$ with Com'' roughly corresponds to the extension of **Jan** with linearity to obtain **GD**. Note also that Com' can be derived using

[2] Attributed to personal communication from Mathias Baaz and Agata Ciabattoni.

$\mathsf{S_c}$, which is not surprising since **Cl** is stronger than **GD**. (A proof is given in Lemma C.9 on page 251.)

It's also noteworthy that alternative versions of hypersequent calculi for **GD** [Avr00] and Galmiche et al [GLWS07] give a framework called **GLC*** for **GD** and $\mathbf{G_k}$ that uses **cyclic hypersequents** as axioms, e.g. $A \Rightarrow B \mid B \Rightarrow C \mid C \Rightarrow A$. The multi-premiss Com''_n rule can be used to show that such axioms are derivable. These systems are noteworthy for having terminating (with respect to root-first proof search) and invertible logical rules for implication, where various combinations of subformulae of the principal formula $A \supset B$ are analysed akin to the system **G4ip** [Dyc92] for **Int**.

First-order and propositional "fuzzy" quantifier extensions to **HG1ip**+Com, as well as modal extensions, are given in [BZ00, BCF03a]. They will not be covered in this thesis, which examines transformations between propositional systems only.

From [CF01b, BCF03a], calculi for k-valued Gödel logics $\mathbf{G_k}$ can be obtained by adding the rule $\mathsf{G_k}$ to **HG1ip**:

$$\frac{\langle \mathcal{H} \mid \Gamma_i, \Gamma_{i+1} \Rightarrow A_i \rangle_{i=1}^{k}}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_k \Rightarrow A_k \mid \Gamma_{k+1} \Rightarrow} \ \mathsf{G_k}$$

Above is a hypercontext-sharing variant of the rule, given for simplicity. The original paper gives a hypercontext-splitting rule.

The logics $\mathbf{BC_k}$ for Intuitionistic Kripke Frames of bounded cardinality can be obtained by adding the rule $\mathsf{BC_k}$ to **HG1ip**

$$\frac{\langle\, \langle \mathcal{H} \mid \Gamma_i, \Gamma_j \Rightarrow A_i \rangle_{i=0}^{k-1} \,\rangle_{j=i+1}^{k}}{\mathcal{H} \mid \Gamma_0 \Rightarrow A_0 \mid \ldots \mid \Gamma_{k-1} \Rightarrow A_{k-1} \mid \Gamma_k \Rightarrow} \ \mathsf{BC_k}$$

while preserving the admissibility of cut. Again, a hypercontext-sharing variant of the rule is given for simplicity. The original paper gives a hypercontext-splitting rule.

Recall that $\mathbf{BC_2}$ is equivalent to **Sm** and $\mathbf{G_3}$. We give the rule below:

$$\frac{\mathcal{H} \mid \Gamma_0, \Gamma_1 \Rightarrow A_0 \quad \mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A_1}{\mathcal{H} \mid \Gamma_0, \Rightarrow A_0 \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow} \ \mathsf{BC_2}$$

EXAMPLE 4.75. *Derivation for the canonical formula of* $\mathbf{BC_2}$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad \dfrac{A, B \Rightarrow B}{A, A \wedge B \Rightarrow B}\,\mathsf{L}\wedge}{\Rightarrow A \mid A \Rightarrow B \mid A \wedge B \Rightarrow}\,\mathsf{BC_2}}{\Rightarrow A \mid A \Rightarrow B \mid A \wedge B \Rightarrow C}\,\mathsf{RW}}{\Rightarrow A \mid \ \Rightarrow A \supset B \mid \ \Rightarrow A \wedge B \supset C}\,\mathsf{R}\supset^2}{\Rightarrow A \vee (A \supset B) \vee (A \wedge B \supset C)}\,\mathsf{R}\vee'^2$$

Note that **HG1ip**+$\mathsf{G_k}$ is equivalent to **HG1ip**+Com+$\mathsf{BC_k}$.

The logics $\mathbf{BW_k}$ for Intuitionistic Kripke Frames of bounded width (at most $k+1$ branches) can be obtained by adding the rule $\mathsf{BW_k}$ to **HG1ip**

$$\frac{\langle\, \langle \mathcal{H} \mid \Gamma_i, \Gamma_j \Rightarrow A_i \rangle\,{}_{i=0}^{k}\, \rangle\,{}_{j=i+1}^{k}}{\mathcal{H} \mid \Gamma_0 \Rightarrow A_0 \mid \ldots \mid \Gamma_{k-1} \Rightarrow A_{k-1} \mid \Gamma_k \Rightarrow A_k}\,\mathsf{BW_{k+1}}$$

while preserving the admissibility of cut. Again, a hypercontext-sharing variant of the rule is given for simplicity. The original paper gives a hypercontext-splitting rule.

4.4.3. Path Hypertableaux. In work by Ciabattoni and Ferrari [CF00], tableau variants of hypersequent calculi, called **hypertableaux**, are introduced, along with rules for $\mathbf{BC_k}$ and $\mathbf{BW_k}$ logics that correspond to the hypersequent rules given above. The tableau framework given corresponds to a multiple conclusioned hypersequent variant of **HG1ip** (which we call **HG2ipm**, as the internal weakening rules are absorbed) with the following rule for implication:

$$\frac{\mathcal{H} \mid \Gamma, A \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \supset B, \Delta}\,\mathsf{R}\supset$$

(The $\mathsf{L}\wedge$ and $\mathsf{R}\vee$ are the same as **HG1ip**.)

It is noted in the paper that the logics $\mathbf{BD_k}$ for Intuitionistic Kripke Frames of bounded depth (chains of length at most k) require that only adjacent components (corresponding to adjacent worlds in a Kripke Frame) can interact. The authors' solution is to use a system where the hypersequents are lists rather than multisets, and omit the external permutation rule, and add the rules

$$\frac{\langle \mathcal{H} \parallel \overbrace{\Gamma_1 \Rightarrow \Delta_1 \parallel \ldots \parallel \Gamma_{i-1} \Rightarrow \Delta_{i-1}}^{i>1} \parallel \Gamma_i, \Gamma_{i+1} \Rightarrow \Delta_i, \Delta_{i+1} \parallel \mathcal{H}' \rangle\,{}_{i=2}^{k}}{\mathcal{H} \parallel \Gamma_1 \Rightarrow \Delta_1 \parallel \ldots \parallel \Gamma_{k+1} \Rightarrow \Delta_{k+1} \parallel \mathcal{H}'}\,\mathsf{BD_k}$$

The formalism is called **path hypertableaux** because the components form a chain, or *path*.

The rule $\mathsf{BD_2}$ in this framework is

$$\frac{\mathcal{H} \parallel \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \parallel \mathcal{H}' \quad \mathcal{H} \parallel \Gamma_1 \Rightarrow \Delta_1 \parallel \Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3 \parallel \mathcal{H}'}{\mathcal{H} \parallel \Gamma_1 \Rightarrow \Delta_1 \parallel \Gamma_2 \Rightarrow \Delta_2 \parallel \Gamma_3 \Rightarrow \Delta_3 \parallel \mathcal{H}'}\ \mathsf{BD_2}$$

EXAMPLE 4.76. *A proof of the canonical formula for* $\mathbf{BD_2}$*:*

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{B \Rightarrow A, B \quad \cfrac{A, B, B \Rightarrow A}{\Rightarrow B \parallel A, B, B \Rightarrow A}\ \mathsf{EW}}{\Rightarrow B \parallel B \Rightarrow A \parallel A, B \Rightarrow}\ \mathsf{BD_2}}{\Rightarrow B \parallel B \Rightarrow A \parallel B \Rightarrow \neg A}\ \mathsf{R\neg}}{\Rightarrow B \parallel B \Rightarrow A \parallel B \Rightarrow A \vee \neg A}\ \mathsf{R\vee_2}}{\Rightarrow B \parallel B \Rightarrow A \vee \neg A \parallel B \Rightarrow A \vee \neg A}\ \mathsf{R\vee_1}}{\Rightarrow B \parallel B \Rightarrow A \vee \neg A}\ \mathsf{EC}}{\Rightarrow B \parallel \Rightarrow B \supset (A \vee \neg A)}\ \mathsf{R\supset}}{\Rightarrow B \vee (B \supset (A \vee \neg A))}\ (\mathsf{R\vee'})$$

CONJECTURE 4.77. We conjecture that the $\mathsf{BD_k}$ rules are sound for hypersequent calculi with external permutation, so long as the restriction is added that $\Gamma_i \subseteq \Gamma_{i+1}$. (Observe that the invertible form of $\mathsf{L\wedge}$ is admissible, and this property is preserved by the right-sided rules.)

REMARK 4.78. See the procedure for obtaining hypersequent rules from geometric rules given later in Section 4.4.5 on page 93.

4.4.4. HLJpm and HG3ipm. The proof theory for various intermediate logics is well developed using single-succedent hypersequent calculi. Using **HG1ip** has some drawbacks: notably that the $\mathsf{R\vee}$ rule is not invertible, nor is $\mathsf{R\vee'}$ for calculi weaker than **HG1ip**+ Com. (An invertible form of $\mathsf{R\wedge}$ is easily shown admissible using contraction.) Since the aim of this work is to show a relationship between hypersequents and labelled calculi, multisuccedent hypersequent calculi were used here instead, as the labelled calculi examined here are multisuccedent.

Mints [Min04] suggests a hyperextension of the multisuccedent calculus **LJpm** (Figure A.4 on page 236) for **Int**, referred to as a "tableau", although with a trivially invertible form of the $\mathsf{R\supset}$ rule. The calculus **HLJpm** is based on the propositional fragment of that, and was used in presentations on earlier work in this thesis [Rot08a, Rot08b, Rot09].

$$\frac{}{\mathcal{H} \mid P \Rightarrow P}\ \mathsf{Ax} \qquad \frac{}{\mathcal{H} \mid \bot \Rightarrow P}\ \mathsf{L}\bot$$

$$\frac{\mathcal{H} \mid \Gamma, A, B \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \wedge B \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A, \Delta \quad \mathcal{H} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{H} \mid \Gamma \Rightarrow A \wedge B, \Delta}\ \mathsf{R}\wedge$$

$$\frac{\mathcal{H} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{H} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \vee B \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A, B, \Delta}{\mathcal{H} \mid \Gamma \Rightarrow A \vee B, \Delta}\ \mathsf{R}\vee$$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{H} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \supset B \Rightarrow \Delta}\ \mathsf{L}\supset \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A \supset B, \Delta \mid \Gamma, A \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \supset B, \Delta}\ \mathsf{R}\supset_\iota$$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \Rightarrow \Delta}\ \mathsf{LW} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow A, \Delta}\ \mathsf{RW} \qquad \frac{\mathcal{H} \mid \Gamma, A, A \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \Rightarrow \Delta}\ \mathsf{LC} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A, A, \Delta}{\mathcal{H} \mid \Gamma \Rightarrow A, \Delta}\ \mathsf{RC}$$

$$\frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EW} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}\ \mathsf{EC}$$

FIGURE 4.3. The hypersequent calculus **HLJpm**.

PROPOSITION 4.79 (Generalised Axiom). **HLJpm** $\vdash \mathcal{H} \mid A, \Gamma \Rightarrow \Delta, A$.

Proof. By induction on the structure of A and the derivation depth. □

A similar hypersequent calculus **HG3ipm** is given in Figure 4.4 on the facing page, which is an extension of **m-G3i** from [TS00] in Appendix A on page 233.

Like **m-G3ip**, the standard internal weakening and contraction rules are admissible. Instances of external weakening can be eliminated from proofs in **HG3ipm** (shown by induction on the derivation height).

PROPOSITION 4.80. *The standard internal structural rules are admissible in* **HG3ipm**.

Proof. By induction on the derivation depth. □

PROPOSITION 4.81 (Generalised Axiom). **HG3ipm** $\vdash \mathcal{H} \mid A, \Gamma \Rightarrow \Delta, A$.

Proof. By induction on the structure of A and the derivation depth. □

LEMMA 4.82. **HLJpm** $\vdash \mathcal{H}$ *iff* **HG3ipm** $\vdash \mathcal{H}$.

Proof. By induction on the derivation depth, similar to Proposition A.6 on page 236. □

$$\frac{}{\mathcal{H} \mid P, \Gamma \Rightarrow \Delta, P}\ \mathsf{Ax} \qquad \frac{}{\mathcal{H} \mid \bot, \Gamma \Rightarrow \Delta}\ \mathsf{L}\bot$$

$$\frac{\mathcal{H} \mid \Gamma, A, B \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \wedge B \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A, \Delta \quad \mathcal{H} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{H} \mid \Gamma \Rightarrow A \wedge B, \Delta}\ \mathsf{R}\wedge$$

$$\frac{\mathcal{H} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{H} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \vee B \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A, B, \Delta}{\mathcal{H} \mid \Gamma \Rightarrow A \vee B, \Delta}\ \mathsf{R}\vee$$

$$\frac{\mathcal{H} \mid A \supset B, \Gamma \Rightarrow A, \Delta \quad \mathcal{H} \mid B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \supset B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\supset \qquad \frac{\mathcal{H} \mid A, \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \supset B}\ \mathsf{R}\supset$$

$$\frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EW} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EC}$$

FIGURE 4.4. The hypersequent calculus **HG3ipm**.

LEMMA 4.83. $\mathbf{HGipm}_{\sqsupset} \vdash \mathcal{H}$ *iff* $\mathbf{HG3ipm} \vdash \mathcal{H}$.

Proof. By induction on the derivation depth. (The proof is written out in Lemma C.11 on page 252.) □

COROLLARY 4.84. **HG3ipm** *is a conservative hyperextension of* **m-G3i** *(Figure A.3 on page 234).*

Proof. Follows from Theorem 4.45 on page 76. Note also the equivalence to other sound and complete calculi for **Int** in Lemma 4.83. □

4.4.5. Obtaining hypersequent rules from geometric frame conditions. Geometric formulae can be translated into external hypersequent rules in a straightforward manner. Recall from Lemma 3.62 on page 61 that every geometric formula has a corresponding geometric rule from Definition 3.60 on page 61:

$$\frac{[\bar{z}/\bar{y}]\bar{A}_1, \bar{A}_0, \Gamma \Rightarrow \Delta \quad \dots \quad [\bar{z}/\bar{y}]\bar{A}_n, \bar{A}_0, \Gamma \Rightarrow \Delta}{\bar{A}_0, \Gamma \Rightarrow \Delta}\ \rho$$

There is a corresponding hypersequent rule ρ_{H} with $n+1$ active components, where each first-order parameter in ρ corresponds to a schematic component in ρ_{H}, i.e. two parameters $x_1 \sim S_1$ and $x_2 \sim S_2$, where $S_1 = \Gamma_1 \Rightarrow \Delta_1$ and $S_2 = \Gamma_2 \Rightarrow \Delta_2$. Then for every relation Rx_1x_2

in the active formulae $\bar{A}_0, \bar{A}_i$ of the ith premiss of ρ (and the conclusion for $\bar{A}_0$), $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_2 \subseteq \Delta_1$,

EXAMPLE 4.85. *For the logics in* $\mathbf{BW_k}$ *(pages 41 and 48), the hypersequent rule*

$$\frac{\langle\, \mathcal{H} \setminus i,j \mid \Gamma_i \Rightarrow \Delta_i, \Delta_j \mid \Gamma_i, \Gamma_j \Rightarrow \Delta_j \,\rangle_{i,j=0\ i\neq j}^{k}}{\mathcal{H}}\ \mathsf{BW_k}$$

where $\mathcal{H} \setminus i,j =_{def} \mathcal{H} \setminus (\Gamma_i \Rightarrow \Delta_i \mid \Gamma_j \Rightarrow \Delta_j)$*, can be obtained.*

EXAMPLE 4.86. *For the logics in* $\mathbf{BTW_k}$ *(pages 41 and 49), the hypersequent rule*

$$\frac{\langle\, \mathcal{H} \mid \Gamma_i, \Gamma_j \Rightarrow\ \mid\ \Rightarrow \Delta_i, \Delta_j \,\rangle_{i,j=0\ i\neq j}^{k}}{\mathcal{H} \mid \Gamma_0 \Rightarrow \Delta_0 \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k}\ \mathsf{BTW_k}$$

can be obtained.

For brevity, all of the hypersequent rules corresponding to the logics in $\mathbf{Int}^*/\mathtt{Geo}$ discussed in Chapter 3 will be omitted. Some extension rules logics in $\mathbf{Int}^*/\mathtt{Geo}$ in the framework $\mathbf{HG3ipm}^*$ for the most commonly-discussed logics are given in Figure 4.5. The other rules can be obtained in a straightforward manner using the procedure outlined above. Their soundness can be checked similar to the cases outlined below.

$$\frac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow}{\mathcal{H} \mid \Gamma \Rightarrow\ \mid \Gamma' \Rightarrow}\ \mathsf{LQ}$$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Delta' \mid \Gamma, \Gamma' \Rightarrow \Delta \quad \mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta, \Delta'}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}\ \mathsf{Com_m}$$

$$\frac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}\ \mathsf{S}$$

FIGURE 4.5. Extension rules for $\mathbf{HG3ipm}^*$.

The LQ rule can be added to **HG3ipm** to give a calculus for **Jan**:

$$\frac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow}{\mathcal{H} \mid \Gamma \Rightarrow\ \mid \Gamma' \Rightarrow}\ \mathsf{LQ}$$

Note that LQ is a special case of the $\mathsf{BTW_1}$ rule (Example 4.86), where $\Delta_1, \Delta_2 = \emptyset$.

LEMMA 4.87. *The* LQ *rule is sound for* **Jan**.

Proof. By contraposition. Let $\mathfrak{M} = \langle W, R, v \rangle$ be a directed Intuitionistic Kripke model (page 3.3.3), and suppose $\mathfrak{M} \nvDash \mathcal{H} \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow$. Then $\mathfrak{M} \nvDash \Gamma \Rightarrow$ and $\mathfrak{M} \nvDash \Gamma' \Rightarrow$. Suppose there exists $x, y \in W$ such that $x \Vdash \bigwedge\Gamma$ and $y \Vdash \bigwedge\Gamma'$. Since $\mathfrak{M}$ is directed, there exists $z \in W$ such that $(x,z) \in R$ and $(y,z) \in R$. By persistence, $z \Vdash \bigwedge\Gamma$ and $z \Vdash \bigwedge\Gamma'$. So $z \Vdash \bigwedge\Gamma, \Gamma'$. So $\mathfrak{M} \nvDash \Gamma, \Gamma' \Rightarrow$ and $\mathfrak{M} \nvDash \mathcal{H} \mid \Gamma, \Gamma' \Rightarrow$. □

LEMMA 4.88. *The internal weakening rules (*LW *and* RW*) are admissible in* **HG3ipm**+ LQ.

Proof. Straightforward induction on the derivation height. □

LEMMA 4.89. *The internal contraction rules (*LC *and* RC*) are admissible in* **HG3ipm**+ LQ.

Proof. Straightforward induction on the derivation height. (The cases for RC are trivial. For LC, note that the rule is easily permutable over instances of LQ.) □

LEMMA 4.90. *The cut rule is admissible in* **HG3ipm**+LQ.

Proof. By induction on the rank defined by the size of the cut formula and the sum of derivation depths of the premisses. (The proof is written out with relevant cases in Lemma C.12 on page 253.) □

The split (S) rule can be added to **HG3ipm** to give a calculus for **Cl**:

$$\frac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}\ \mathsf{S}$$

LEMMA 4.91. *The* S *rule is sound for* **Cl**.

Proof. By contraposition. Let $\mathfrak{M} = \langle W, R, v \rangle$ be a symmetric Intuitionistic Kripke model (page 47), and suppose $\mathfrak{M} \nvDash \mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'$. Then $\mathfrak{M} \nvDash \Gamma \Rightarrow \Delta$ and $\mathfrak{M} \nvDash \Gamma' \Rightarrow \Delta'$. So there exists $x, y \in W$ such that $x \Vdash \bigwedge\Gamma$, $x \nVdash \bigvee\Delta$, $y \Vdash \bigwedge\Gamma'$ and $y \nVdash \bigvee\Delta'$. Since both $(x,y) \in R$ and $(y,x) \in R$, $x \Vdash \bigwedge\Gamma, \Gamma'$, $x \nVdash \bigvee\Delta, \Delta'$ (and similarly for y). So $\mathfrak{M} \nvDash \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ and $\mathfrak{M} \nvDash \mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. □

LEMMA 4.92. *The internal weakening rules (*LW *and* RW*) are admissible in* **HG3ipm**+ S.

Proof. Straightforward induction on the derivation height. □

LEMMA 4.93. *The internal contraction rules (*LC *and* RC*) are admissible in* **HG3ipm**+S.

Proof. Straightforward induction on the derivation height. □

LEMMA 4.94. *The cut rule is admissible in* **HG3ipm**+S.

Proof. By induction on the rank defined by the size of the cut formula and the sum of the derivation depths of the premisses, similar to the case for the LQ rule (Lemma C.12). The case where the left premiss of a cut is the conclusion of an instance of S is similar. □

The $\mathsf{Com_m}$ rule (a multisuccedent variant of Com') can be added to **HG3ipm** to give a calculus for **GD**:

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Delta' \mid \Gamma, \Gamma' \Rightarrow \Delta \quad \mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta, \Delta'}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'} \ \mathsf{Com_m}$$

Note that the $\mathsf{Com_m}$ rule is the same of $\mathsf{BW_1}$ (Example 4.85 on page 94).

NOTATION 4.95 (Linear Intuitionistic Kripke Models). We will use $\mathcal{K}_{\mathbf{GD}}$ to denote linear intuitionistic Kripke models.

LEMMA 4.96. *The* $\mathsf{Com_m}$ *rule is sound for* **GD**.

Proof. By contraposition. Let $\mathfrak{M} = \langle W, R, v\rangle \in \mathcal{K}_{\mathbf{GD}}$ and suppose $\mathfrak{M} \nvDash \mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'$. Then $\mathfrak{M} \nvDash \Gamma \Rightarrow \Delta$ and $\mathfrak{M} \nvDash \Gamma' \Rightarrow \Delta'$. So there exists $x, y \in W$ such that $x \Vdash \bigwedge\Gamma$, $x \nVdash \bigvee\Delta$, $y \Vdash \bigwedge\Gamma'$ and $y \nVdash \bigvee\Delta'$. Either $(x,y) \in R$ or $(y,x) \in R$. If $(x,y) \in R$, then $x \nVdash \bigvee\Delta, \Delta'$ and $y \Vdash \bigwedge\Gamma, \Gamma'$. So $\mathfrak{M} \nvDash \mathcal{H} \mid \Gamma \Rightarrow \Delta, \Delta' \mid \Gamma, \Gamma' \Rightarrow \Delta$. Otherwise $(y,x) \in R$, so $y \nVdash \bigvee\Delta, \Delta'$ and $x \Vdash \bigwedge\Gamma, \Gamma'$. So $\mathfrak{M} \nvDash \mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta, \Delta'$. □

LEMMA 4.97. *The internal weakening rules (*LW *and* RW*) are admissible in* **HG3ipm**+$\mathsf{Com_m}$.

Proof. Straightforward induction on the derivation height. □

LEMMA 4.98. *The internal contraction rules (*LC *and* RC*) are admissible in* **HG3ipm**+$\mathsf{Com_m}$.

Proof. Straightforward induction on the derivation height. □

PROPOSITION 4.99. *The multisuccedent form of the* $\mathsf{S_I}$ *rule*

$$\frac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta}\ (\mathsf{S_I})$$

is admissible in **HG3ipm**+$\mathsf{Com_m}$.

Proof. By derivation, using $\mathsf{Com_m}$ and EW. □

LEMMA 4.100. *The cut rule is admissible in* **HG3ipm**+$\mathsf{Com_m}$.

Proof. By induction on the rank defined by the size of the cut formula, the number of instances of $\mathsf{Com_m}$ in the premisses, and the sum of the derivation depths of the premisses. (The proof is given in Lemma C.13 on page 253.) □

PROPOSITION 4.101. $\mathsf{Com'}$ *is admissible in* **HG3ipm**+$\mathsf{Com_m}$.

Proof. By weakening. □

Although we have only provided rules for a few logics in the framework **HG3ipm***, we expect that multisuccedent forms of rules for other logics in **Int***/$\mathtt{Geo}$, such as $\mathbf{G_k}$, $\mathbf{BC_k}$ or $\mathbf{BW_k}$ can be adapted from the single-succedent forms from **HG1ip*** by adding components to the premisses to show the accessibility relations between succedents, as has been done for the $\mathsf{Com_m}$ rule for **GD**. They will be omitted from this thesis for brevity.

REMARK 4.102. Work by Ciabattoni et al [CGT08] gives syntactic conditions of single-succedent hypersequent rules that admit cut. An area for future work is to extend these conditions to multisuccedent calculi, that would eliminate the need for the explicit proofs of cut admissibility which are given above for the rules in Figure 4.5 on page 94. (These properties are discussed in more detail later on page 220.)

4.4.6. Other Systems. We briefly note the existence of other hypersequent calculi for **Int** and extensions of **Int** below.

Kurokawa [Kur07, Kur09] gives a multisuccedent form of **HG3ipm** called **mLIC** for a hybrid logic of **Int** plus classical atoms introduced in [Kur04]. It has a restricted form of classical split:

$$\frac{\mathcal{H} \mid \Gamma_1, \Gamma_2^\circ \Rightarrow \Delta_1, Dl_2^\circ}{\mathcal{H} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2^\circ \Rightarrow \Delta_2^\circ}\ \mathsf{CS}$$

where $\Gamma_2^\circ, \Delta_2^\circ$ contain only "classical atoms". (How classical atoms differ from other atomic variables is not defined.) It is unclear how this logic relates to semiconstructive logics such as **GSc** (page 42).

Corsi and Tassi [CT07] introduce a system called **SIC**, where hypersequents without external permutation are viewed as "stacks" of components (Figure 4.6). The intention of the system is to have a calculus with both invertible rules—to eliminate backtracking in proof search, which is called a "metarule" while having terminating root-first proof search and the subformula property (minus the markings).

$$\frac{}{\mathcal{H} \parallel \Gamma, P \Rightarrow P, \Delta}\ \mathsf{Id.} \qquad \frac{}{\mathcal{H} \parallel \Gamma, \bot \Rightarrow \Delta}\ \mathsf{L}\bot$$

$$\frac{\mathcal{H} \parallel \Gamma, (A \supset B)^* \Rightarrow A, \Delta \quad \mathcal{H} \parallel \Gamma, B \Rightarrow \Delta}{\mathcal{H} \parallel \Gamma, A \supset B \Rightarrow \Delta}\ \mathsf{L}\supset \qquad \frac{\mathcal{H} \parallel \Gamma, A^\dagger \Rightarrow B, \Delta}{\mathcal{H} \parallel \Gamma, A^\dagger \Rightarrow A \supset B, \Delta}\ \mathsf{a.f.}$$

$$\frac{\mathcal{H} \parallel \Sigma, \Pi^\dagger, \Gamma_p, A_1^\dagger, A_1, \Rightarrow B_1 \parallel \ \dots \ \parallel \Sigma, \Pi^\dagger, \Gamma_p, A_n^\dagger, A_n, \Rightarrow B_n}{\mathcal{H} \parallel \Sigma^*, \Pi^\dagger, \Gamma_p \Rightarrow \Delta_q, A_1 \supset B_1, \dots, A_n \supset B_n}\ \mathsf{push_n}$$

$$\frac{\mathcal{H}}{\mathcal{H} \parallel \Sigma^*, \Pi^\dagger, \Gamma_p \Rightarrow \Delta_q}\ \mathsf{pop}$$

FIGURE 4.6. The hypersequent calculus for the implicational fragment of **SIC**.

(For Figure 4.6, Σ^* and $\Pi^\dagger$ contain formulae marked with $*$ and $\dagger$, respectively, and Γ_p, Δ_q contain only atomic formulae.)

Since the rules are said to be invertible, in root-first proof search one analyses the right-most component until it is no longer analysable, and pops the component from the stack if it is not an axiom. The implication rules work by marking analysed formulae with symbols, which involves extra bookkeeping. Since the system need only analyse the first component until it is popped (weakened), implementations may be optimised for pattern matching.

EXAMPLE 4.103. *A derivation in* **SIC**.

$$\frac{\dfrac{\dfrac{(A \supset B)^*, A^\dagger, A \Rightarrow A, B \quad B, A^\dagger, A \Rightarrow B}{A \supset B, A^\dagger, A \Rightarrow B}\ \mathsf{L}\supset}{(A \supset B)^* \Rightarrow A, A \supset B} \qquad \dfrac{A^\dagger, A, B \Rightarrow B}{B \Rightarrow A \supset B}}{A \supset B \Rightarrow A \supset B}\ \mathsf{L}\supset$$

4.5. Conclusion

We have introduced the notation and terminology of hypersequents, and provided a survey of various hypersequent calculi from the literature focusing on calculi for **Int** and frameworks for some logics in **Int***/Geo. We note that a common feature of the structural rules of these frameworks are the implicit relationship between components which corresponds to the semantics of the logics. For the single-conclusioned extensions to **HG1ip** (Figure 4.2 on page 84), the accessibility relation is indicated by subset relations between antecedents of components, as shown in Figure 4.7. For extensions to the multisuccedent calculus **HG3ipm** that we have introduced, we extend this by adding the converse accessibility relation between succedents of components, e.g. in the $\mathsf{Com_m}$ rule.

Hypersequent Rule	Characteristic Frame Property
$\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow \ \mid \Gamma, \Gamma' \Rightarrow}{\mathcal{H} \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow}\ \mathsf{LQ}_\iota$	$\exists z.(Rxz \wedge Ryz)$
$\dfrac{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow A \mid \Gamma, \Gamma' \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \mid \Gamma' \Rightarrow B}\ \mathsf{S}'_\mathsf{c}$	$\forall xy.Rxy \supset Ryx$
$\dfrac{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \quad \mathcal{H} \mid \Gamma_2 \Rightarrow A_2}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_2 \mid \Gamma_2 \Rightarrow A_1}\ \mathsf{Com}''_\mathsf{cs}$	$\forall xy.Rxy \vee Ryx$
$\dfrac{\langle\, \langle \mathcal{H} \mid \Gamma_i, \Gamma_j \Rightarrow A_i \rangle_{i=0}^{k-1}\, \rangle_{j=i+1}^{k}}{\mathcal{H} \mid \Gamma_0 \Rightarrow A_0 \mid \ldots \mid \Gamma_{k-1} \Rightarrow A_{k-1} \mid \Gamma_k \Rightarrow}\ \mathsf{BC_k}$	$\forall x_0, \ldots, x_k. \bigwedge_{i=1}^{k} Rx_0x_i \supset \bigvee_{j \neq i} x_j = x_i$
$\dfrac{\langle\, \langle \mathcal{H} \mid \Gamma_i, \Gamma_j \Rightarrow A_i \rangle_{i=0}^{k}\, \rangle_{j=i+1}^{k}}{\mathcal{H} \mid \Gamma_0 \Rightarrow A_0 \mid \ldots \mid \Gamma_k \Rightarrow A_k}\ \mathsf{BW_k}$	$\forall x, y_0, \ldots, y_k. \bigvee_{j \neq i} Ry_iy_j$

FIGURE 4.7. Hypersequent rules extending **HG1ip** and their corresponding frame properties.

A procedure for obtaining multisuccedent hypersequent rules for logics in **Int***/Geo was given, with specific examples for some of the logics. Note that the rules are similar to the single-succedent rules in Figure 4.7.

In the next chapter, we examine labelled sequent calculi, with an emphasis on systems for **Int** and logics in **Int***/Geo.

CHAPTER 5

Labelled Sequent Calculi

5.1. Overview

5.1.1. Synopsis. In this chapter, we introduce the notation and terminology of labelled sequent calculi. We will begin with preliminary definitions of simply labelled sequent calculi, and then introduce various extensions to the formalism through a brief survey. In Section 5.2, we will provide formal definitions for labelled sequent calculi, and then introduce two simply labelled sequent calculi for **Int**, from the literature. In Section 5.3, we provide formal definitions for relational sequent calculi, and introduce a framework of relational sequent calculi for **Int***/`Geo`, as well as another relational calculus for **Int**.

We also discuss a variant of relational sequent calculi for modal logics, and introduce rules for a framework for **Int***/`Geo` by using the Gödel translation into **S4**

(Definition 3.3 on page 43). In Section 5.4 we briefly look at prefix calculi, and provide a prefix calculus for **Int** that is also based on a calculus for modal logic, again by using the Gödel translation into **S4**. In Section 5.5 we introduce the notion of equivalence modulo permutation of labels for sequents.

5.1.2. Preliminaries. Labelled sequents are an extension of Gentzen-style sequents by annotating logical formulae with expressions called **labels**, (where the annotated formulae are called **labelled formulae**), and adding a new kind of formula built up from labels, called a **relational formula**.

Formally, we can define a language of labels `Lab` that is constructed from a denumerable set of atomic labels and a finite set of n-ary **label connectives**. A labelled formula is an ordered pair $\langle$`Form`, `Lab`$\rangle$. We also define a language of **(prime) relational formula** in `Rel` from a finite set of n-ary relation symbols and labels. (We do not call this set atomic, since the labels may not be atomic.) A **labelled sequent** is an expression of the form:

$$\mathtt{Rel}^*;\langle \mathtt{Form},\ \mathtt{Lab}\rangle^* \Rightarrow \mathtt{Rel}^*;\langle \mathtt{Form},\ \mathtt{Lab}\rangle^*$$

One could extend the language further, by allowing logical connectives to combine relational formulae, or even relational formulae and logical formulae, e.g. in [Bla00].

The language of labelled sequents can be thought of as an alternative notation for first-order logic with functions. Indeed, the motivation for many labelled systems is their correspondence with relational models, e.g. [Vig00, Neg07, PU09]. However, the labelled sequents only contain the points and relations from a relational model, and not the full first-order translation, which may include quantifiers and additional relations that are implicit in the logical operators.

By including semantics as part of the formalisms, labelled calculi are said to be "syntactically impure," e.g. [Wan94, Avr96]. However, when one considers that labels correspond to locations in "pure" formalisms such as hypersequents, then the criticism has less weight. The difference between a "semantic" rule such as

$$\frac{x \leq y, \Sigma; A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, B^y}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^y} \ \mathsf{R}\supset_{\leq}$$

and a "syntactic" rule such as

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma, A \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \supset B} \ \mathsf{R}\supset'$$

(aside from the difference that they are not quite the same rule, as the former is invertible) is that the labelled rule *names* the location of formulae explicitly, and includes a new kind of formula that explicitly states the relationship between locations.

5.1.3. A Brief Survey of Labelled Calculi. There are many variants of labelled sequent calculi in the literature. Aside from obvious notational differences, they also vary in the richness of the labelling languages. The simplest form, which we call a **simply labelled calculus** (discussed in detail in Section 5.2 later), consists of a labelling language restricted to atoms, and no relational formulae. The earliest example of any labelled calculus appears to be a simply labelled calculus by Kanger [Kan57] for **S5**, which was introduced in the 1950s as a "spotted calculus". (Kanger's system was discussed in Section 1.3 on page 15.) Other examples are Maslov's [Mas67] calculus for first-order **Int** with equality, and (more recently) de Paiva and Pereira's [dP05] calculus for **Int**, both of which will be discussed later in this chapter. (These latter two systems label formulae with sets but the interpretation of labels renders them equivalent to simply labelled sequents.) Other simply labelled calculi are given for the logics **A** and **Ł** by Metcalfe in [MOG05] (also discussed briefly in Section 1.3 on page 15, where the notation uses compound labels as shorthand).

One can extend the language of simply labelled sequents in a variety of ways. One method is to extend the labelling language itself, often with a simple composition operator. This technique is used in calculi for modal logics by Fitch [Fit66] and later by Fitting [Fit72, Fit83], Mints [Min97] and for subintuitionistic logics by Ishigaki and Kikuchi [IK07]. Note the notational differences between these calculi: Fitch, Fitting and Mints omit the composition operator in labels, and Mints gives a system as hypersequents with labels annotating the sequent operator of each component rather than individual formulae. Ishigaki and Kikuchi label formulae with lists of numbers. These particular systems happen to share the feature that an accessibility relation can be determined by comparing two labels, and will be discussed later in Section 5.4.

Another extension of simply labelled sequents is to impose a structure on the atomic labels, e.g., by using integers (which are totally ordered). This technique is used in [GS08]. As this technique is not used for the calculi we examine in this thesis, it will not be examined further. Note that many calculi as they are presented use numbers for labels, but only as a notational convenience (particularly for defining a freshness condition).

A common extension to simply labelled calculi adds relational formulae to the sequent. This is done for binary relations by Viganò [Vig00], Negri [Neg05, Neg07] and Pinto and Uustalu [PU09], although the systems by Viganò allow single relational formula in the succedent. Gabbay [Gab96] gives systems with higher-arity relations, and, in the method for deriving a relational semantics from an arbitrary logic, allows for multiple kinds of relational formulae.

Other extensions combine some these features. In [Gab96] are given systems which define an "algebra of labels" that provide a framework for a variety of substructural logics. This is extended further in the **compiled labelled deductive systems** of [BGLR04], where the structural rules of various logics are transformed into label manipulation rules, allowing the system to be embedded into **Cl**. These systems will not be discussed here.

In [Bla00], a framework of tableaux for modal logics is introduced where labels are treated as formulae in their own right, and labelled formulae, represented as $x : A$, are considered as formulae where an operator ":" that asserts A at state x. This kind of system will not be discussed here.

Another method of extending labelled calculi, sometimes called called a **connection system**, e.g. [AW07, GM07], is to incorporate *variables* in the labels. This requires one to resolve constraints on the labels of initial sequents. These systems will likewise not be discussed here.

The labelled calculi that we will be examining in the chapter use labels and relational formulae to control proof search. There are variants of labelled calculi which use labels to extract proofs in weaker logics that we will not be examining. For example, labelled calculi for **Cl** which use the labels to extract intuitionistic proofs [BG00, Har01].

The variations of labelled calculi make it difficult to provide a separate section with a survey of existing calculi for intermediate logics. So they will be introduced as subsections of the various types of calculi discussed in this chapter.

A study of the hierarchy of labelled systems that examines the minimum structure required for cut-free labelled calculi for various logics is an area for future work.

5.2. Simply Labelled Sequent Calculi

5.2.1. Definitions. We start with a formal definition of simply labelled sequents:

Definition 5.1 (Labelled Formula). Let $\mathtt{Lab}_0$ be a denumerable set of atomic labels, and let $x \in \mathtt{Lab}_0$ and $A \in \mathtt{Form}_1$. Then $A^x \in \mathtt{Form}_1 \times \mathtt{Lab}_0$ is a **labelled formula**.

The **propositional fragment of labelled formulae** is in $\mathtt{Prop} \times \mathtt{Lab}_0$. (Because most of the calculi examined here are for propositional logics, we will assume the set of labelled formulae is limited to the propositional fragment.

Remark 5.2. The subformulae of a labelled formula are not labelled. (In the simply labelled calculus for **S5** in [Kan57], atomic formulae are labelled, however, all atoms in a formulae are "homogeneously spotted" with the same label.)

Remark 5.3. The notation for labelled formulae varies in the literature. Labelled formulae are often written as $x : A$ or $A : x$, possibly with additional punctuation surrounding the labels, particularly when formulae are labelled with sets of labels, e.g. $A^{\{x,y,z\}}$ in [Mas67].

Notation 5.4. Labels will be denoted by lowercase Roman letters w, x, y, z, possibly with prime marks (e.g. x') or subscripts. The uppercase Roman letters M, N will be used to denote sets of labels.

Definition 5.5 (Simply Labelled Sequents). Let $\mathtt{Prop} \times \mathtt{Lab}_0{}^*$ be the set of **simply labelled multisets**, which are finite multisets of labelled formulae. A simply labelled multiset is denoted by $\underline{\Gamma}^M$, where the **set of labels** for all formulae in the multiset is $M' \subseteq M$. (When the set M is irrelevant, $\underline{\Gamma}$ is written.) A **uniformly labelled multiset** (or **unilabelled multiset**, for short) Γ^x is a multiset of uniformly labelled (by x) formulae.

This definition is extended to **simply labelled sequents** in the natural manner. The set of simply labelled sequents is denoted by $\mathtt{SLS}$, and the set of unilabelled sequents labelled by x is denoted by $\mathtt{SLS} \mathbin{/\!/} x$. (See Definition 5.22 on page 108 for a formal definition of $/\!/$.)

Remark 5.6. We bring the reader's attention to a misnomer: by "labelled multiset" (or sequent or calculus) we mean that the constituent formulae are labelled, and not that the multiset (or sequent or calculus) as a whole is labelled.

Definition 5.7 (Schematic Simply Labelled Sequents). Let $\mathtt{MultisetVar}$ be the denumerable **set of multiset variables**, e.g. Γ, Δ. Then a **schematic simply labelled multiset** is a multiset of type $(\langle \mathtt{Prop} + \mathtt{MultisetVar},\ \mathtt{Lab}_0 \rangle + \langle \mathtt{MultisetVar},\ \dagger \rangle)^*$, where $\dagger$ is a unit type that differentiates Γ^x from $\underline{\Gamma}$. (For our purposes, we consider Γ^x and $\underline{\Gamma}$ to different variables, although we will avoid using both in the same context.)

This definition is extended to **schematic simply labelled sequents** in the natural manner. The set of schematic simply labelled sequents is denoted by $\mathtt{SLS}^\mu$, and the set of schematic unilabelled sequents labelled by x is denoted by $\mathtt{SLS}^\mu \mathbin{/\!/} x$. (See Definition 5.22 on page 108 for a formal definition of $/\!/$.)

A **simply labelled calculus** is a calculus of (schematic) simply labelled sequents.

TERMINOLOGY 5.8. For brevity, we will use the term "labelled multiset" (or sequent or calculus) to mean only a simply labelled multiset (or sequent or calculus) in the rest of this section.

NOTATION 5.9. The following alternative notation for labelled sequents is used (following the notation for components in Note 4.7 on page 70):

$$\underline{S}^M =_{def} \underline{\Gamma}^M \Rightarrow \underline{\Delta}^M \tag{67}$$

$$\underline{S}_1^M, \underline{S}_2^N =_{def} \underline{\Gamma}_1^M, \underline{\Gamma}_2^N \;\Rightarrow\; \underline{\Delta}_1^M, \underline{\Delta}_2^N \tag{68}$$

The sets of labels M, N may be omitted when they are irrelevant.

DEFINITION 5.10 (Sequent Union). We define the **union of (labelled) sequents** as:

$$\underline{S}_1 \sqcup \underline{S}_2 =_{def} \underline{S}_1, \underline{S}_2$$

PROPOSITION 5.11. $\sqcup$ *is commutative and associative.*

Proof. From the definition. □

NOTATION 5.12 (Label Function). Let the label function be defined as

$$\pi_{\texttt{lab}}\, A^x =_{def} x \tag{69}$$

Then the label set function on labelled multisets is defined as

$$\texttt{lab}(\underline{\Gamma}) =_{def} \bigcup \{\, \pi_{\texttt{lab}} \circledast \underline{\Gamma} \,\} \tag{70}$$

That is, $\texttt{lab}(\underline{\Gamma})$ denotes the set of labels that the labelled formulae in $\underline{\Gamma}$ are annotated with. This notation is extended naturally for labelled sequents.

We extend the terminology for sequent calculi from [TS00] to its labelled sequent analogue below:

DEFINITION 5.13 (Active and Principal Labels). Let ρ be a schematic rule of a labelled calculus, and let $\underline{\Gamma}$ be a multiset of the active (resp. principal) formulae of ρ. Then $\texttt{lab}(\underline{\Gamma})$ is the set of **active labels** (resp. **principal labels**) of ρ.

DEFINITION 5.14 (Context Labels). Let ρ be a schematic rule of a labelled calculus. A metavariable $\underline{\Gamma}$ which occurs in a premiss of ρ and in the conclusion of ρ, with no annotations to indicate that the variable has different values between the conclusion and premiss beyond relabelling (i.e. that the variables are equivalent modulo permutation of labels, discussed in Section 5.5 on page 130), is called a **context** of ρ. The labelled formulae in $\underline{\Gamma}$ are called **context formulae** of ρ.

Let $\underline{\Delta}$ be a multiset of the active and principal formulae in ρ such that $\mathtt{lab}(\underline{\Delta})$ is the set of active and principal labels. Then $\mathtt{lab}(\underline{\Gamma}) \setminus \mathtt{lab}(\underline{\Delta})$ is the set of **context labels** of ρ.

If the context formulae of the conclusion of ρ is the same as the context formulae of every premiss of ρ, then ρ is called **context-sharing**. Otherwise ρ is called **context-splitting**.

Example 5.15. *In the following rules*

$$\frac{A^x, \underline{\Gamma} \Rightarrow \underline{\Delta} \quad B^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}{(A \vee B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\vee \qquad \frac{A^x, \underline{\Gamma} \Rightarrow \underline{\Delta} \quad B^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}'}{(A \vee B)^x, \underline{\Gamma}, \underline{\Gamma}' \Rightarrow \underline{\Delta}, \underline{\Delta}'}\ \mathsf{L}\vee_{\mathsf{cs}}$$

The $\mathsf{L}\vee$ *rule is context-sharing, and the* $\mathsf{L}\vee_{\mathsf{cs}}$ *rule is context-splitting. The active formulae are* A^x *and* B^x *in the left and right premisses of both rules, respectively, and the principal formulae of both rules are* $(A \vee B)^x$*. The sets of active and principal labels are* $\{x\}$*. The context formulae are* $\underline{\Gamma}, \underline{\Delta}$ *for* $\mathsf{L}\vee$ *and* $\underline{\Gamma}, \underline{\Gamma}', \underline{\Delta}, \underline{\Delta}'$ *for* $\mathsf{L}\vee_{\mathsf{cs}}$*. The context labels are* $\mathtt{lab}(\underline{\Gamma}, \underline{\Delta})$ *for* $\mathsf{L}\vee$ *and* $\mathtt{lab}(\underline{\Gamma}, \underline{\Gamma}', \underline{\Delta}, \underline{\Delta}')$ *for* $\mathsf{L}\vee_{\mathsf{cs}}$.

Remark 5.16. Note that some of the context formulae in a rule ρ may be labelled by active or principal labels. Unlike the situation with hypersequents, there is no distinction between side formulae which occur in an active or principal component, and side formulae which occur in a hypercontext.

Definition 5.17 (Single-Labelled and Multilabelled Rules). A rule ρ is called **single-labelled** (also **uniformly labelled**, or **unilabelled** for short) iff every premiss has only one active label that is equal to the (only) principal label of the conclusion. Otherwise ρ is called **multilabel**. (Single-labelled rules correspond to the internal rules of hypersequent calculi, and multilabelled rules correspond to the external rules of hypersequent calculi.)

Example 5.18. *In Example 5.15, both rules are unilabelled. The* LQ *rule,*

$$\frac{\Gamma_1^z, \Gamma_2^z, \underline{\Gamma}' \Rightarrow \underline{\Delta}}{\Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}}\ \mathsf{LQ}$$

is an example of a multi-labelled rule.

Remark 5.19. A notable difference between hypersequent and labelled sequent calculi is that the distinction between internal and external rules disappears for the latter with respect to the standard structural rules of weakening and contraction. If the weakening rules in a labelled calculus have no restrictions on the labels, then the labelled form of external weakening rule is redundant. Likewise, if label substitution is unrestricted, then the labelled form of external contraction rule is redundant. (This will be shown later for the calculus **LG3ipm**, Figure 7.1 on page 157).

Definition 5.20 (Labelling Rules). Rules in a labelled calculus which act only on the labels, and not on formulae, are called **labelling rules**. (These roughly correspond to structural rules of a hypersequent calculus.)

Example 5.21. *The* LQ *rule from Example 5.18 on the preceding page is an example of a labelling rule.*

Definition 5.22 (Slice). Let $\underline{\Gamma}^M$ be a labelled multiset. A **slice** of $\underline{\Gamma}^M$ by the label x, written as $\underline{\Gamma}^M /\!\!/ x$, is defined inductively as:

$$\begin{aligned} \emptyset /\!\!/ x &= \emptyset \\ (A^x, \underline{\Gamma}) /\!\!/ x &= A^x, (\underline{\Gamma} /\!\!/ x) \\ (A^y, \underline{\Gamma}) /\!\!/ x &= \underline{\Gamma} /\!\!/ x && \text{where } y \neq x \end{aligned}$$

This notion is extended to labelled sequents in the obvious way.

Remark 5.23. Clearly the slice of a multiset or sequent is unilabelled.

Notation 5.24. The notation $\lambda x.\underline{\Gamma} /\!\!/ x$ is used to represent a function that takes a label x and returns $\underline{\Gamma} /\!\!/ x$. So $(\lambda x.\underline{\Gamma}^M /\!\!/ x) \circledast M$ denotes the set of all slices of $\underline{\Gamma}^M$. (This notation is extended to labelled sequents in the obvious way.)

Definition 5.25 (Anti-Slice). The **anti-slice** of labelled multiset $\underline{\Gamma}$ is defined as:

$$\underline{\Gamma} \setminus x =_{def} \underline{\Gamma} \setminus (\underline{\Gamma} /\!\!/ x)$$

This definition is extended naturally for labelled sequents.

PROPOSITION 5.26.

$$\bigsqcup((\lambda x.\underline{S} \mathbin{/\!/} x) \circledast \mathtt{lab}(\underline{S})) = \underline{S}$$

Proof. From Definitions 5.22 and 5.10. □

DEFINITION 5.27 (Single-Succedent and Multisuccedent Labelled Sequent). A labelled sequent $\underline{\Gamma} \Rightarrow \underline{\Delta} \in \mathrm{SLS}$ is a **single-succedent labelled sequent** if for all labels $x \in \mathtt{lab}(\underline{\Gamma}, \underline{\Delta})$, $|\underline{\Delta} \mathbin{/\!/} x| \leq 1$. Otherwise it is a **multisuccedent labelled sequent**.

NOTATION 5.28. We use $\mathtt{SLS}_1$ to denote the set of single-succedent labelled sequents.

NOTATION 5.29 (Compound Labels). We may label a formula (or multiset) with a string of labels, e.g., A^{xyz}, as shorthand for a multiset of multiple occurrences of the formula (or multisets) labelled by each label in the string, e.g. A^x, A^y, A^z.

DEFINITION 5.30. Let the **label stripping** function be defined as follows:

$$\pi_{\mathtt{form}}\, A^x =_{def} A \tag{71}$$

DEFINITION 5.31 (General Semantics of Simply Labelled Sequents). Let $\Gamma^x \Rightarrow \Delta^x$ be a unilabelled sequent, and let $\mathfrak{I}$ be an interpretation. Then we define $\mathfrak{I} \vDash \Gamma^x \Rightarrow \Delta^x$ iff $\mathfrak{I} \vDash (\pi_{\mathtt{form}} \circledast \Gamma^x) \Rightarrow (\pi_{\mathtt{form}} \circledast \Delta^x)$.

A labelled sequent $\underline{S}$ is said to be **true in an interpretation** $\mathfrak{I}$, written $\mathfrak{I} \vDash \underline{S}$, iff there exists a label $x \in \mathtt{lab}(\underline{S})$ such that $\mathfrak{I} \vDash \underline{S} \mathbin{/\!/} x$.

Likewise, a labelled sequent $\underline{S}$ is said to be **false in an interpretation** $\mathfrak{I}$, written $\mathfrak{I} \nvDash \underline{S}$, iff for all labels $x \in \mathtt{lab}(\underline{S})$, $\mathfrak{I} \nvDash \underline{S} \mathbin{/\!/} x$.

REMARK 5.32. Note that the semantics given in Definition 5.31 is *generic*, and not dependent on the logic.

DEFINITION 5.33 (Standard Translation of Simply Labelled Sequents). The standard translation of unilabelled sequents into formulae is

$$\mathtt{form}\,(\Gamma^x \Rightarrow \Delta^x) =_{def} \bigwedge(\pi_{\mathtt{form}} \circledast \Gamma^x) \supset \bigvee(\pi_{\mathtt{form}} \circledast \Delta^x) \tag{72}$$

This is extended to multilabelled sequents by using the general semantics in Definition 5.31:

$$\mathtt{form}\,\underline{S} =_{def} \bigvee \mathtt{form}\,((\lambda x.\underline{S} \mathbin{/\!/} x) \circledast \mathtt{lab}(\underline{S})) \tag{73}$$

Definition 5.34 (Standard Semantics of Simply Labelled Sequents). Let $\underline{S}$ be a labelled sequent and $\mathfrak{I}$ an interpretation. Then $\mathfrak{I} \models \underline{S}$ iff $\mathfrak{I} \models \mathtt{form}(\underline{S})$.

Remark 5.35 (Conventional Sequent Semantics). Not all labelled sequent calculi have the disjunctive general semantics from Definition 5.31. Some calculi, such as the system for **S5** in [Kan57, §8.3], have a **conventional sequent semantics**, with a translation to formulae that is

$$\mathtt{form}'\,(\underline{\Gamma} \Rightarrow \underline{\Delta}) =_{def} \bigwedge(\pi_{\mathtt{form}} \circledast \underline{\Gamma}) \supset \bigvee(\pi_{\mathtt{form}} \circledast \underline{\Delta})$$

This translation does not take into account that labels correspond to different points in a Kripke model. (In [Kan57], literally, "the marks are extraneous... and we shall associate no meaning with them".) In such systems, the labels function only to control proof search. In the case of **S5**, implication is classical, and the accessibility relation in the underlying Kripke model is *symmetric*, i.e., let $\mathfrak{M} = \langle W, R, v\rangle$ be the Kripke model such that for all points $x, y \in W$, $(x, y) \in R$.

In Chapter 6, these two translations will be shown as equivalent for formulae in **Int***/$\mathtt{Geo}$ using techniques from correspondence theory to translate them into $\mathtt{PSF}$.

Definition 5.36 (Meta-Slice). Let $\underline{\Gamma} \in \mathsf{SLS}^{\mu}$ be a schematic labelled multiset. A **meta-slice** of $\underline{\Gamma}$ is (informally) either

(1) A meta-variable $\underline{\Gamma}' \in \underline{\Gamma}$ denoting arbitrary multilabelled multisets, or
(2) A unilabelled multiset of metavariables and formulae.

This definition is extended to schematic labelled sequents naturally.

Remark 5.37. Unlike slices (Definition 5.22 on page 108), meta-slices may contain a *single* metavariable that denotes arbitrary multilabelled multisets or sequents. For example, the meta-slices of $\underline{\Gamma}', \underline{\Delta}', \Gamma^x, A^x, A^y$ are (1) $\underline{\Gamma}'$, (2) $\underline{\Delta}'$, (3) Γ^x, A^x and (4) A^y.

Definition 5.38 (Trivially Invertible Rule). Let ρ be a labelled rule. Then ρ is **trivially invertible** iff every meta-slice that occurs in the conclusion occurs in every premiss.

Lemma 5.39. *Let* **LGS** *be a labelled sequent calculus with weakening and contraction rules. Then a rule ρ and its trivially invertible form of the rule (Definition 5.38) ρ_ι are interderivable.*

Proof. Straightforward. □

REMARK 5.40. We do not define a canonical trivially invertible form of a rule. A rule such as

$$\frac{(A \supset B)^x, \Gamma^x, \underline{\Gamma'} \Rightarrow \underline{\Delta} \setminus x, A^x \quad B^x, \Gamma^x, \underline{\Gamma'} \Rightarrow \underline{\Delta}}{(A \supset B)^x, \Gamma^x, \underline{\Gamma'} \Rightarrow \underline{\Delta}} \ \mathsf{L}{\supset}'$$

can (for our purposes) have multiple trivially invertible forms, for example, by a trivially invertible, multilabelled left premiss and a right premiss unchanged from the original, by a variants where the right premiss contains $(A \supset B)^x$, or even by having a multilabelled right premiss as well.

We examine some labelled sequent calculi for **Int** from the literature below. In Chapter 7 another labelled calculus for **Int**, called **LG3ipm** (Figure 7.1 on page 157), will be derived from a hypersequent calculus.

5.2.2. System O. Maslov [Mas67, Mas69] introduced a labelled sequent calculus for first-order **Int** with equality, called System **O** (from the Russian word *obratimyī*, for "invertible" [Min91, p. 403]) where formulae are annotated with sets (represented as lists) of natural numbers. (The labelled sequents are called "supersequents".) The principal formulae of the identity axiom are annotated with lists that have a common element, and some of the rules manipulate the labelling lists by adding or removing elements from those lists. Two "reduction rules" are specified for the premisses of rules: (1) formulae annotated with empty sets can be eliminated, and (2) two occurrences of the same formulae annotated by different sets can be simplified into one occurrence of the formula annotated by a union of those sets. Thus the notation can be simplified so that the sequents contain multiple instances of formulae annotated by a single number, which can be treated as a simply labelled sequent calculus. The propositional fragment of **O** is shown in Figure 5.1 on the next page with updated notation.

What is noteworthy about **O** is that there are no structural rules, and all of the rules are invertible, generally by using trivially invertible variants. An unexpected feature of the system is that eigenlabel condition on the $\mathsf{R}\vee$ and $\mathsf{L}\supset$ rules rather than the $\mathsf{R}\supset$ rule.[1] (This appears to come from the Beth semantics for **Int**, see Section 3.3.4 on page 50, although

[1]We are grateful to Alexander Konovalov for assistance in translating portions of the original paper [Mas67], which clarified questions about the eigenlabel conditions on these rules in [Mas69].

$$\dfrac{}{A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x}\ \mathsf{Ax} \qquad \dfrac{}{\bot^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\bot$$

$$\dfrac{A^{x_1}, \ldots, A^{x_n}, B^{x_1}, \ldots, B^{x_n}, \underline{\Gamma} \Rightarrow \underline{\Delta}}{(A \wedge B)^{x_1}, \ldots, (A \wedge B)^{x_n}, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\wedge_{\mathsf{O}} \qquad \dfrac{\underline{\Gamma} \Rightarrow A^x, \underline{\Delta} \quad \underline{\Gamma} \Rightarrow B^x, \underline{\Delta}}{\underline{\Gamma} \Rightarrow A \wedge B^x, \underline{\Delta}}\ \mathsf{R}\wedge$$

$$\dfrac{\underline{\Gamma}, A^x \Rightarrow \underline{\Delta} \quad \underline{\Gamma}, B^x \Rightarrow \underline{\Delta}}{\underline{\Gamma}, A \vee B^x \Rightarrow \underline{\Delta}}\ \mathsf{L}\vee$$

$$\dfrac{\Gamma^{xy}, \underline{\Gamma}' \Rightarrow \underline{\Delta}, (A \vee B)^x, A^y}{\Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}, (A \vee B)^x}\ \mathsf{R}\vee_{\mathsf{O}_1} \qquad \dfrac{\Gamma^{xy}, \underline{\Gamma}' \Rightarrow \underline{\Delta}, (A \vee B)^x, B^y}{\Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}, (A \vee B)^x}\ \mathsf{R}\vee_{\mathsf{O}_2}$$

$$\dfrac{(A \supset B)^{xy}, \Gamma^{xy}, \underline{\Gamma}' \Rightarrow \underline{\Delta}, A^y \quad B^x, (A \supset B)^x, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}}{(A \supset B)^x, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}}\ \mathsf{L}\supset_{\mathsf{O}}$$

where $x\#\underline{\Gamma}'$ and $y\#\underline{\Gamma}', \underline{\Delta}$ in $\mathsf{R}\vee_{\mathsf{O}_1}$, $\mathsf{R}\vee_{\mathsf{O}_2}$ and $\mathsf{L}\supset_{\mathsf{O}}$.

$$\dfrac{A^{x_1}, \ldots, A^{x_n}, \underline{\Gamma} \Rightarrow \underline{\Delta}, B^{x_1}, \ldots, B^{x_n}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^{x_1}, \ldots,, (A \supset B)^{x_n}}\ \mathsf{R}\supset_{\mathsf{O}}$$

FIGURE 5.1. The simply labelled calculus **O**.

they are not explicitly discussed in the article.) The $\mathsf{R}\supset_{\mathsf{O}}$ and $\mathsf{L}\wedge_{\mathsf{O}}$ rules also allow for multiple occurrences of formulae to be analysed in once inference.

PROPOSITION 5.41. *The rules of* **O** *preserve the single-succedent property of their premisses.*

Proof. By inspection of the rules. □

REMARK 5.42. **O** is given as a single-succedent calculus, and indeed, it is required for the invertibility of the $\mathsf{R}\supset_{\mathsf{O}}$ rule. (Without the restriction, the labelled sequent $\Rightarrow A^x, \neg A^x$ would be derivable.)

LEMMA 5.43 (External Weakening). *The rule*

$$\dfrac{\underline{\Gamma}' \Rightarrow \underline{\Delta}'}{\underline{\Gamma}', \Gamma^x \Rightarrow \underline{\Delta}', \Delta^x}\ (\mathsf{EW})$$

where $x\#\underline{\Gamma}'.\underline{\Delta}'$*, is admissible in* **O**.

Proof. See [Mas69]. □

PROPOSITION 5.44. *The unilabelled forms of* $\mathsf{L}\wedge_{\mathsf{O}}$ *and* $\mathsf{R}\supset_{\mathsf{O}}$

$$\frac{\underline{\Gamma}, A^x, B^x \Rightarrow \underline{\Delta}}{\underline{\Gamma}, A \wedge B^x \Rightarrow \underline{\Delta}}\ \mathsf{L}\wedge \qquad \frac{A^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', B^x}{\underline{\Gamma}' \Rightarrow \underline{\Delta}', (A \supset B)^x}\ \mathsf{R}\supset$$

are interderivable with $\mathsf{L}\wedge_{\mathsf{O}}$ *and* $\mathsf{R}\supset_{\mathsf{O}}$ *in* **O**.

Proof. The unilabelled rules are special forms of the multilabelled rules. And the multilabelled rules are derivable by multiple instances of the unilabelled rules. □

LEMMA 5.45. *The rules of* **O** *are invertible.*

Proof. See [Mas69]. The invertibility of the $\mathsf{R}\vee_{\mathsf{O}_1}$, $\mathsf{R}\vee_{\mathsf{O}_2}$ and $\mathsf{L}\supset$ is given as a corollary of Lemma 5.43. □

THEOREM 5.46. *Let* $\underline{S} \in \mathsf{SLS}_1$. $\mathbf{O} \vdash \underline{S}$ *iff there exists* $x \in \mathtt{lab}(\underline{S})$ *such that* $\mathbf{G3i} \vdash \underline{S} \,/\!/\, x$.

Proof. See [Mas69]. The proof is in reference to the **G3** system in [Kle52]. Note that in [Mas69], a slice $\underline{S} \,/\!/\, x$ is called a sequent that is "conjugate" to $\underline{S}$ in label x. □

REMARK 5.47. The single succedent restriction on **O** is important for the soundness of the system. Otherwise the sequent $\Rightarrow A^x, \neg A^x$ would be derivable.

EXAMPLE 5.48. *A sample derivation in* **O***:*

$$\dfrac{\dfrac{\dfrac{\ldots, A^{xy} \Rightarrow A^y, B^x \quad B^x, \ldots, A^x \Rightarrow B^x}{(A \supset B)^x, \ldots, A^x \Rightarrow B^x}\ \mathsf{L}\supset_{\mathsf{O}}}{(A \supset B)^{wx}, \ldots, A^{wx} \Rightarrow B^x, C^w}\ (\mathsf{EW}) \qquad C^w, (A \supset B)^w, \ldots, A^w \Rightarrow C^w}{\dfrac{(A \supset B)^w, (B \supset C)^w, A^w \Rightarrow C^w}{\Rightarrow ((A \supset B) \supset (B \supset C) \Rightarrow (A \supset C))^w}\ \mathsf{R}\supset^+_{\mathsf{O}}}\ \mathsf{L}\supset_{\mathsf{O}}$$

where the principal formula that also occurs in the premisses of instances of $\mathsf{L}\supset_{\mathsf{O}}$ *is denoted by "…" for brevity.*

5.2.3. System FIL (Full Intuitionistic Logic). de Paiva and Pereira [dP05] introduce a labelled calculus for **Int** with simple labels on formulae in the antecedent and sets of simple labels on formulae in the succedent (which corresponds to multiple instances of formulae with simple labels). The motivation is a multisuccedent system with multisuccedent $\mathsf{R}\supset$ rules that track the dependency of formulae.

The original system used natural numbers as labels, which allowed for deterministic choice between two labels in rules with multiple active formulae (such as LC or $\mathsf{L}\wedge$) by using min and max functions. Instead, we arbitrarily choose one of the labels in the rules from the version adapted to the notation for simply labelled sequents in Figure 5.2.

$$\dfrac{}{A^x \Rightarrow A^x}\;\mathsf{Ax} \qquad \dfrac{}{\bot^x \Rightarrow, \Delta^x}\;\mathsf{L}\bot_{\mathsf{F}}$$

$$\dfrac{A^x, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{(A \wedge B)^x, \underline{\Gamma} \Rightarrow [y/x]\underline{\Delta}}\;\mathsf{L}\wedge_{\mathsf{F}} \qquad \dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^x \quad \underline{\Gamma} \Rightarrow \underline{\Delta}, B^y}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^x, (A \wedge B)^y}\;\mathsf{R}\wedge_{\mathsf{F}}$$

where $x \neq y$.

$$\dfrac{A^x, \underline{\Gamma} \Rightarrow \underline{\Delta} \quad B^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}'}{(A \vee B)^z, \underline{\Gamma}, \underline{\Gamma}' \Rightarrow [z/x]\underline{\Delta}, [z/y]\underline{\Delta}'}\;\mathsf{L}\vee_{\mathsf{F}} \qquad \dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, B^y}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \vee B)^x, (A \vee B)^y}\;\mathsf{R}\vee_{\mathsf{F}}$$

where $x \neq y$ and z does not occur in the premisses of $\mathsf{L}\vee_{\mathsf{F}}$.

$$\dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^y \quad B^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}'}{(A \supset B)^x, \underline{\Gamma}, \underline{\Gamma}' \Rightarrow \underline{\Delta}, [y/x]\underline{\Delta} \mathbin{/\!/} x, \underline{\Delta}'}\;\mathsf{L}\supset_{\mathsf{F}} \qquad \dfrac{A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, B^x, B^y}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^x}\;\mathsf{R}\supset_{\mathsf{F}}$$

where $x \neq y$ and y does not occur in the conclusion of $\mathsf{R}\supset_{\mathsf{F}}$.

$$\dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, B_1^{y_1}, \ldots, B_n^{y_n}}{A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, B_1^{y_1}, \ldots, B_n^{y_n}, B_1^x, \ldots, B_n^x}\;\mathsf{LEW} \qquad \dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^x}\;\mathsf{REW}$$

where $x \neq y_i$, x is fresh for the premisses of LW and RW, and $n \geq 1$ for LW.

$$\dfrac{\underline{\Gamma}, A^x, A^y \Rightarrow \underline{\Delta}}{\underline{\Gamma}, A^x \Rightarrow [x/y]\underline{\Delta}}\;\mathsf{LC} \qquad \dfrac{\underline{\Gamma} \Rightarrow A^x, A^x, \underline{\Delta}}{\underline{\Gamma} \Rightarrow A^x, \underline{\Delta}}\;\mathsf{RC}$$

FIGURE 5.2. A variant of the simply labelled calculus **FIL**.

REMARK 5.49. In [dP05], the $\mathsf{R}\wedge_{\mathsf{F}}$ and $\mathsf{L}\vee_{\mathsf{F}}$ rules are given as

$$\dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^{x_1}, \ldots, A^{x_m} \quad \underline{\Gamma} \Rightarrow \underline{\Delta}, A^{y_1}, \ldots, B^{y_n}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^{x_1}, \ldots, (A \wedge B)^{x_m}, (A \wedge B)^{y_1}, \ldots, (A \wedge B)^{y_n}}\;\mathsf{R}\wedge'_{\mathsf{F}}$$

and

$$\dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^{x_1}, \ldots, A^{x_m}, B^{y_1}, \ldots, B^{y_n}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \vee B)^{x_1}, \ldots, (A \vee B)^{x_m}, (A \vee B)^{y_1}, \ldots, (A \vee B)^{y_n}}\;\mathsf{R}\vee'_{\mathsf{F}}$$

These are derivable using the rules given in Figure 5.2.

PROPOSITION 5.50 (**FIL** is a Single-Antecedent Calculus). *Let* **FIL** $\vdash \underline{\Gamma} \Rightarrow \underline{\Delta}$. *For each label* $x \in \mathtt{lab}(\underline{\Gamma}, \underline{\Delta})$, $|\underline{\Gamma} \mathbin{/\!/} x| \leq 1$.

Proof. By induction on the derivation depth. □

The system **FIL** has some interesting properties. The only form of weakening allowed is external weakening. The most unusual feature is that the system can be characterised as a *single-antecedent* calculus (Proposition 5.50 on the preceding page). This seems odd, considering that a motivation for the calculus is the single succedent nature of the $\mathsf{R}{\supset}$ rule for other multisuccedent calculi, such as **m-G3i** (Figure A.3 on page 234). Such a calculus seems to have no obvious correspondence with natural deduction systems.

EXAMPLE 5.51. *A sample derivation in* **FIL***:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^z \Rightarrow A^z}{A^z \Rightarrow A^z, (A\supset C)^x}\,\mathsf{REW} \qquad \dfrac{B^x\Rightarrow B^x}{B^x\Rightarrow B^x, C^y}\,\mathsf{REW}}{(A\supset B)^x, A^z \Rightarrow B^x, C^y, (A\supset C)^x}\,\mathsf{L{\supset}_F} \qquad C^y\Rightarrow C^y}{(A\supset B)^x, ((B\supset C)^y, A^z \Rightarrow C^z, C^y, (A\supset C)^x}\,\mathsf{L{\supset}_F}}{(A\supset B)^x, (B\supset C)^y \Rightarrow (A\supset C)^y, (A\supset C)^x}\,\mathsf{R{\supset}_F}}{(A\supset B)^x \Rightarrow ((B\supset C)\supset (A\supset C))^x}\,\mathsf{R{\supset}_F}}{(A\supset B)^x \Rightarrow ((B\supset C)\supset (A\supset C))^x, ((B\supset C)\supset (A\supset C))^w}\,\mathsf{REW}}{\Rightarrow ((A\supset B)\supset (B\supset C)\Rightarrow (A\supset C))^w}\,\mathsf{R{\supset}_F}
$$

5.3. Simple Relational Labelled Sequent Calculi

5.3.1. Definitions. We examine an extension of labelled sequents, which we call (for brevity), **(simple) relational sequents**, although they are more properly called "simply labelled sequents with relations".[2]

NOTATION 5.52. Because labelled sequents and relational sequents have similar representations, they will be differentiated from labelled sequent rules with a relation symbol as a subscript, e.g. $\mathsf{R}{\supset}_{\leq}$, when rules share the same names or the difference is not clear from the context.

DEFINITION 5.53 (Relational Formulae). Let $\mathtt{Lab}_0$ be a denumerable set of simple labels in $\mathtt{Lab}$, and let $\mathtt{Rel}$ be a denumerable set of relation symbols. Then a **relational formula** is a tuple $\langle R, x_1, \ldots, x_n\rangle$ in $\mathtt{Rel}\times\mathtt{Lab}_0^n$, where $R \in \mathtt{Rel}$ and $x_1, \ldots, x_n \in \mathtt{Lab}_0$. A **binary relational formula** is a tuple $\langle \leq, x, y\rangle$ in $\mathtt{Rel}\times\mathtt{Lab}_0^2$, represented as $x \leq y$.

REMARK 5.54. No assumption is made about the kind of relation that $\leq$ denotes, e.g. whether it is an ordering relation.

[2] We are not referring to the relational sequents (also called "sequents of relations") from [BF99], where inequalities such as $\leq$ or $<$ are used in place of the sequent arrow in calculi for fuzzy logics.

NOTATION 5.55. We generally use the Greek letter Σ as a schematic variable to denote multisets of relational formulae.

REMARK 5.56. In [IK07, PU09], a different notation is used, where the metavariables that denote multisets of relational formulae annotate the sequent arrow rather than residing in the antecedent.

DEFINITION 5.57 (Relational Sequent). Let Σ be a multiset of 0 or more relational formulae, called the **relational context**, and $\underline{\Gamma} \Rightarrow \underline{\Delta}$ be a simply labelled sequent, called the **logical context**. Then $\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$ is a **relational sequent**.

NOTATION 5.58. We use $x \leq y \leq z$ as an abbreviation for $x \leq y, y \leq z$.

NOTATION 5.59. The set of relational sequents is denoted by $\mathtt{RLS}$.

DEFINITION 5.60 (Relational Rules). **Relational rules** are rules in a simple relational sequent calculus with active or principal relational formulae. A relational rule where the active and principal formulae are only relational formulae is called a **pure relational rule**. (In [Neg07, DN10], pure relational rules are called "order rules".)

EXAMPLE 5.61. *The following are relational rules:*

$$\frac{x \leq y, \Sigma; A^y, \underline{\Gamma}_1 \Rightarrow \underline{\Delta}_1, B^y}{\Sigma; \underline{\Gamma}_1 \Rightarrow \underline{\Delta}_1, (A \supset B)^x} \ \mathsf{R}{\supset_{\leq}} \qquad \frac{x \leq y, y \leq z, x \leq z, \Sigma; \underline{S}}{x \leq y, y \leq z, \Sigma; \underline{S}} \ \mathsf{trans}$$

The trans *rule is a pure relational rule.*

DEFINITION 5.62 (Sequent Union). The definition of sequent union is extended for relational sequents:

$$(\Sigma_1; \underline{S}_1) \sqcup (\Sigma_2; \underline{S}_2) =_{def} (\Sigma_1 \cup \Sigma_2); (\underline{S}_1 \sqcup \underline{S}_2)$$

Because the language of relational sequents is an extension of labelled sequents, much of the terminology for the latter is naturally extended to the former naturally.

NOTATION 5.63 (Label Function). The label function is extended for relational formulae:

$$\pi_{\mathtt{lab}}\ x \leq y =_{def} \{x, y\} \tag{74}$$

Then the label set function on relational sequents is defined as

$$\mathtt{lab}(\Sigma; \underline{S}) =_{def} \bigcup (\pi_{\mathtt{lab}} \circledast \Sigma) \cup \mathtt{lab}(\underline{S}) \tag{75}$$

We may abuse the notation and mix relational contexts and labelled multisets together as the arguments of `lab` where the meaning is clear, e.g. $\mathtt{lab}(\Sigma, \underline{\Gamma})$.

Definition 5.64 (Active and Principal Labels). Let ρ be a schematic rule of a relational calculus, and let $\underline{\Gamma}$ be a multiset of the active (resp. principal) formulae, (including the relational formulae). of ρ. Then $\mathtt{lab}(\underline{\Gamma})$ is the set of **active labels** (resp. **principal labels**) of ρ.

Definition 5.65 (Context Labels). Let ρ be a schematic rule of a relational calculus. A metavariable $\underline{\Gamma}$ (resp. Σ) which occurs in a premiss of ρ and in the conclusion of ρ, with no annotations to indicate that the variable has different values between the conclusion and premiss beyond relabelling (i.e. that the variables are equivalent modulo permutation of labels, discussed in Section 5.5 on page 130), is called a **context** of ρ. The formulae in $\underline{\Gamma}$ (resp. Σ) are called **logical context formulae** (resp. **relational context formulae**) of ρ.

Let $\Sigma, \underline{\Delta}$ be a multiset of the active and principal formulae in ρ such that $\mathtt{lab}(\Sigma, \underline{\Delta})$ is the set of active and principal labels. Then $\mathtt{lab}(\underline{\Gamma}) \setminus \mathtt{lab}(\Sigma, \underline{\Delta})$ is the set of **context labels** of ρ.

Definition 5.66 (Single-Labelled and Multilabelled rules). A rule ρ is called **single-labelled** (or **unilabelled**, for short) iff every premiss has only one active label, (including those in the relational context), that is identical to the (only) principal label (also including those in the relational context) in the conclusion. Otherwise ρ is called **multilabelled**.

Example 5.67. *The rule below*

$$\frac{x \preceq y, \Sigma; \underline{\Gamma} \Rightarrow (A \vee B)^x, A^y, \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow A \vee B^x, \underline{\Delta}} \ \mathsf{R}{\vee}_{1\preceq}$$

is a multilabelled rule, with active labels x, y and principal label x.

Definition 5.68 (General Semantics of Relational Sequents). Let $\mathfrak{I} = \langle W, R, v \rangle$ be a Kripke model. A relational context Σ **describes** $\mathfrak{I}$, written $\mathfrak{I} \vDash \Sigma$, iff for all relational formulae $x \preceq y \in \Sigma$, $(x, y) \in R$. A labelled formula A^x is **true** in $\mathfrak{I}$ iff $\mathfrak{I}, x \Vdash A$. (Note that we are assuming a one-to-one correspondence between labels and the names of points in a model.)

A relational sequent $\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$ is **true** in a Kripke model $\mathfrak{I}$, written $\mathfrak{I} \vDash \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$, iff $\mathfrak{I} \vDash \Sigma$ and $\mathfrak{I} \vDash \bigwedge\!\!\!\bigwedge \underline{\Gamma}$ implies $\mathfrak{I} \vDash \bigwedge\!\!\!\bigwedge \underline{\Delta}$.

A relational sequent is **false** in a Kripke model $\Im$, written $\Im \nvDash \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$, iff $\Im \vDash \Sigma$, $\Im \vDash \bigwedge\!\!\!\bigwedge \underline{\Gamma}$ and $\Im \nvDash \bigvee\!\!\!\bigvee \underline{\Delta}$.

Remark 5.69. Unlike hypersequents and labelled sequents, the semantics of relational sequents requires an interpretation of the relational context. A natural means of defining such an interpretation is to assume the interpretation is a Kripke model.

Remark 5.70. There is no standard translation of arbitrary relational sequents to formulae, e.g. it is not obvious what formula the sequent $x \le y; (A \vee B)^x \Rightarrow A^x, B^y$ corresponds to. Methods for translating arbitrary relational sequents into simply labelled sequents are discussed in Chapter 8. In Definition 5.94 on page 123, we suggest extensions of logic based on relational calculus as a means of specifying a set of logical formulae that a calculus is sound and complete for.

5.3.2. Corresponding Graphs. If we suppose $\le$ to be a partial order (which it is for calculi for logics in **Int***/`Geo`), then we can apply the terminology of directed graphs to relational sequents. We introduce the terminology here, adapting graph theoretic terminology from [Die05].

Definition 5.71 (Corresponding Graph). Let $\Sigma; \underline{S}$ be a relational sequent. Let $\mathcal{G} = \langle V, E \rangle$ be the (multi-)digraph where $V = \mathtt{lab}(\Sigma, \underline{S})$ is the set of vertices, and $(x, y) \in E$ iff $x \le y \in \Sigma$ is the set of edges. $\mathcal{G}$ is called the **corresponding graph** of $\Sigma; \underline{S}$.

Remark 5.72. Because the contraction of relational formulae is shown admissible in the calculi given below, we can consider the corresponding graph to be a digraph rather than a multi-digraph without loss of generality.

Definition 5.73 (Connected). Let $\Sigma; \underline{S}$ be a relational sequent. $\Sigma; \underline{S}$ is called **connected** iff the corresponding graph $\mathcal{G}$ is (weakly) connected. A relational sequent is **disconnected** iff it is not connected.

Proposition 5.74. *Let $\Sigma; \underline{S}$ be a connected relational sequent. Then either (1) $\Sigma = \emptyset$ and $\mathtt{lab}(S)$ is a singleton; (2) or $\mathtt{lab}(\underline{S}) \subseteq \mathtt{lab}(\Sigma)$.*

Proof. The first case is trivial. For the second case, suppose $x \in \mathtt{lab}(\underline{S})$ and $x \notin \mathtt{lab}(\Sigma)$. Then the corresponding graph would not be connected. $\square$

Definition 5.75 (Cyclic). Let $\Sigma;\underline{S}$ be a relational sequent. $\Sigma;\underline{S}$ is called **cyclic** if the corresponding graph $\mathcal{G}$ contains one or more cycles—that is, if $\mathcal{G}$ contains a path from a vertex to itself. $\Sigma;\underline{S}$ is called **acyclic** if it is not cyclic.

Notation 5.76. We use $x_1 \leq \ldots \leq x_n \leq x_1$ as an abbreviation for a cycle of labels $x_1, \ldots, x_n$.

Notation 5.77. Let RLS_c be the set of cyclic relational sequents, and RLS_a be the set of acyclic relational sequents, such that $\mathsf{RLS} = \mathsf{RLS}_c + \mathsf{RLS}_a$.

Definition 5.78 (Transitive Closure). Let Σ be a (multi)set of relational formulae. The **transitive closure** Σ^+ of Σ is defined as

(1) if $x \leq y \in \Sigma$, then $x \leq y \in \Sigma^+$;
(2) if $x \leq y, y \leq z \in \Sigma^+$, then $x \leq z \in \Sigma^+$.

Definition 5.79 (Reflexive Transitive Closure). Let Σ be a (multi)set of relational formulae. The **reflexive transitive closure** Σ^* of Σ is defined as

(1) if $x \leq y \in \Sigma^+$, then $x \leq y \in \Sigma^*$ (where Σ^+ is defined in 5.78 above);
(2) if $x \in \mathtt{lab}(\Sigma)$, then $x \leq x \in \Sigma^*$.

Definition 5.80 (Minimal and Maximumal Labels). Let Σ be a (multi)set of relational formulae. A label $x \in \mathtt{lab}(\Sigma)$ is **minimal** (resp. **maximal**) in Σ iff $x \in \mathtt{lab}(\Sigma)$ and for all labels $y \in \mathtt{lab}(\Sigma)$, $y \leq x \notin \Sigma^+$ (resp. $x \leq y \notin \Sigma^+$).

Definition 5.81 (Rooted). Let Σ be a (multi)set of relational formulae. Σ is called **rooted** iff there exists $x \in \mathtt{lab}(\Sigma)$ such that x is minimal and for all labels $y \in \mathtt{lab}(\Sigma)$, $x \leq y \in \Sigma^*$.

Definition 5.82 (Directed). Let Σ be a (multi)set of relational formulae. Σ is called **directed** iff there exists $x \in \mathtt{lab}(\Sigma)$ such that x is maximal and for all labels $y \in \mathtt{lab}(\Sigma)$ such that $y \neq x$, $y \leq x \in \Sigma^+$.

Definition 5.83 (Width). Let Σ be a (multi)set of relational formulae. Let $x, y \in \mathtt{lab}(\Sigma)$ such that neither $x \leq y \in \Sigma^+$ nor $y \leq x \in \Sigma^+$ be called disconnected. Σ has **width** n iff there are at most n disconnected labels in Σ. (Note that Σ as a whole may be connected.)

Definition 5.84 (Linear). Let Σ be a (multi)set of relational formulae. Σ is called **linear** iff for every pair of distinct labels $x, y \in \mathtt{lab}(\Sigma)$, either $x \leq y \in \Sigma^+$ or $y \leq x \in \Sigma^+$.

Definition 5.85 (Symmetric). Let Σ be a (multi)set of relational formulae. Σ is called **symmetric** iff for every pair of distinct labels $x, y \in \mathtt{lab}(\Sigma)$, both $x \leq y \in \Sigma^+$ and $y \leq x \in \Sigma^+$.

Remark 5.86. A relational sequent may be valid in a Kripke frame that is directed, linear, symmetric while the relational context of that sequent is not directed, linear or symmetric.

Terminology 5.87. The graph-theoretic terminology applied to relational sequents can be extended to cover multisets of relational formulae (which can be thought of as the relational contexts of empty sequents) and vice versa.

We examine some relational sequent calculi from the literature below. In Chapter 9 another framework of simple relational sequent calculus for **Int***/Geo, **RG3ipm*** (Figure 9.2 on page 204) will be derived from the derived simply labelled framework.

5.3.3. The Framework G3I* for Int*/Geo. Dyckhoff and Negri [Neg07, DN10] introduce a framework of relational sequent calculi **G3I*** for logics in **Int***/Geo, by using a **G3**-style base calculus that admits weakening, contraction and cut, even with the addition of relational rules that have a particular form (called geometric rules, discussed in Section 3.5.3 on page 60).

The base calculus **G3I** for **Int** is given in Figure 5.3 on the next page (with updated notation), and the extension rules from **G3I*** are given in Figure 5.4 on page 122 for some of the logics in **Int***/Geo. The rules dir (directedness), lin (linearity) and sym (symmetry) are the distinguishing rules for the logics **Jan**, **GD** and **Cl**, respectively. The logic **Sm** can be obtained using the BD_2 (see Remark 5.90) and lin rules, and **GSc** can be obtained using the BD_2 and BTW_2 rules.

Remark 5.88. Note that the name **G3I** should not be confused with the calculus **G3i** from [TS00].

Remark 5.89. The notation for relational sequents in [Neg07, DN10], makes no distinction between the Σ and $\underline{\Gamma}$ context variables in the antecedent.

$$\frac{}{x \leq y, \Sigma; P^x, \underline{\Gamma} \Rightarrow P^y, \underline{\Delta}}\ \mathsf{Ax}_{\leq} \qquad \frac{}{\Sigma; \bot^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\bot$$

where P is atomic.

$$\frac{\Sigma; \underline{\Gamma}, A^x, B^x \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma}, (A \wedge B)^x \Rightarrow \underline{\Delta}}\ \mathsf{L}\wedge \qquad \frac{\Sigma; \underline{\Gamma} \Rightarrow A^x, \underline{\Delta} \quad \Sigma; \underline{\Gamma} \Rightarrow B^x, \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow (A \wedge B)^x, \underline{\Delta}}\ \mathsf{R}\wedge$$

$$\frac{\Sigma; \underline{\Gamma}, A^x \Rightarrow \underline{\Delta} \quad \Sigma; \underline{\Gamma}, B^x \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma}, (A \vee B)^x \Rightarrow \underline{\Delta}}\ \mathsf{L}\vee \qquad \frac{\Sigma; \underline{\Gamma} \Rightarrow A^x, B^x, \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow (A \vee B)^x, \underline{\Delta}}\ \mathsf{R}\vee$$

$$\frac{x \leq y, \Sigma; (A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y \quad x \leq y, \Sigma; (A \supset B)^x, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; (A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}{\supset_{\leq}}$$

$$\frac{x \leq y, \Sigma; A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, B^y}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^x}\ \mathsf{R}{\supset_{\leq}}$$

where y does not occur in the conclusion of $\mathsf{R}{\supset_{\leq}}$.

$$\frac{x \leq x, \Sigma; \underline{S}}{\Sigma; \underline{S}}\ \mathsf{refl} \qquad \frac{x \leq y, y \leq z, x \leq z, \Sigma; \underline{S}}{x \leq y, y \leq z, \Sigma; \underline{S}}\ \mathsf{trans}$$

FIGURE 5.3. The simple relational calculus **G3I**.

REMARK 5.90. From Figure 5.4 on the next page, we see that the sym rule is a special case of the BD_1 rule, and that BD_2 is

$$\frac{y \leq x, x \leq y \leq z, \Sigma; \underline{S} \quad z \leq y, x \leq y \leq z, \Sigma; \underline{S}}{x \leq y \leq z, \Sigma; \underline{S}}\ \mathsf{BD}_2$$

In [DN10], only the rule BD_2 is given. We have extrapolated the rule for BD_k, as well as the rules of BC_k, BW_k and BTW_k from the semantics (Section 3.3 on page 44).

THEOREM 5.91 (Cut Admissibility). *The cut rule*

$$\frac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x \quad \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}\ (\mathsf{cut})$$

is admissible in **G3I***.

Proof. See [DN10]. □

$$\frac{\langle x_{i+1} \leq x_i, x_1 \leq x_2, \ldots, x_{k-1} \leq x_k, \underline{\Gamma} \Rightarrow \underline{\Delta} \rangle_{i=1}^{k-1}}{x_1 \leq x_2, \ldots, x_{k-1} \leq x_k, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}} \; \mathsf{BD_k}$$

$$\frac{\langle x_i \leq x_o, x_0 \leq x_1, \ldots, x_0 \leq x_k, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta} \rangle_{i=1}^{k}}{x_0 \leq x_1, \ldots, x_0 \leq x_k, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}} \; \mathsf{BC_k}$$

$$\frac{\langle x_i \leq x_1, \ldots, x_i \leq x_k, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta} \rangle_{i=1, i \neq k}^{k}}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}} \; \mathsf{BW_k}$$

$$\frac{\langle x_1 \leq z_i, \ldots, x_k \leq z_i, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta} \rangle_{i=1}^{k}}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}} \; \mathsf{BTW_k}$$

where $z_1, \ldots, z_k$ do not occur free in $\underline{\Gamma}, \underline{\Delta}$ for $\mathsf{BTW_k}$.

$$\frac{w \leq x, w \leq y, x \leq z, y \leq z, \Sigma; \underline{S}}{w \leq x, w \leq y, \Sigma; \underline{S}} \; \mathsf{dir}$$

where $z \# \Sigma, \underline{S}$.

$$\frac{x \leq y, \Sigma; \underline{S} \quad y \leq x, \Sigma; \underline{S}}{\Sigma; \underline{S}} \; \mathsf{lin} \qquad \frac{y \leq x, x \leq y, \Sigma; \underline{S}}{x \leq y, \Sigma; \underline{S}} \; \mathsf{sym}$$

FIGURE 5.4. Extension rules for **G3I***.

THEOREM 5.92 (Soundness and Completeness). **G3I** *is sound and complete for* **Int***, that is,* **G3I** $\vdash A$ *iff* $A \in$ **Int**. *Similarly for the various extensions in* **G3I*** *and their corresponding logics in* **Int***/`Geo`.

Proof. See [DN10]. □

REMARK 5.93. Recall that a (propositional) logic $\mathbf{S} \subset$ `Prop` is closed set under modus ponens and substitution of formulae for propositional variables. Formalisms such as sequents or hypersequents are not strictly in the logic **S**, because they contain symbols that are not part of the language of **S**. However, when they have translations into formulae in `Prop`, then one can abuse notation by using the translation to say that the sequent or hypersequent is in **S** if its translation into propositional formulae is in **S**, e.g. $\mathcal{H} \in \mathbf{S}$ iff $(\texttt{form}\,\mathcal{H}) \in \mathbf{S}$.

As noted above in Remark 5.70 on page 118, there is no standard translation for relational sequents into formulae. This makes it problematic when discussing whether a particular sequent is in a logic. The solution is to define an extended logic based on the relational calculi for that logic.

DEFINITION 5.94. We define the following sets of "logics":

$$\Sigma; \underline{S} \in \mathbf{Int}_{\leq} \text{ iff } \mathbf{G3I} \vdash \Sigma; \underline{S} \tag{76}$$

$$\Sigma; \underline{S} \in \mathbf{BD_{k}}_{\leq} \text{ iff } \mathbf{G3I} + \mathsf{BD_k} \vdash \Sigma; \underline{S} \tag{77}$$

$$\Sigma; \underline{S} \in \mathbf{BC_{k}}_{\leq} \text{ iff } \mathbf{G3I} + \mathsf{BC_k} \vdash \Sigma; \underline{S} \tag{78}$$

$$\Sigma; \underline{S} \in \mathbf{BW_{k}}_{\leq} \text{ iff } \mathbf{G3I} + \mathsf{BW_k} \vdash \Sigma; \underline{S} \tag{79}$$

$$\Sigma; \underline{S} \in \mathbf{BTW_{k}}_{\leq} \text{ iff } \mathbf{G3I} + \mathsf{BTW_k} \vdash \Sigma; \underline{S} \tag{80}$$

$$\Sigma; \underline{S} \in \mathbf{G_{k}}_{\leq} \text{ iff } \mathbf{G3I} + \mathsf{lin} + \mathsf{BC_k} \vdash \Sigma; \underline{S} \tag{81}$$

$$\Sigma; \underline{S} \in \mathbf{GSc}_{\leq} \text{ iff } \mathbf{G3I} + \mathsf{BD_2} + \mathsf{BTW_2} \vdash \Sigma; \underline{S} \tag{82}$$

Other "logics" in the framework are defined similarly.

COROLLARY 5.95. *Let* $\mathbf{S}_{\leq}$ *be "logic" extended from* $\mathbf{S}$ *in Definition 5.94. Then for all* $A \in \texttt{Prop}$, $(\Rightarrow A) \in \mathbf{S}_{\leq}$ *if* $A \in \mathbf{S}$.

Proof. By completeness of calculi in $\mathbf{G3I}^*$. □

REMARK 5.96. For Definition 5.94 and Corollary 5.95, note that **Jan**, **GD**, **Sm** and **Cl** are special cases of the above logics.

PROPOSITION 5.97. *The following weakening and contraction rules*

$$\frac{\Sigma; \underline{S}}{x \leq y, \Sigma; \underline{S}} \ (\mathsf{LW} \leq) \qquad \frac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma}, A^x \Rightarrow \underline{\Delta}} \ (\mathsf{LW}) \qquad \frac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow A^x, \underline{\Delta}} \ (\mathsf{RW})$$

$$\frac{x \leq y, x \leq y, \Sigma; \underline{S}}{x \leq y, \Sigma; \underline{S}} \ (\mathsf{LC} \leq) \qquad \frac{\Sigma; \underline{\Gamma}, A^x, A^x \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma}, A^x \Rightarrow \underline{\Delta}} \ (\mathsf{LC}) \qquad \frac{\Sigma; \underline{\Gamma} \Rightarrow A^x, A^x, \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow A^x, \underline{\Delta}} \ (\mathsf{RC})$$

are depth-preserving admissible in $\mathbf{G3I}^*$.

Proof. See [DN10]. □

PROPOSITION 5.98. *The logical rules of* **G3I** *are invertible.*

Proof. See [DN10]. □

LEMMA 5.99 (Label substitution). *The following label substitution rule*

$$\frac{\Sigma; \underline{S}}{[y/x]\Sigma; \underline{S}}\ \text{[y/x]}$$

is admissible in **G3I***.

Proof. By induction on the derivation height. □

LEMMA 5.100 (Folding Rules). *The **folding rules**:*

$$\frac{x \leq y, \Sigma; A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ (\mathsf{L} \leq) \qquad \frac{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ (\mathsf{R} \leq)$$

are admissible in **G3I***.

Proof. By cut. For $\mathsf{L} \leq$:

$$\frac{x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y \quad x \leq y, \Sigma; A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ (\mathsf{cut})$$

For $\mathsf{R} \leq$:

$$\frac{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y \quad x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ (\mathsf{cut})$$

The inverted "unfolding" rules are admissible by weakening. □

An alternative proof that the folding rules are admissible in **G3I** by induction on the derivation depth is given in Lemma G.7 on page 278.

5.3.4. The System L. Pinto and Uustalu [PU09] introduce a similar labelled calculus **L** for bi-intuitionistic logic (**BiInt**), an extension of **Int** with an exclusion operator that is the dual of implication. The intuitionistic fragment is in Figure 5.5 on the next page, with updated notation. This calculus is similar to the systems **G3I** and more similar to **RG3ipm′** (in Figure 9.1 on page 195) and **RG3ipm** (Figure 9.2 on page 204) introduced later as relational extensions of a labelled calculus.

What is noteworthy about **L** compared to **G3I** is that the persistence property of the logic is incorporated into the the folding rules, rather than absorbed into the axiom.

PROPOSITION 5.101. *The following weakening and contraction rules*

$$\frac{\Sigma; \underline{S}}{x \leq y, \Sigma; \underline{S}}\ (\mathsf{LW} \leq) \quad \frac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma}, A^x \Rightarrow \underline{\Delta}}\ (\mathsf{LW}) \quad \frac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow A^x, \underline{\Delta}}\ (\mathsf{RW})$$

$$\frac{}{\Sigma;\underline{\Gamma},A^x \Rightarrow A^x,\underline{\Delta}}\ \mathsf{Ax} \qquad \frac{}{\Sigma;\bot^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\ \mathsf{L}\bot$$

$$\frac{\Sigma;\underline{\Gamma},A^x,B^x\Rightarrow\underline{\Delta}}{\Sigma;\underline{\Gamma},(A\wedge B)^x\Rightarrow\underline{\Delta}}\ \mathsf{L}\wedge \qquad \frac{\Sigma;\underline{\Gamma}\Rightarrow A^x,\underline{\Delta}\quad \Sigma;\underline{\Gamma}\Rightarrow B^x,\underline{\Delta}}{\Sigma;\underline{\Gamma}\Rightarrow(A\wedge B)^x,\underline{\Delta}}\ \mathsf{R}\wedge$$

$$\frac{\Sigma;\underline{\Gamma},A^x\Rightarrow\underline{\Delta}\quad \Sigma;\underline{\Gamma},B^x\Rightarrow\underline{\Delta}}{\Sigma;\underline{\Gamma},(A\vee B)^x\Rightarrow\underline{\Delta}}\ \mathsf{L}\vee \qquad \frac{\Sigma;\underline{\Gamma}\Rightarrow A^x,B^x,\underline{\Delta}}{\Sigma;\underline{\Gamma}\Rightarrow(A\vee B)^x,\underline{\Delta}}\ \mathsf{R}\vee$$

$$\frac{x\leq y,\Sigma;(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^y\quad x\leq y,\Sigma;(A\supset B)^x,B^y,\underline{\Gamma}\Rightarrow\underline{\Delta}}{\Sigma;(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\ \mathsf{L}{\supset_{\leq}}$$

$$\frac{x\leq y,\Sigma;A^y,\underline{\Gamma}\Rightarrow\underline{\Delta},B^y}{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(A\supset B)^x}\ \mathsf{R}{\supset_{\leq}}$$

where y does not occur in the conclusion of $\mathsf{R}{\supset_{\leq}}$.

$$\frac{x\leq x,\Sigma;\underline{S}}{\Sigma;\underline{S}}\ \mathsf{refl} \qquad \frac{x\leq y,y\leq z,x\leq z,\Sigma;\underline{S}}{x\leq y,y\leq z,\Sigma;\underline{S}}\ \mathsf{trans}$$

$$\frac{x\leq y,\Sigma;\underline{\Gamma},A^x,A^y\Rightarrow\underline{\Delta}}{x\leq y,\Sigma;\underline{\Gamma},A^x\Rightarrow\underline{\Delta}}\ \mathsf{L}\leq \qquad \frac{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^x,A^y}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^y}\ \mathsf{R}\leq$$

FIGURE 5.5. The simple relational calculus for the intuitionistic fragment of **L**.

$$\frac{x\leq y,x\leq y,\Sigma;\underline{S}}{x\leq y,\Sigma;\underline{S}}\ (\mathsf{LC}\leq) \qquad \frac{\Sigma;\underline{\Gamma},A^x,A^x\Rightarrow\underline{\Delta}}{\Sigma;\underline{\Gamma},A^x\Rightarrow\underline{\Delta}}\ (\mathsf{LC}) \qquad \frac{\Sigma;\underline{\Gamma}\Rightarrow A^x,A^x,\underline{\Delta}}{\Sigma;\underline{\Gamma}\Rightarrow A^x,\underline{\Delta}}\ (\mathsf{RC})$$

are depth preserving (except for LC *and* RC*) admissible in* **L***.*

Proof. See [PU09]. Note that the contraction rules are derived from $\mathsf{L}\leq$ and $\mathsf{R}\leq$ using refl. □

We note that the dual folding rules make cut admissibility difficult to prove. Suppose a premiss of the cut rule is a conclusion of the $\mathsf{L}\leq$ rule, with the cut formula as the principal formula:

$$\frac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^x \qquad \dfrac{x\leq y,\Sigma';A^x,A^y,\underline{\Gamma'}\Rightarrow\underline{\Delta'}}{x\leq y,\Sigma';A^x,\underline{\Gamma'}\Rightarrow\underline{\Delta'}}\ \mathsf{L}\leq}{x\leq y,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma'}\Rightarrow\underline{\Delta},\underline{\Delta'}}\ (\mathsf{cut}) \tag{83}$$

We can replace this with two cuts

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}\,\textsf{(RW)}}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\,\textsf{R}\le \qquad \dfrac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x \quad x \le y, \Sigma'; A^x, A^y, \underline{\Gamma'} \Rightarrow \underline{\Delta'}}{x \le y, \Sigma, \Sigma'; A^y, \underline{\Gamma}, \underline{\Gamma'} \Rightarrow \underline{\Delta}, \underline{\Delta'}}\,\textsf{(cut)}}{x \le y, x \le y, \Sigma, \Sigma, \Sigma'; \underline{\Gamma}, \underline{\Gamma}, \underline{\Gamma'} \Rightarrow \underline{\Delta}, \underline{\Delta}, \underline{\Delta'}}\,\textsf{(cut)}}{x \le y, \Sigma, \Sigma'; \underline{\Gamma}, \underline{\Gamma}, \underline{\Gamma'} \Rightarrow \underline{\Delta}, \underline{\Delta}, \underline{\Delta'}}\,\textsf{(LC}\le\textsf{)}^+}{x \le y, \Sigma, \Sigma'; \underline{\Gamma}, \underline{\Gamma'} \Rightarrow \underline{\Delta}, \underline{\Delta}, \underline{\Delta'}}\,\textsf{(LC)}^*}{x \le y, \Sigma, \Sigma'; \underline{\Gamma}, \underline{\Gamma'} \Rightarrow \underline{\Delta}, \underline{\Delta'}}\,\textsf{(RC)}^* \tag{84}$$

but it is not clear that the rank is smaller on the lower cut. In [Pin09], a lemma (which we call the "Monotonicity Mid-Sequent Lemma") is used:

Lemma 5.102 (Monotonicity Mid-Sequent Lemma). *Let δ be a proof of $\Sigma; \underline{S}$ in* **L**. *Then there exists a proof δ' of $\Sigma; \underline{S}$ in* **L** *where all instances of the* $\textsf{L}\le$ *and* $\textsf{R}\le$ *rules occur below axioms, or an instance of* $\textsf{L}{\supset_\le}$ *where the principal formula of* $\textsf{L}\le_\le$ *is not the principal formula of* $\textsf{L}{\supset_\le}$.

Proof. Claimed in [Pin09]. □

Using the Monotonicity Mid-Sequent Lemma for (84) above, the left premiss of the lower cut can be assumed to be on an axiom, and so eliminated. We have not formalised the cut elimination procedure from [Pin09], but instead use a variant of multicut (which we call "polycut") for a similar calculus **RG3ipm′** (Figure 9.1 on page 195) in Theorem 9.15 on page 197.

Theorem 5.103 (Interderivability). *Let* $\Sigma; \underline{S} \in \mathrm{RLS}$. $\mathbf{L} \vdash \Sigma; \underline{S}$ *iff* $\mathbf{G3I} \vdash \Sigma; \underline{S}$.

Proof. By induction on the derivation depths.

From left-to-right: the axiom of **L** is derivable by refl, and the folding rules are admissible in **G3I** by Lemma 5.100 on page 124.

From right-to-left: the axiom of **G3I** is derivable in **L** by use of the $\textsf{L}\le$ rule. □

Corollary 5.104. *The system* **L** *is complete for* $\mathbf{Int}_\le$.

Proof. Follows from the Interderivability Theorem above and Definition 5.94 on page 123. □

5.3.5. Other Labelled Calculi. Simpson gives cut-free simple relational calculi for intuitionistic modal logics in [Sim94, §7.2]. Without the modal rules, the calculi are simply labelled forms of **G1ip** with conventional sequent semantics. (The non-modal rules do not have active or principal relational formulae.)

5.3.6. Alternative Notation. Viganò [Vig00] gives a framework for modal logics, where relational rules are handled in separate branches. This is achieved by allowing a single relational formula on the right. (The intention is to separate reasoning about formulae and reasoning about relations.) For example, the modal rules and relational rules (called "Horn sequent rules") from the system **S**(**K**) in Figure 5.6.

$$\frac{\Sigma \Rightarrow x \leq y \quad \Sigma; \Gamma, A^y \Rightarrow \Delta}{\Sigma; \Gamma, \Box A^x \Rightarrow \Delta}\ \mathsf{L}\Box \qquad \frac{x \leq y, \Sigma; \Gamma \Rightarrow A^y, \Delta}{\Sigma; \Gamma \Rightarrow \Box A^x, \Delta}\ \mathsf{R}\Box$$

where y does not occur in the conclusion of $\mathsf{R}\Box$.

$$\frac{}{x \leq y \Rightarrow x \leq y}\ \mathsf{Ax_v} \qquad \frac{}{\Rightarrow x \leq x}\ \mathsf{refl_v} \qquad \frac{}{\Rightarrow x \leq f(x)}\ \mathsf{ser_v}$$

$$\frac{\Sigma \Rightarrow x \leq y}{\Sigma \Rightarrow y \leq x}\ \mathsf{sym_v} \qquad \frac{\Sigma \Rightarrow x \leq y \quad \Sigma \Rightarrow y \leq z}{\Sigma \Rightarrow x \leq z}\ \mathsf{trans_v} \qquad \frac{\Sigma \Rightarrow x \leq y \quad \Sigma \Rightarrow x \leq z}{\Sigma \Rightarrow z \leq y}\ \mathsf{eucl_v}$$

$$\frac{\Sigma \Rightarrow x \leq y \quad \Sigma \Rightarrow x \leq z}{\Sigma \Rightarrow y \leq g(x,y,z)}\ \mathsf{dir_1} \qquad \frac{\Sigma \Rightarrow x \leq y \quad \Sigma \Rightarrow x \leq z}{\Sigma \Rightarrow z \leq g(x,y,z)}\ \mathsf{dir_2}$$

$$\frac{\Sigma \Rightarrow y \leq z}{w \leq x, \Sigma \Rightarrow y \leq z}\ \mathsf{LW}\leq_{\mathsf{v}} \qquad \frac{w \leq x, w \leq x, \Sigma \Rightarrow y \leq z}{w \leq x, \Sigma \Rightarrow y \leq z}\ \mathsf{LC}\leq_{\mathsf{v}}$$

where f, g are Skolem function constants.

FIGURE 5.6. Modal and relational rules from the system **S**(**K**).

Although no calculi for logics in **Int***/Geo are explicitly given, one could apply the Gödel Translation (recall Definition 3.3 on page 43) to derive rules for **Int** in Figure 5.7, using the same relational rules from Figure 5.6.

$$\frac{\Sigma \Rightarrow x \leq y \quad \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y \quad \Sigma; B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{\Sigma; (A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}{\supset}_{\mathsf{v}\leq} \qquad \frac{x \leq y, \Sigma; A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, B^y}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^x}\ \mathsf{R}{\supset}_{\leq}$$

where y does not occur in the conclusion of $\mathsf{R}{\supset}_{\leq}$.

FIGURE 5.7. Intuitionistic rules derived from rules of **S**(**K**).

This notation is an interesting variant from the notation in the other calculi given, and so worth mentioning. Note that the left premisses of the $\mathsf{L}\Box$ and $\mathsf{L}{\supset}_{\mathsf{v}\leq}$ rules are clearly not relational sequents as defined in this chapter. The intention is, recalling terminology from Wansing [Wan94], to present rules which are *separated*, *symmetric* and *explicit* ([Vig00], p. 141). However, the rule implicitly requires that $x \leq y$ occur in Σ, or be derivable using relational rules. This is equivalent to the relational rules given for other calculi above which require the $x \leq y$ to be in the relational context of the premiss and conclusion.

It's not clear that separating relational branches (where relational rules are applied) from logical branches in proofs provides any computational advantages. One must incorporate techniques in root-first proof search when using the $\mathsf{syn_v}$ or $\mathsf{trans_v}$ rules, for instance, to avoid loops or useless rule applications. Although the system **S**(**K**) is said by [Vig00] to be cut-free (no explicit cut admissibility proof is given, only a correspondence with a normalisable natural deduction system, citing [Pot77]), it is not obvious that a form of cut is applicable to relational branches, or if so, what the structure of a cut-admissibility proof for relational branches would be.

5.4. Prefix Calculi

Prefix calculi were introduced by Fitch [Fit66] as tableaux calculi for modal logics, and extended by Fitting in [Fit72, Fit83]. They are similar to relational calculi in that the accessibility relations between labels is made explicit, but differ by extending the language of labels to be strings of simple labels rather where the relation is determined by comparing labels for prefix relations.

Definition 5.105 (Prefix labels). A **prefix label** is a list of 1 or more simple labels in Lab. The metavariable σ will be used to denote prefix labels. $\sigma\sigma'$ denotes a prefix label, with σ as a **prefix**.

Definition 5.106 (Accessibility). Let σ be a prefix label. Generally, σ' is **accessible** from σ, written as $\sigma \leq \sigma'$, if σ is a prefix of σ'—that is, if $\sigma' = \sigma, x_1, \ldots, x_n$ for $n \geq 0$ and $\{x_1, \ldots, x_n\} \subset \mathsf{Lab}$.

A prefix σ' is **immediately accessible** from σ if $\sigma' = \sigma x$.

Remark 5.107. Prefix label notation should not be confused with the compound label notation for simply labelled sequents in Note 5.29 on page 109.

$$\frac{}{A^{\sigma},\underline{\Gamma} \Rightarrow \underline{\Delta}, A^{\sigma'}}\ \mathsf{Ax} \qquad \frac{}{\bot^{\sigma},\underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\bot$$

where $\sigma \leq \sigma'$ in Ax.

$$\frac{\underline{\Gamma}, A^{\sigma}, B^{\sigma} \Rightarrow \underline{\Delta}}{\underline{\Gamma}, A \wedge B^{\sigma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\wedge \qquad \frac{\underline{\Gamma} \Rightarrow A^{\sigma}, \underline{\Delta} \quad \underline{\Gamma} \Rightarrow B^{\sigma}, \underline{\Delta}}{\underline{\Gamma} \Rightarrow A \wedge B^{\sigma}, \underline{\Delta}}\ \mathsf{R}\wedge$$

$$\frac{\underline{\Gamma}, A^{\sigma} \Rightarrow \underline{\Delta} \quad \underline{\Gamma}, B^{\sigma} \Rightarrow \underline{\Delta}}{\underline{\Gamma}, A \vee B^{\sigma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\vee \qquad \frac{\underline{\Gamma} \Rightarrow A^{\sigma}, B^{\sigma}, \underline{\Delta}}{\underline{\Gamma} \Rightarrow A \vee B^{\sigma}, \underline{\Delta}}\ \mathsf{R}\vee$$

$$\frac{(A \supset B)^{\sigma}, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^{\sigma'} \quad B^{\sigma'}, \underline{\Gamma} \Rightarrow \underline{\Delta}}{(A \supset B)^{\sigma}, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\supset \qquad \frac{A^{\sigma x}, \underline{\Gamma} \Rightarrow \underline{\Delta}, B^{\sigma x}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^{\sigma}}\ \mathsf{R}\supset$$

where σ' occurs in $\underline{\Gamma}, \underline{\Delta}$ or σ' is immediately accessible from σ in $\mathsf{L}\supset$, and where $\sigma x \# \underline{\Gamma}, \underline{\Delta}$ in $\mathsf{R}\supset$.

FIGURE 5.8. A prefix calculus for **Int**.

PROPOSITION 5.108 (Partial Order). *The accessibility relation between prefixes is a partial order—that is, it is (1) reflexive, i.e., $\sigma \leq \sigma$; (2) transitive, i.e., $\sigma \leq \sigma''$ if $\sigma \leq \sigma'$ and $\sigma' \leq \sigma''$; and (3) antisymmetric, i.e. $\sigma = \sigma'$ if $\sigma \leq \sigma'$ and $\sigma' \leq \sigma$.*

Proof. Straightforward from the definition of accessibility. □

We are not aware of any published prefix calculi for superintuitionistic logics in **Int***/`Geo`. Fitting [Fit83] suggests translating a prefix calculus for **S4** into **Int**, using the Gödel translation (Definition 3.3 on page 43) of formulae as an exercise for the reader. A prefix calculus for the propositional fragment of **Int** obtained by that method is given in Figure 5.8, adapted for the notation given above. (In [Fit83], strings of natural numbers are used.)

Ishigaki and Kikuchi give a similar prefix system to Figure 5.8, which they call a "tree sequent", for the subintuitionistic predicate logic $\mathbf{K}^{\mathbf{I}}$ (transitive and persistent, but not reflexive) called $\mathbf{TK}^{\mathbf{I}}$ in [IK07]. (They also give various modal extensions which will not be discussed here.)

We note that the notation for prefix calculi is not suitable for expressing relations that correspond to superintuitionistic logics in **Int***/`Geo`, such as linearity or symmetry. Because the accessibility relation is determined by comparing two labels, one cannot express arbitrary relations. This makes translations from relational calculi to prefix calculi to be impossible.

In [Fit83], it is suggested that one can assume such properties hold for arbitrary prefix labels, e.g. for symmetry to assume that the accessibility relation holds between arbitrary prefix labels. For example,

$$\dfrac{\dfrac{\dfrac{}{A^{xy} \Rightarrow A^{x}}\ \mathsf{sym}\ (\mathsf{xy},\mathsf{x})}{\Rightarrow A^{x}, \neg A^{x}}\ \mathsf{R}\neg}{\Rightarrow (A \vee \neg A)^{x}}\ \mathsf{R}\vee \tag{85}$$

It is unclear how rules such as dir that require fresh labels are implemented in prefix calculi.

We note that such fiat rules make the point of using prefix sequents instead of relational sequents unclear. The advantage of absorbing relational formulae into prefix labels is less clear when one allows a set of additional relations beyond what is in the prefixes. This also adds complexity to proof *checking* as well as proof search.

5.5. Equivalence of Labelled Sequents

An equivalence relation for labelled (including relational or prefix) sequents, called here **equivalence modulo permutation of labels**, that is akin to α-equivalence for λ terms, is introduced below. Ideas and notation are borrowed from work on from nominal logic [Pit03]. This equivalence allows us to consider sequents such as $x \leq y, A^{x} \Rightarrow A^{y}$ and $w \leq z, A^{w} \Rightarrow A^{z}$ to belong to the same equivalence class.

Although labels appear to behave like implicitly bound variables, they are not subject to the same restrictions on substitution for logics where label merging (corresponding to instances of internal weakening and external contraction in hypersequents) is admissible. In some cases, labels also behave like free variables in first-order logic. (Indeed, there is a correspondence shown later in Chapter 6.)

Definition 5.109 (Permutation of Labels). A **permutation π of labels** is a one-to-one function (called a **permutation** for short) π of a finite set of labels T onto itself. The application of a permutation to a label $x \in T$ is written as $\pi \cdot x$.

Let π be a permutation A^{x} be a labelled formula. We define the application of a permutation π to a labelled formula as $\pi \cdot A^{x} =_{def} A^{\pi x}$. This is extended naturally to labelled multisets:

$$\pi \cdot \emptyset =_{def} \emptyset$$

$$\pi \cdot (A^x, \underline{\Gamma}) =_{def} (\pi \cdot A^x), (\pi \cdot \underline{\Gamma})$$

and to labelled sequents:

$$\pi \cdot (\underline{\Gamma} \Rightarrow \underline{\Delta}) =_{def} (\pi \cdot \underline{\Gamma}) \Rightarrow (\pi \cdot \underline{\Delta})$$

Let $Rx_1, \ldots, x_n$ be an n-ary relational formula. We define the application of a permutation π to an n-ary relational formula as $\pi \cdot Rx_1, \ldots, x_n =_{def} R(\pi \cdot x_1), \ldots, (\pi \cdot x_n)$.

Clearly, $\pi \cdot (x \leq y) =_{def} (\pi \cdot x) \leq (\pi \cdot y)$.

This is extended naturally to multisets of relational formulae.

Terminology 5.110. When discussing permutations of labels, explicit reference to the set T of labels will be omitted when it is obvious from the context.

Notation 5.111. Permutations will be represented by π (with a possible subscript or prime mark), so that the notation for the application of a permutation, e.g. $\pi \cdot \underline{\Gamma}$, will not be confused with the notation for n instances of schematic variable, e.g. $n \cdot \underline{\Gamma}$.

Proposition 5.112. $\pi \cdot (\underline{\Gamma} \cup \underline{\Gamma}') = (\pi \cdot \underline{\Gamma}) \cup (\pi \cdot \underline{\Gamma}')$.

Proof. By induction on the size of $\underline{\Gamma}'$. The base case is trivial. For the induction step, we assume that $\pi \cdot (\underline{\Gamma} \cup \underline{\Gamma}') = (\pi \cdot \underline{\Gamma}) \cup (\pi \cdot \underline{\Gamma}')$ holds for smaller multisets. Using Definition 5.109,

$$\begin{aligned}
\pi \cdot \underline{\Gamma} \cup \pi \cdot (\underline{\Gamma}', A^x) &= \pi \cdot \underline{\Gamma} \cup (\pi \cdot \underline{\Gamma}', \pi \cdot A^x) \\
&= \pi \cdot \underline{\Gamma} \cup (\pi \cdot \underline{\Gamma}' \cup \pi \cdot A^x) \\
&= (\pi \cdot \underline{\Gamma} \cup \pi \cdot \underline{\Gamma}') \cup \pi \cdot A^x \\
&= \pi \cdot (\underline{\Gamma} \cup \underline{\Gamma}'), \pi \cdot A^x \\
&= \pi \cdot ((\underline{\Gamma} \cup \underline{\Gamma}') \cup \{A^x\}) \\
&= \pi \cdot (\underline{\Gamma} \cup (\underline{\Gamma}' \cup \{A^x\})) = \pi \cdot (\underline{\Gamma} \cup (\underline{\Gamma}', A^x))
\end{aligned}$$

□

COROLLARY 5.113. $\pi \cdot (\underline{S} \sqcup \underline{S}') = (\pi \cdot \underline{S}) \sqcup (\pi \cdot \underline{S}')$.

Proof. From Definition 5.10 on page 106. □

DEFINITION 5.114 (Composition of Permutations). The **composition** of two permutations π and π' is denoted by $\pi \circ \pi'$. We define the application of a composed permutation to a labelled formula as

$$(\pi \circ \pi') \cdot A^x =_{def} \pi \cdot (\pi' \cdot A^x)$$

This definition is extended naturally to labelled multisets and labelled sequents, as well as to relational formulae and multisets of relational formulae.

PROPOSITION 5.115. *The set of permutations on a given set of of labels Y forms a group under composition—that is, composition is closed and associative, each permutation π has an inverse π^{-1}, and that the composition of a permutation with its inverse is equivalent to the identity permutation π_{id}.*

Proof. See [Pit03]. □

REMARK 5.116. In [Pit03], it is shown that permutations can be represented by lists of pairs of names, e.g. $(x\ y)$, such that

$$(x\ y) \cdot z =_{def} \begin{cases} y & \text{if } z = x \\ x & \text{if } z = y \\ z & \text{otherwise} \end{cases}$$

and $\pi(x\ y) \cdot z =_{def} \pi \cdot ((x\ y) \cdot z)$. Composition then corresponds with appending lists of pairs.

DEFINITION 5.117 (π-permutable). Let π be a permutation, and x, y be labels. Then $x \rightarrow_\pi y$ iff $\pi \cdot x = y$.

This definition is extended to labelled multisets and labelled sequents in the obvious way.

PROPOSITION 5.118 (Permutation Substitution). *Let π be a permutation such that $\underline{\Gamma} \rightarrow_\pi \underline{\Delta}$, $x \rightarrow_\pi y$ and $x' \rightarrow_\pi y'$. Then $[x/x']\underline{\Gamma} \rightarrow_\pi [y/y']\underline{\Delta}$.*

Proof. Follows from Definition 5.117 on the preceding page. □

PROPOSITION 5.119. *If* $\underline{\Gamma} \rightarrow_{\pi} \underline{\Delta}$, *then* $\underline{\Delta} \rightarrow_{\pi^{-1}} \underline{\Gamma}$.

Proof. Follows from Proposition 5.115 on the facing page. □

DEFINITION 5.120 (Equivalence modulo permutation). Two multisets of labelled formula, $\underline{\Gamma}, \underline{\Delta}$ are **equivalent modulo permutation**, written as $\underline{\Gamma} \approx \underline{\Delta}$, iff there exists π such that $\underline{\Gamma} \rightarrow_{\pi} \underline{\Delta}$.

Two sequents $\underline{\Gamma}_1 \Rightarrow \underline{\Delta}_1$ and $\underline{\Gamma}_2 \Rightarrow \underline{\Delta}_2$ are **equivalent modulo permutation**, written as $(\underline{\Gamma}_1 \Rightarrow \underline{\Delta}_1) \approx (\underline{\Gamma}_2 \Rightarrow \underline{\Delta}_2)$, iff there exists π such that $\underline{\Gamma}_1 \rightarrow_{\pi} \underline{\Gamma}_2$ and $\underline{\Delta}_1 \rightarrow_{\pi} \underline{\Delta}_2$.

This notation is extended naturally to relational and prefix sequents.

PROPOSITION 5.121. *Let* $\underline{S}_1 \approx \underline{S}_2$, *where* $\underline{S}_1, \underline{S}_2 \# \underline{S}_3$. *Then* $\underline{S}_1 \sqcup \underline{S}_3 \approx \underline{S}_2 \sqcup \underline{S}_3$.

Proof.

$$\begin{aligned}
\exists \pi . \pi \cdot \underline{S}_1 &= \underline{S}_2 \\
\pi \cdot \underline{S}_3 &= \underline{S}_3 \\
\pi \cdot (\underline{S}_1 \sqcup \underline{S}_3) &= (\pi \cdot \underline{S}_1) \sqcup (\pi \cdot \underline{S}_3) \\
&= \underline{S}_2 \sqcup \underline{S}_3
\end{aligned}$$

□

LEMMA 5.122 (Equivalence Relation). $\approx$ *is an equivalence relation on labelled sequents. That is,* $\approx$ *is (a) reflexive, (b) symmetric and (c) transitive.*

Proof. (a) Using the identity permutation. (b) Using the inverse permutation (Proposition 5.119). (c) By composition of permutations (Proposition 5.115 on the facing page). □

PROPOSITION 5.123. *If* $\underline{\Gamma} \approx \underline{\Delta}$, *then* $|\underline{\Gamma}| = |\underline{\Delta}|$.

Proof. From Definition 5.120. □

DEFINITION 5.124 (Subset Modulo Permutation). $\underline{\Gamma} \subsetsim \underline{\Delta}$ iff there exists $\underline{\Gamma}'$ such that $\underline{\Gamma}' \approx \underline{\Gamma}$ and $\underline{\Gamma}' \subseteq \underline{\Delta}$.

LEMMA 5.125 (Preorder). $\subsetsim$ *is a preorder—i.e., it is (a) reflexive, and (b) transitive.*

Proof. Straightforward. □

LEMMA 5.126. *If* $\underline{\Gamma} \underset{\sim}{\subseteq} \underline{\Delta}$ *and* $\underline{\Delta} \underset{\sim}{\subseteq} \underline{\Gamma}$, *then* $\underline{\Gamma} \approx \underline{\Delta}$.

Proof. Suppose $\underline{\Gamma} \underset{\sim}{\subseteq} \underline{\Delta}$ and $\underline{\Delta} \underset{\sim}{\subseteq} \underline{\Gamma}$. By Definition 5.124 on the previous page, there exists $\underline{\Gamma}'$ and $\underline{\Delta}'$ such that $\underline{\Gamma}' \approx \underline{\Gamma}$, $\underline{\Gamma}' \subseteq \underline{\Delta}$, $\underline{\Delta}' \approx \underline{\Delta}$ and $\underline{\Delta}' \subseteq \underline{\Gamma}$. So there exists $\underline{\Gamma}''$ and $\underline{\Delta}''$ such that $\underline{\Gamma}' \cup \underline{\Gamma}'' = \underline{\Delta}$ and $\underline{\Delta}' \cup \underline{\Delta}'' = \underline{\Gamma}$. Then $\underline{\Gamma}' \approx \underline{\Delta}' \cup \underline{\Delta}''$ and $\underline{\Delta}' \approx \underline{\Gamma}' \cup \underline{\Gamma}''$. From Proposition 5.123 on the preceding page, $\underline{\Gamma}'' = \underline{\Delta}'' = \emptyset$. So $\underline{\Gamma}' \approx \underline{\Delta}'$, which means $\underline{\Gamma} \approx \underline{\Delta}$. □

REMARK 5.127. M. J. Gabbay has suggested [Gab10] that there may be a connection between the formal properties of labels, particularly with respect to equivalence modulo permutation, and the fresh name quantifier И from [GP01] or the ∇ quantifier from [MT02]. This is an area for future investigation.

5.6. Conclusion

In this chapter we have introduced the notation and terminology of simply labelled and relational calculi, along with example calculi for logics in **Int***/Geo. We have also superficially examined a variant of relational calculi from [Vig00] that restricts relation rules to separate branches of the proof, as well as prefix calculi, which absorb relations into the labels themselves. This is only a superficial survey of the kinds of labelled calculi discussed in this thesis. We do not have a theory to describe the relative strength of various kinds of labelled sequent calculi, which we consider a topic for a separate thesis in itself.

We also introduced the notation of a relational logic by defining a logic in terms of relational sequents that are derived by a particular calculus, e.g. $\mathbf{Int}_{\leq}$ as the set of all relational sequents derivable by **G3I**. This notion is "good enough" to discuss whether a relational sequent calculus for a particular logic such as **Int** is *complete* for the relational logic—that is, can it derive sequents such as $x \leq y; (A \vee B)^x \Rightarrow A^y, B^x$. An alternative method of defining a logic such as $\mathbf{Int}_{\leq}$ by extending the language of **Int** to have labelled formulae, and to incorporate relational formulae as atomic formulae. A corresponding Hilbert system for $\mathbf{Int}_{\leq}$ would be defined by using labelled forms of the axioms for **Int** and adding corresponding axioms for the persistence property (i.e., $x \leq y \wedge A^x \supset A^y$), reflexivity and transitivity. It is not clear that such an axiomatisation would be advantageous over the simpler method of defining $\mathbf{Int}_{\leq}$ with respect to a calculus that is known to be sound and complete.

We have also introduced a notion of equivalence modulo permutation of labels, which is akin to α-equivalence for bound variables. While this is an unsurprising result, we believe it to be novel, and to suggest a more general notion of equivalence of variables that includes unbound variables. (As will be shown in Chapters 6 and 8, there is a correspondence between labels and unbound first-order variables when translating sequents into first-order formulae corresponding to the truth conditions on intermediate Kripke frames.) Equivalence modulo permutation will be useful for showing the correspondence between hypersequents and simply labelled sequents in the next chapter.

CHAPTER 6

The Relationship between Hypersequents and Labelled Sequents

6.1. Overview

Hypersequent and some simply labelled sequent calculi have the same disjunctive general semantics, where components correspond to labels, so an equivalence between these two formalisms is not surprising. There are simply labelled sequent calculi that have a conventional sequent semantics. It turns out that disjunctive and conventional general semantics are classically equivalent, but not constructively equivalent. We use techniques from correspondence theory to translate both structures into first-order formulae (introduced in Chapter 3) that correspond to the structure's truth condition on Kripke frames, and show that the first-order formulae resulting from the translations based on disjunctive and conventional semantics are equivalent.

This chapter is organised as follows: in Section 6.2 we define translation functions between hypersequents and sequents of partially shielded formulae (PSF), and in Section 6.3 we extend these to translation functions between simply labelled sequents and sequents of PSF.

We show in Section 6.4 that the translation of formulae to PSF is correct (that is, a formula is true in a model iff the translation of that formula is true in the model). We then introduce a novel proof to show that if a hypersequent is derivable in **HG3ipm***, then its translation to a sequent of PSF is derivable in **G3c/PSF***.

Because the translations have the same structure, we can combine these translations into translation functions between hypersequents and simply labelled sequents in Section 6.5. We also define an alternative translation function based on the conventional semantics of simply labelled sequents, and use the translation to sequents of PSF and the calculus **G3c/PSF*** to show that they are interderivable. We then adapt this translation to a simpler translations of hypersequents and simply labelled sequents to sequents of PSF.

TERMINOLOGY 6.1. As with previous chapters, for brevity, we will use the term "labelled multiset" (or sequent or calculus) to refer to simply labelled multiset (or sequent or calculus).

6.2. Translation of Hypersequents into Partially Shielded Formulae

Here we introduce functions for translating hypersequents and labelled sequents into PSF, and show that the translations are equivalent. (Recall the definitions of SPSF and PSF Section 3.5.)

DEFINITION 6.2 (Unary Bottom and Top). Let $Q \in \mathtt{Pred}_1$ be a fixed unary predicate symbol. Then we define the following unary predicates:

$$\hat{\bot} x =_{def} Qx \wedge \bot \tag{86}$$

$$\hat{\top} x =_{def} Qx \vee \top \tag{87}$$

Clearly $\hat{\bot} x$ and $\hat{\top} x$ are in PSF.

Recall the notation for numeric subscripts of conjunction and disjunction given in Note 2.29 on page 28.

REMARK 6.3. The semantics of PSF formulae with respect to Kripke frames from Definition 3.41 on page 55 can be extended to cover the above notation and predicates:

(1) $\mathfrak{M} \vDash A \wedge_i B$ where $i \in \mathbb{N}$ iff $\mathfrak{M} \vDash A \wedge B$;
(2) $\mathfrak{M} \vDash A \vee_i B$ where $i \in \mathbb{N}$ iff $\mathfrak{M} \vDash A \vee B$;
(3) $\mathfrak{M} \nvDash \hat{\bot} x$;
(4) $\mathfrak{M} \vDash \hat{\top} x$.

We acknowledge that formulae with annotated connectives such as $A \vee_i B$ are not in PSF, but we consider this only a minor abuse of notation.

The validity of a formula in PSF (Definition 3.42 on page 56) is extended similarly.

PROPOSITION 6.4. *There is a bijection between* Var *and* $\mathtt{Pred}_1$.

Proof. By sets are denumerable. □

Definition 6.5. We define the following subset of $\mathtt{SPSF}$:

$$\mathtt{SPSF}_x =_{def} \left\{ A \in \mathtt{SPSF} \ : \ |FV(A)| = 1 \right\} \tag{88}$$

Definition 6.6. We inductively define the translation function $\mathtt{psf}$ from $\mathtt{Term}_0 \times \mathtt{Prop}$ to $\mathtt{SPSF}_x$:

$$\mathtt{psf}\ x\ A =_{def} \begin{cases} \hat{P}x & \text{if } A = P \in \mathtt{Var}, \\ \hat{\bot}x & A = \bot, \\ (\mathtt{psf}\ x\ B) \wedge_0 (\mathtt{psf}\ x\ C) & A = B \wedge C, \\ (\mathtt{psf}\ x\ B) \vee_0 (\mathtt{psf}\ x\ C) & A = B \vee C, \\ \forall y.(\mathcal{R}xy \wedge (\mathtt{psf}\ y\ B)) \supset (\mathtt{psf}\ y\ C) & A = B \supset C, \end{cases}$$

where $\hat{P}$ is chosen from an infinite supply of possible symbols in $\mathtt{Pred}_1$ so that $\hat{P} \neq \mathcal{Q}$ (from Definition 6.2 on the facing page), and y is chosen from an infinite supply of possible variables in $\mathtt{Term}_0$ so that it does not occur bound in either $(\mathtt{psf}\ y\ B)$ or $(\mathtt{psf}\ y\ C)$.

Remark 6.7. $\bot$ is translated to $\hat{\bot}x$ so that the inverse function (Definition 6.17 on page 141) can recover the component or label that corresponds to the parameter x.

Example 6.8. *Let* $A = (B \wedge C) \supset (B \vee D)$*. Then* $\mathtt{psf}\ x\ A$ *is*

$$\forall y.(\mathcal{R}xy \wedge (\hat{B}y \wedge \hat{C}y)) \supset (\hat{B}y \vee \hat{D}y)$$

Proposition 6.9. *Let* $A \in \mathtt{Prop}$*. Then* $FV(\mathtt{psf}\ x\ A) = \{x\}$*.*

Proof. By induction on the structure of A. □

We use the definition of $\mathtt{psf}$ to define a translation of sequents and hypersequents into $\mathtt{PSF}_{\bar{x}}$, below:

NOTATION 6.10 (List Function). Let $\Gamma \in \mathtt{T}^*$ be a multiset of elements $\alpha_1, \ldots, \alpha_n$. Then we define (as a notational convenience) the function from multisets to list

$$\mathtt{list}_{i=1}^{n}\ \Gamma =_{def} \alpha_1, \ldots, \alpha_n$$

that preserves the order of representation of the multiset (as per Remark 2.23 on page 26) in the list.

DEFINITION 6.11. We define the translation function $\mathtt{psf}_{\Rightarrow}$ from $\mathtt{Term}_0 \times \mathtt{Seq}$ to $\mathtt{SPSF}_x$, using functions from $\mathtt{Term}_0 \times \mathtt{Prop}^*$ to $\mathtt{SPSF}_x$:

$$\mathtt{psf}_{\wedge}\ x\,\Gamma =_{def} \begin{cases} \hat{\top} x & \text{if } \Gamma = \emptyset \\ \bigwedge\!\!\!\bigwedge_1 (\mathtt{psf}\ x) \circledast\ \Gamma & \text{otherwise} \end{cases} \tag{89}$$

$$\mathtt{psf}_{\vee}\ x\,\Delta =_{def} \begin{cases} Qx \wedge_1 \bot & \text{if } \Delta = \emptyset \\ \bigwedge\!\!\!\bigwedge_1 (\mathtt{psf}\ x) \circledast\ \Delta & \text{otherwise} \end{cases} \tag{90}$$

$$\mathtt{psf}_{\Rightarrow}\ x\ (\Gamma \Rightarrow \Delta) =_{def} \forall y.(\mathcal{R}xy \wedge (\mathtt{psf}_{\wedge}\ y\ \Gamma)) \supset (\mathtt{psf}_{\vee}\ y\ \Delta) \tag{91}$$

where y is chosen from an infinite supply of possible variables in $\mathtt{Term}_0$ so that it does not occur bound in either ($\mathtt{psf}_{\wedge}\ y\ \Gamma$) or ($\mathtt{psf}_{\vee}\ y\ \Delta$).

DEFINITION 6.12. We define the following subset of $\mathtt{PSF}$:

$$\mathtt{PSF}_{\bar{x}} =_{def} A_1 x_1 \vee_2 \ldots \vee_2 A_n x_n \tag{92}$$

where $n \geq 1$ and all x_i are distinct and all $A_i x_i \in \mathtt{SPSF}_x$. (Recall the notation for subscripts given in Note 2.29 on page 28.)

REMARK 6.13. Note that a formula in $\mathtt{PSF}_{\bar{x}}$ is not necessarily in $\mathtt{SPSF}_x$, as it may have more than one free variable.

DEFINITION 6.14. We define the translation function $\mathtt{psf}_{|}$ from $\mathtt{Term}_0^+ \times \mathtt{Seq}^+$ to $\mathtt{PSF}_{\bar{x}}$:

$$\mathtt{psf}_{|}\ \bar{x}\ (S_1 \mid \ldots \mid S_n) =_{def} (\mathtt{psf}_{\Rightarrow}\ x_1\ S_1) \vee_2 \ldots \vee_2 (\mathtt{psf}_{\Rightarrow}\ x_n\ S_n) \tag{93}$$

where $\bar{x} = x_1, \ldots, x_n$ is a *list* of *distinct* variables. (From Remark 2.23 on page 26, the order of variables in $\bar{x}$ is determined by the order of representation of the hypersequent $S_1 \mid \ldots \mid S_n$.)

For brevity, we may represent the definition of $\mathtt{psf}_{\mid}$ as

$$\mathtt{psf}_{\mid}\ \bar{x}\ (S_1 \mid \ldots \mid S_n) =_{def} \mathbb{V}2_{i=1}^{n}(\mathtt{psf}_{\Rightarrow}\ x_i\ S_i) \tag{94}$$

is proofs.

Example 6.15. *Let $\mathcal{H} = A, C \Rightarrow B \mid B \Rightarrow A \mid \Rightarrow C$ be a hypersequent. Then is*

$$\begin{aligned} \mathtt{psf}_{\mid}\ \mathcal{H} =& (\forall y_1.((\mathcal{R}x_1y_1 \wedge \hat{A}y_1 \wedge_1 \hat{C}y_1) \supset \hat{B}y_1)) \vee_2 \\ & (\forall y_2.((\mathcal{R}x_2y_2 \wedge \hat{B}y_2) \supset \hat{A}y_2)) \vee_2 \\ & (\forall y_3.((\mathcal{R}x_3y_3 \wedge \hat{\top}y_3) \supset \hat{C}y_3)) \end{aligned}$$

Remark 6.16. For simplicity, the standard translation of hypersequents into formulae (Definition 4.12 on page 70) could be used to translate hypersequents into their corresponding first-order formulae for intuitionistic Kripke models in $\mathcal{K}_{\mathbf{Int}}$:

$$\mathtt{psf}'_{\mid}\ \mathcal{H} =_{def} \mathtt{psf}\ \hat{x}\ (\mathtt{form}\ \mathcal{H})$$

However, the distinctions between components is lost in $\mathtt{psf}'_{\mid}$. There is no difference between the translation of $\Rightarrow A$ and $\top \Rightarrow A$. Logically (and semantically), the two translations are equivalent, but in order to preserve the exact syntax of the hypersequent in the translation (particularly if the inverse translation function defined later in Definition 6.17 will be used as an intermediary for converting between formalisms), explicit functions will need to be defined for translating antecedents and succedents.

DEFINITION 6.17. We define the inverse function $\mathtt{psf}^{-1}$ from $\mathtt{SPSF}_x$ to $\mathtt{Term}_0 \times \mathtt{Prop}$:

$$\mathtt{psf}^{-1}\ Ax =_{def} \begin{cases} \langle x, P\rangle & \text{if } A\hat{x} = \hat{P}x \\ \langle x, \bot\rangle & A\hat{x} = \hat{\bot}x \\ \langle x, (\pi_2\ \mathtt{psf}^{-1}\ Bx) \wedge (\pi_2\ \mathtt{psf}^{-1}\ Cx)\rangle & Ax = Bx \wedge_0 Cx \\ \langle x, (\pi_2\ \mathtt{psf}^{-1}\ Bx) \vee (\pi_2\ \mathtt{psf}^{-1}\ Cx)\rangle & Ax = Bx \vee_0 Cx \\ \langle x, (\pi_2\ \mathtt{psf}^{-1}\ By) \supset (\pi_2\ \mathtt{psf}^{-1}\ Cy)\rangle & Ax = \forall y.(\mathcal{R}xy \wedge By) \supset Cy \end{cases} \tag{95}$$

Below we show that $\mathtt{psf}^{-1}$ is the inverse of $\mathtt{psf}$:

LEMMA 6.18. *Let* $x \in \mathtt{Term}_0$ *and* $A \in \mathtt{Prop}$. *Then* $\mathtt{psf}^{-1}\ (\mathtt{psf}\ x\ A) = \langle x,\ A\rangle$.

Proof. By induction on the structure of A. (The proof is written out in Lemma D.1 on page 257.) □

COROLLARY 6.19. *Let* $x \in \mathtt{Term}_0$. *Then* $(\pi_2\ \circ \mathtt{psf}^{-1}) \circ (\mathtt{psf}\ x) = id$.

Proof. Follows from Lemma 6.18. □

DEFINITION 6.20. We define the inverse function $\mathtt{psf}_{\Rightarrow}^{-1}$ from $\mathtt{SPSF}_x$ to $\mathtt{Term}_0 \times \mathtt{Seq}$ using functions from $\mathtt{SPSF}_x$ to $\mathtt{Term}_0 \times \mathtt{Prop}^*$.

$$\mathtt{psf}_{\wedge}^{-1}\ Ax =_{def} \begin{cases} \langle x,\ \emptyset\rangle & \text{if } Ax = \hat{\top}x \\ \langle x,\ \pi_2 \circledast \mathtt{psf}^{-1} \circledast (\bigwedge\!\!\!\!\bigwedge_1^{-1}\ Ax)\rangle & Ax = B_1x \wedge_1 \ldots \wedge_1 B_nx \end{cases} \tag{96}$$

$$\mathtt{psf}_{\vee}^{-1}\ Ax =_{def} \begin{cases} \langle x,\ \emptyset\rangle & \text{if } Ax = \hat{\bot}x \\ \langle x,\ \pi_2 \circledast \mathtt{psf}^{-1} \circledast (\bigvee\!\!\!\!\bigvee_1^{-1}\ Ax)\rangle & Ax = B_1x \vee_1 \ldots \vee_1 B_nx \end{cases} \tag{97}$$

$$\mathtt{psf}_{\Rightarrow}^{-1}\ (\mathcal{R}xy \wedge By) \supset Cy =_{def} \langle x,\ (\pi_2\ \mathtt{psf}_{\wedge}^{-1}\ By) \Rightarrow (\pi_2\ \mathtt{psf}_{\vee}^{-1}\ Cy)\rangle \tag{98}$$

DEFINITION 6.21. Let $A\bar{x} - B_1x_1 \vee_2 \ldots \vee_2 B_nx_n$. We define an inverse function $\mathtt{psf}_{|}^{-1}$ from $\mathtt{PSF}_{\bar{x}}$ to $\mathtt{Term}^+ \times \mathtt{Seq}^+$.

$$\begin{aligned} \mathtt{psf}_{|}^{-1}\ B_1x_1 \vee_2 \ldots \vee_2 B_nx_n =_{def} \langle\ & (\pi_1\ \mathtt{psf}_{\Rightarrow}^{-1}\ B_1x_1), \ldots, (\pi_1\ \mathtt{psf}_{\Rightarrow}^{-1}\ B_nx_n), \\ & (\pi_2\ \mathtt{psf}_{\Rightarrow}^{-1}\ B_1x_1) \mid \ldots \mid (\pi_2\ \mathtt{psf}_{\Rightarrow}^{-1}\ B_nx_n)\ \rangle \end{aligned} \tag{99}$$

For notational convenience, we may represent the definition of $\mathtt{psf}_{|}^{-1}$ as

$$\mathtt{psf}_{|}^{-1}\ \mathbb{W}2_{i=1}^{n} B_i x_i =_{def} \langle \mathtt{list}_{i=1}^{n}(\pi_1\ \mathtt{psf}_{\Rightarrow}^{-1}\ B_i x_i),\ |_{i=1}^{n}(\pi_2\ \mathtt{psf}_{\Rightarrow}^{-1}\ B_i x_i)\rangle \tag{100}$$

We now show that the inverse translations are inverse functions of the corresponding translation functions:

LEMMA 6.22. *Let* $x \in \mathtt{Term}_0$ *and* $\Gamma \in \mathtt{Prop}^*$. *Then* $\mathtt{psf}_{\wedge}^{-1}\ \mathtt{psf}_{\wedge}\ x\, \Gamma = \langle x,\ \Gamma\rangle$.

Proof. Straightforward. (The proof is written out in Lemma D.2 on page 257.) □

LEMMA 6.23. *Let* $x \in \mathtt{Term}_0$ *and* $\Delta \in \mathtt{Prop}^*$. *Then* $\mathtt{psf}_{\vee}^{-1}\ \mathtt{psf}_{\vee}\ x\, \Delta = \langle x,\ \Delta\rangle$.

Proof. Similar to Lemma 6.22. □

LEMMA 6.24. *Let* $x \in \mathtt{Term}_0$ *and* $S \in \mathtt{Seq}$. *Then* $\mathtt{psf}_{\Rightarrow}^{-1}\ (\mathtt{psf}_{\Rightarrow}\ x\, S) = \langle x,\ S\rangle$.

Proof. Let $S = (\Gamma \Rightarrow \Delta)$. Then

$$\begin{aligned}
\mathtt{psf}_{\Rightarrow}^{-1}\ (\mathtt{psf}_{\Rightarrow}\ x\, (\Gamma \Rightarrow \Delta)) &= \mathtt{psf}_{\Rightarrow}^{-1}\ \forall y.(\mathcal{R}xy \wedge (\mathtt{psf}_{\wedge}\ y\, \Gamma)) \supset (\mathtt{psf}_{\vee}\ y\, \Delta)\\
&= \langle x,\ (\pi_2\ \mathtt{psf}_{\wedge}^{-1}\ (\mathtt{psf}_{\wedge}\ y\, \Gamma)) \Rightarrow (\pi_2\ \mathtt{psf}_{\vee}^{-1}\ (\mathtt{psf}_{\vee}\ y\, \Delta))\rangle\\
&= \langle x,\ (\pi_2\ \langle y,\ \Gamma\rangle) \Rightarrow (\pi_2\ \langle y,\ \Delta\rangle)\rangle\\
&= \langle x,\ (\Gamma \Rightarrow \Delta)\rangle
\end{aligned}$$

□

LEMMA 6.25. *Let* $\bar{x} \in \mathtt{Term}_0^+$ *and* $\mathcal{H} \in \mathtt{Seq}^+$. *Then* $\mathtt{psf}_{|}^{-1}\ (\mathtt{psf}_{|}\ \bar{x}\, \mathcal{H}) = \langle \bar{x},\ \mathcal{H}\rangle$.

Proof. Let $\bar{x} = x_1, \ldots, x_n$ and $\mathcal{H} = (S_1 \mid \ldots \mid S_n)$. Then

$$\begin{aligned}
\mathtt{psf}_{|}^{-1}\ \mathtt{psf}_{|}\ \mathcal{H} &= \mathtt{psf}_{|}^{-1}\ \mathbb{W}2_{i=1}^{n}(\mathtt{psf}_{\Rightarrow}\ x_i\, S_i)\\
&= \langle \mathtt{list}_{i=1}^{n}(\pi_1\ \mathtt{psf}_{\Rightarrow}^{-1}\ (\mathtt{psf}_{\Rightarrow}\ x_i\, S_i)),\ |_{i=1}^{n}(\pi_2\ \mathtt{psf}_{\Rightarrow}^{-1}\ (\mathtt{psf}_{\Rightarrow}\ x_i\, S_i))\rangle\\
&= \langle \mathtt{list}_{i=1}^{n}(\pi_1\ \langle x_i,\ S_i\rangle),\ |_{i=1}^{n}(\pi_2\ \langle x_i,\ S_i\rangle)\rangle\\
&= \langle \bar{x},\ \mathcal{H}\rangle
\end{aligned}$$

□

6.3. Translation of Labelled Sequents into Partially Shielded Formulae

These translations can be easily extended to labelled sequents, as defined below:

NOTATION 6.26. If x is a simple label, we use $\hat{x}$ to denote the corresponding first-order parameter.

PROPOSITION 6.27. *There is a bijection between the set* $\mathtt{Lab}$ *of simple labels and the set* $\mathtt{Term}_0$ *of variables.*

Proof. Both sets are denumerable. □

The bijection between simple labels and variables (Proposition 6.27) cab be used to define translation functions between labelled formulae (or sequents) and $\mathtt{PSF}_{\bar{x}}$.

DEFINITION 6.28. We define a translation function from $\mathtt{Prop} \times \mathtt{Lab}$ to $\mathtt{SPSF}_x$:

$$\mathtt{psf'}_{\dagger}\ A^x =_{def} \mathtt{psf}\ \hat{x}\ A \tag{101}$$

DEFINITION 6.29. We similar define translation functions from $\mathtt{SLS} \mathbin{/\!/} x$ to $\mathtt{SPSF}_x$ and from $\mathtt{SLS}$ to $\mathtt{PSF}_{\bar{x}}$:

$$\mathtt{psf}_{\dagger\Rightarrow}\ (\Gamma^x \Rightarrow \Delta^x) =_{def} \mathtt{psf}_{\Rightarrow}\ \hat{x}\ ((\pi_{\mathtt{form}} \circledast \Gamma^x) \Rightarrow (\pi_{\mathtt{form}} \circledast \Delta^x)) \tag{102}$$

$$\mathtt{psf}_{\dagger}\ \underline{S} =_{def} \mathbb{W}_2\ (\mathtt{psf}_{\dagger\Rightarrow} \circledast (\lambda x.\underline{S} \mathbin{/\!/} x) \circledast \mathtt{lab}(\underline{S})) \tag{103}$$

REMARK 6.30. We presume that the label x is known for for the $\mathtt{psf}_{\dagger\Rightarrow}$ function when either $\Gamma^x = \emptyset$ or $\Delta^x = \emptyset$. We discuss the use of a special labelled symbol to represent empty slices in Appendix E on page 263.

REMARK 6.31. Note that these functions do not require a pair of arguments, because the variable is determined by the label. Also note that equivalence modulo permutation of labels (Section 5.5 on page 130) corresponds to a kind of alpha equivalence on free variables. By binding free variables,

$$\mathtt{psf}_{\forall\dagger}\ \underline{S} =_{def} \forall \bar{x}.(\mathtt{psf}_{\dagger}\ \underline{S}) \tag{104}$$

where $\bar{x} = \mathtt{FV}(\mathtt{psf}_{\dagger}\ \underline{S})$, we can give an alternative definition of equivalence modulo permutation of labels for labelled sequents:

$$\underline{S}_1 \approx \underline{S}_2 \Leftrightarrow (\mathtt{psf}_{\forall\dagger}\ \underline{S}_1) =_{\alpha} (\mathtt{psf}_{\forall\dagger}\ \underline{S}_2) \tag{105}$$

where $E_1 =_{\alpha} E_2$ denotes that the expressions E_1 and E_2 are α-equivalent.

Definition 6.32. We define the **label application function** $\mathtt{aplab}$ from $\mathtt{Term}_0, \times\mathtt{Prop}$ to $\mathtt{Prop} \times \mathtt{Lab}$, and extend it naturally for sequents:

$$\mathtt{aplab}\ \hat{x}\ A =_{def} A^x \tag{106}$$

$$\mathtt{aplab}_{\Rightarrow}\ \hat{x}\ (\Gamma \Rightarrow \Delta) =_{def} ((\mathtt{aplab}\ \hat{x})\ \circledast\ \Gamma) \Rightarrow ((\mathtt{aplab}\ \hat{x})\ \circledast\ \Delta) \tag{107}$$

Definition 6.33. We define the inverse function $\mathtt{psf}'^{-1}_{\dagger}$ from $\mathtt{SPSF}_x$ to $\mathtt{Prop} \times \mathtt{Lab}$:

$$\mathtt{psf}'^{-1}_{\dagger} A\hat{x} =_{def} \mathtt{aplab}\ (\mathtt{psf}^{-1}\ A\hat{x}) \tag{108}$$

Lemma 6.34. $\mathtt{psf}'^{-1}_{\dagger}\mathtt{psf}'_{\dagger}A^x = A^x$.

Proof. Straightforward. (The proof is written out in Lemma D.3 on page 258.) □

We use the label application functions to define inverse functions:

Definition 6.35. We give the inverse function $\mathtt{psf}^{-1}_{\dagger\Rightarrow}$ from $\mathtt{SPSF}_x$ to $\mathtt{SLS} /\!\!/ x$, and the inverse function from $\mathtt{PSF}_{\bar{x}}$ to SLS:

$$\mathtt{psf}^{-1}_{\dagger\Rightarrow} A\hat{x} =_{def} \mathtt{aplab}_{\Rightarrow}\ (\mathtt{psf}^{-1}_{\Rightarrow}\ A\hat{x}) \tag{109}$$

$$\mathtt{psf}^{-1}_{\dagger}\ A\bar{x} =_{def} \bigsqcup (\mathtt{psf}^{-1}_{\dagger\Rightarrow}\ \circledast\ \mathbb{W}^{-1}_2 A\bar{x}) \tag{110}$$

We now show that the inverse translations are inverse functions of their corresponding translations:

Lemma 6.36. $\mathtt{psf}^{-1}_{\dagger\Rightarrow}\ \mathtt{psf}_{\dagger\Rightarrow}S^x = S^x$.

Proof. Straightforward. (The proof is written out in Lemma D.4 on page 258.) □

Lemma 6.37. $\mathtt{psf}^{-1}_{\dagger}\ \mathtt{psf}_{\dagger}\ \underline{S} = \underline{S}$.

Proof. Straightforward. (The proof is written out in Lemma D.5 on page 259.) □

6.4. Correctness of the Translation to Partially Shielded Formula

Here we show that the function $\mathtt{psf}$ is a *correct* translation from hypersequents to $\mathtt{PSF}$, by showing that if a formula is true in a model, then its translation into $\mathtt{PSF}$ is also true in that model.

THEOREM 6.38 (Model-Theoretic Correctness). *Let* $\mathfrak{M} = \langle W, R, v\rangle$ *be an intuitionistic Kripke model. Then* $\mathfrak{M} \vDash A$ *iff* $\mathfrak{M} \vDash \mathtt{psf}\ A$*, for all* $\hat{x} \in \mathtt{Term}_0$. $\mathfrak{M} \vDash \mathtt{psf}\ \hat{x}\ A'$.

Proof. By induction on the structure of A. (The proof is written out in Theorem D.6 on page 259.) □

Below we use the calculus **G3c/PSF*** (Figure 3.2 on page 57) to give show the correctness of the translation of hypersequents, by showing that for any proof of a hypersequent $\mathcal{H}$ in **HG3ipm*** (Figure 4.4 on page 93), there is a proof of its translation $\mathtt{psf}\ \mathcal{H}$ in **G3c/PSF***. (As far as we are aware, there is no similar style proof that a translation of formulae to their corresponding first-order formulae is correct.)

THEOREM 6.39 (Proof-Theoretic Correctness). *If* **HG3ipm*** $\vdash \mathcal{H}$, **G3c/PSF*** $\vdash \mathtt{psf}\ \mathcal{H}$.

Proof. By induction on the derivation depth. (The proof is written out in Theorem D.7 on page 260.) □

REMARK 6.40. Theorem 6.39 does not include a proof of the converse, because there is no translation of sequents with $\mathcal{R}$-formulae into hypersequents or simply labelled sequents.

6.5. Semantic Correspondence between Hypersequents and Labelled Sequents

Because hypersequents and simply labelled sequents have the same disjunctive semantics, and their translations to $\mathtt{PSF}_{\bar{x}}$ have the same form, we can compose their translations to and inverse translations from $\mathtt{PSF}_{\bar{x}}$ so that the following holds for solid hypersequents (Definition 4.9 on page 70):

$$\mathtt{psf}^{-1}\ \mathtt{psf}_{\dagger}\ \mathtt{psf}_{\dagger}^{-1}\ \mathtt{psf}\ \hat{x}\ \mathcal{H} = \mathtt{psf}^{-1}\ \mathtt{psf}\ \hat{x}\ \mathcal{H} = \mathcal{H} \tag{111}$$

We have not addressed the issue of empty components in hypersequents here. That will be addressed when a syntactic translation between hypersequents and labelled sequents is given in Chapter 7. (We note that semantically the issue is not important, and it does not affect Theorem 6.42 above.)

DEFINITION 6.41 (Semantic Translation Functions). We also can compose the translation functions to obtain semantic translation functions between the set $\mathtt{Seq}^+ \setminus (\Rightarrow)$ of solid hypersequents, and SLS:

$$\mathtt{hs\_to\_sls}\ \mathcal{H} =_{def} \mathtt{psf}_{\dagger}^{-1}\ \mathtt{psf}_{|}\ \hat{x}\ \mathcal{H} \tag{112}$$

$$\mathtt{sls\_to\_hs}\ \underline{S} =_{def} \mathtt{psf}_{|}^{-1}\ \mathtt{psf}_{\dagger}\ \underline{S} \tag{113}$$

THEOREM 6.42 (Semantic Correspondence). *Let* $\mathfrak{M} = \langle W, R, v\rangle \in \mathcal{K}_{\mathbf{Int}}$. *Then*

$$\mathfrak{M} \vDash \mathcal{H} \textit{ iff } \mathfrak{M} \vDash \mathtt{hs\_to\_sls}\ \mathcal{H} \tag{114}$$

$$\mathfrak{M} \vDash \underline{S} \textit{ iff } \mathfrak{M} \vDash \mathtt{sls\_to\_hs}\ \underline{S} \tag{115}$$

Proof. For (114), if $\mathfrak{M} \vDash \mathcal{H}$, then there exists a component $S_i \in \mathcal{H}$ (the ith component) such that $\mathfrak{M} \vDash S_i$. Then $\mathfrak{M} \vDash \mathtt{psf}_{\Rightarrow}\ \hat{x}\ S_i$ (recalling Definition 3.41 on page 55). Let $\underline{S'} = \mathtt{hs\_to\_sls}\ \mathcal{H}$. There exists a slice $\underline{S'} /\!/ y \approx \mathtt{psf}_{\Rightarrow}\ \hat{x}_i\ S_i$, so $\mathfrak{M} \vDash \underline{S'}$. Likewise, if $\mathfrak{M} \nvDash \mathcal{H}$, then for all components $S_j \in \mathcal{H}$, $\mathfrak{M} \nvDash S_j$, and $\mathfrak{M} \nvDash \mathtt{psf}_{\Rightarrow}\ \hat{x}_j\ S_j \approx \underline{S'} /\!/ y_j$.

For (115) the proof is similar. □

6.5.1. Conventional Translation. In Chapter 5 it was noted that some labelled sequent calculi such as the system for **S5** in [Kan57] have a **conventional sequent semantics** (Remark 5.35 on page 110). So let us suppose a labelled sequent calculus for **Int***/Geo similarly has the conventional semantics.

DEFINITION 6.43 (Conventional Translation). Let $\bar{x} = x_1, \ldots, x_n$ and $\bar{y} = y_1, \ldots, y_n$ be vectors in $\mathtt{Term}_0^+$, and $A \in \mathtt{PSF}$. We define a function from $\mathtt{Term}_0^+ \times \mathtt{Term}_0^+ \times \mathtt{Prop}$ to PSF for constructing a shield:

$$\mathtt{shield}\ \bar{x}\ \bar{y}\ A =_{def} \mathcal{R}x_1y_1 \wedge \ldots \wedge \mathcal{R}x_ny_n \wedge A$$

The **conventional translation** from SLS to $\mathtt{PSF}_{\bar{x}}$ is:

$$\mathtt{psf}_{\ddagger}(\underline{\Gamma} \Rightarrow \underline{\Delta}) =_{def} \forall \bar{y}.(((\mathtt{shield}\ \bar{x}\ \bar{y}) \wedge \bigwedge\!\!\!\!\bigwedge_1(\mathtt{psf}_{\dagger} \circledast \underline{\Gamma})) \supset \bigvee\!\!\!\!\bigvee_1(\mathtt{psf}_{\dagger} \circledast \underline{\Delta}))$$

where $\bar{y} \in \mathtt{Term}_0^+$ is the vector of variables that corresponds to the set of labels $\mathtt{lab}(\underline{\Gamma}, \underline{\Delta})$, and $\bar{x}$ is a vector of variables chosen from $\mathtt{Term}_0$ so that they are not bound in either $(\mathtt{psf}_\dagger \circledast \underline{\Gamma})$ or $(\mathtt{psf}_\dagger \circledast \underline{\Delta})$.

Note that unlike the labelled sequents in [Kan57], we assign a semantic meaning to labels.

Because the formulae in $\mathtt{PSF}$ that correspond to the truth conditions of intuitionistic Kripke models are classical, we can (ignoring quantifiers) use the following classical equivalence

$$\bigvee_{i=1}^{n}(A_i \supset B_i) \equiv (\bigwedge_{i=1}^{n} A_i) \supset (\bigvee_{i=1}^{n} B_i) \tag{116}$$

to show that the standard and conventional translations are equivalent. The proof of (116) is straightforward. However, formulae of the form $\neg A$ are not in $\mathtt{PSF}$, so we show that the two forms are interderivable in the $\mathtt{PSF}$ fragment of $\mathtt{Form}_1$ below.

THEOREM 6.44. *Let* $\underline{S} = \Gamma_1^{x_1}, \ldots, \Gamma_n^{x_n} \Rightarrow \Delta_1^{x_1}, \ldots, \Delta_n^{x_n}$ *be a labelled sequent. Then* $\mathbf{G3c/PSF} \vdash \mathtt{psf}_\dagger\ \underline{S}$ *iff* $\mathbf{G3c/PSF} \vdash \mathtt{psf}_\ddagger\ \underline{S}$.

Proof. Recall that the rules of **G3c/PSF** are invertible (Lemma 3.55 on page 59).

Let $\mathtt{psf}_\dagger\ \Gamma_1^{x_1}, \ldots, \Gamma_n^{x_n} \Rightarrow \Delta_1^{x_1}, \ldots, \Delta_n^{x_n} = \bigvee_{i=1}^{n} \forall y_i.(\mathcal{R}x_iy_i \wedge \bigwedge\Gamma'\{y_i\}) \supset (\bigvee\Delta'\{y_i\})$. From left to right:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \bigvee_{i=1}^{n} \forall y_i.(\mathcal{R}x_iy_i \wedge \bigwedge\Gamma_i'\{y_i\}) \supset (\bigvee\Delta_i'\{y_i\})}{\Rightarrow \forall y_1.(\mathcal{R}x_1y_1 \wedge \bigwedge\Gamma_1'\{y_1\}) \supset (\bigvee\Delta_1'\{y_1\}), \ldots, \forall y_n.(\mathcal{R}x_ny_n \wedge \bigwedge\Gamma_n'\{y_n\}) \supset (\bigvee\Delta_n'\{y_n\})}\,(\mathsf{R}\vee^{-1})^*}{\mathcal{R}x_1z_1, \ldots, \mathcal{R}x_nz_n, \bigwedge\Gamma_1'\{z_1\}, \ldots, \bigwedge\Gamma_n'\{z_n\} \Rightarrow \bigvee\Delta_1'\{z_1\}, \ldots, \bigvee\Delta_n'\{z_n\}}\,(\mathsf{R}\forall\supset^{-1})^+}{\mathcal{R}x_1z_1, \ldots, \mathcal{R}x_nz_n, \bigwedge\Gamma_1'\{z_1\}, \ldots, \bigwedge\Gamma_n'\{z_n\} \Rightarrow \bigvee(\Delta_1'\{z_1\}, \ldots, \Delta_n'\{z_n\})}\,\mathsf{R}\vee^*}{\mathcal{R}x_1z_1, \bigwedge(\mathcal{R}x_2z_2 \ldots, \mathcal{R}x_nz_n, \Gamma_1'\{z_1\}, \ldots, \Gamma_n'\{z_n\}) \Rightarrow \bigvee(\Delta_1'\{z_1\}, \ldots, \Delta_n'\{z_n\})}\,\mathsf{L}\wedge^*}{\Rightarrow \forall y_1, \ldots, y_n.(\bigwedge(\mathcal{R}x_1y_1 \ldots, \mathcal{R}x_ny_n, \Gamma_1'\{y_1\}, \ldots, \Gamma_n'\{y_n\}) \Rightarrow \bigvee(\Delta_1'\{y_1\}, \ldots, \Delta_n'\{y_n\}))}\,\mathsf{R}\forall\supset$$

which is equivalent to $\mathtt{psf}_\ddagger\ \Gamma_1^{x_1}, \ldots, \Gamma_n^{x_n} \Rightarrow \Delta_1^{x_1}, \ldots, \Delta_n^{x_n}$.

The proof from right-to-left uses the inverse derivation. □

A simpler translation of labelled sequents can be defined, using the root rule from $\mathbf{G3c/PSF}^*$ to eliminate extraneous $\mathcal{R}$-formulae.

DEFINITION 6.45. An alternative translation of labelled sequents to $\mathtt{PSF}$ is:

$$\mathtt{psf}_\ddagger'\ (\underline{\Gamma} \Rightarrow \underline{\Delta}) =_{def} (\mathtt{psf}_\dagger' \circledast \underline{\Gamma}) \Rightarrow (\mathtt{psf}_\dagger' \circledast \underline{\Delta})$$

LEMMA 6.46. $\mathbf{G3c/PSF}^* \vdash \mathtt{psf}_\dagger\ \underline{\Gamma} \Rightarrow \underline{\Delta}$ *iff* $\mathbf{G3c/PSF}^* \vdash \mathtt{psf}'_\ddagger\ \underline{\Gamma} \Rightarrow \underline{\Delta}$.

Proof. From left to right:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \bigvee_{i=1}^{n} \forall y_i.(\mathcal{R}x_i y_i \wedge \bigwedge\!\!\bigwedge \Gamma'_i\{y_i\}) \supset (\bigvee\!\!\bigvee \Delta'_i\{y_i\})}{\Rightarrow \forall y_1.(\mathcal{R}x_1 y_1 \wedge \bigwedge\!\!\bigwedge \Gamma'_1\{y_1\}) \supset (\bigvee\!\!\bigvee \Delta'_1\{y_1\}), \ldots, \forall y_n.(\mathcal{R}x_n y_n \wedge \bigwedge\!\!\bigwedge \Gamma'_n\{y_n\}) \supset (\bigvee\!\!\bigvee \Delta'_n\{y_n\})}\,(\mathsf{R}\vee^{-1})^*}{\mathcal{R}x_1 z_1, \ldots, \mathcal{R}x_n z_n, \bigwedge\!\!\bigwedge \Gamma'_1\{z_1\}, \ldots, \bigwedge\!\!\bigwedge \Gamma'_n\{z_n\} \Rightarrow \bigvee\!\!\bigvee \Delta'_1\{z_1\}, \ldots, \bigvee\!\!\bigvee \Delta'_n\{z_n\}}\,(\mathsf{R}\forall\supset^{-1})^+}{\mathcal{R}x_1 z_1, \ldots, \mathcal{R}x_n z_n, \bigwedge\!\!\bigwedge \Gamma'_1\{z_1\}, \ldots, \bigwedge\!\!\bigwedge \Gamma'_n\{z_n\} \Rightarrow \Delta'_1\{z_1\}, \ldots, \Delta'_n\{z_n\}}\,(\mathsf{R}\vee^{-1})^*}{\mathcal{R}x_1 z_1, \ldots, \mathcal{R}x_n z_n, \Gamma'_1\{z_1\}, \ldots, \Gamma'_n\{z_n\} \Rightarrow \Delta'_1\{z_1\}, \ldots, \Delta'_n\{z_n\}}\,(\mathsf{L}\wedge^{-1})}{\Gamma'_1\{z_1\}, \ldots, \Gamma'_n\{z_n\} \Rightarrow \Delta'_1\{z_1\}, \ldots, \Delta'_n\{z_n\}}\,\mathsf{root}^+}{\Gamma'_1\{x_1\}, \ldots, \Gamma'_n\{x_n\} \Rightarrow \Delta'_1\{x_1\}, \ldots, \Delta'_n\{x_n\}}\,[\bar{\mathsf{x}}/\bar{\mathsf{z}}]$$

The proof from right-to-left uses the inverse derivation. □

We note that the inverse function $\mathtt{psf'}^{-1}_\ddagger$ is straightforward to define:

DEFINITION 6.47. We define a function from $\mathsf{PSF}_{\bar{x}}$ to SLS:

$$\mathtt{psf'}^{-1}_\ddagger\ (\Gamma \Rightarrow \Delta) =_{def} (\mathtt{psf}^{-1}_\dagger \circledast \Gamma) \Rightarrow (\mathtt{psf}^{-1}_\dagger \circledast \Delta)$$

LEMMA 6.48. $\mathtt{psf'}^{-1}_\ddagger\ (\mathtt{psf'}_\ddagger\ \underline{S}) = \underline{S}$.

Proof. Straightforward, using Lemma 6.34 on page 145. □

DEFINITION 6.49. We define an alternative translation from hypersequents in $\mathtt{Seq}^+$ to $\mathsf{PSF}_{\bar{x}}$:

$$\mathtt{psf'}\ \mathcal{H} =_{def} \mathtt{psf'}_\ddagger\ \mathtt{psf}^{-1}_\ddagger\ (\mathtt{psf}\ \hat{x}\ \mathcal{H}) \tag{117}$$

$$\mathtt{psf'}^{-1}\ (\Gamma \Rightarrow \Delta) =_{def} \mathtt{psf}^{-1}\ \mathtt{psf}_\ddagger\ \mathtt{psf'}^{-1}_\ddagger\ (\Gamma \Rightarrow \Delta) \tag{118}$$

LEMMA 6.50. $\mathbf{G3c/PSF}^* \vdash \mathtt{psf}\ \mathcal{H}$ *iff* $\mathbf{G3c/PSF}^* \vdash \mathtt{psf'}\ \mathcal{H}$.

Proof. Similar to Lemma 6.46. □

PROPOSITION 6.51.

$$\mathtt{psf'}\ (\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n) = \left(\bigcup_{i=1}^{n} \mathtt{psf}\ \hat{x}_i \circledast \Gamma_i\right) \Rightarrow \left(\bigcup_{i=1}^{n} \mathtt{psf}\ \hat{x}_i \circledast \Delta_i\right)$$

Proof. Straightforward. □

6.6. Conclusion

We have introduced translations of hypersequents and labelled sequents to sequents in PSF and used the translations as intermediary translations between the two formalisms. We have also used the translations to PSF to show that the disjunctive semantics of hypersequents and labelled sequents is classically equivalent to the conventional semantics of standard sequents. This equivalence is needed for showing a relationship between hypersequents and relational sequents (which have a conventional semantics) later in Chapter 8.

In the translations we have not directly addressed the issue of equivalence modulo permutation of labels. We believe that it was not needed in this chapter, because of the correspondence between simple labels and first-order variables. We note that equivalence modulo permutation of labels roughly corresponds with a notion of α-equivalence on free variables, which is an area for future investigation. Although there is the alternative definition of equivalence modulo permutation of labels given in Remark 6.31 on page 144 that does add quantifiers to bind these variables. Indeed, there is a view that free variables are universally quantified implicitly, and one can view equivalence modulo permutation of labels as a means of arguing for that view.

We have not addressed the issue of translating hypersequents with empty components into simply labelled sequents. Semantically, the issue is irrelevant for logics in $\mathbf{Int}^*/\mathsf{Geo}$. It is addressed in the next chapter and Appendix E on page 263, however.

We have outlined a straightforward semantic proof of the correctness of the translation, by showing that a formula is true in a model if and only if the translation of the formula into PSF is true in that model. A similar proof can be found in [Min00, §8.3]. We have also outlined of a proof-theoretic correctness proof, which we believe is a novel contribution. We have shown that if a hypersequent is derivable, then its translation is derivable in $\mathbf{G3c}/\mathbf{PSF}^*$. We do not have a proof that a sequent derived in $\mathbf{G3c}/\mathbf{PSF}^*$ has a corresponding hypersequent, because we have no translation of arbitrary sequents of PSF into hypersequents.

With the semantic correspondence shown between formalisms, we can now examine the syntactic translation between them in the next chapter, although we note that a syntactic translation was implicit in the semantic translations we have used.

CHAPTER 7

Translating Between Hypersequents and Labelled Sequents

7.1. Overview

In this chapter we show that hypersequents and labelled sequents are different notations of the same *abstract* formalism, where the latter names the components with labels on the formulae.

In Section 7.2, a *syntactic* translation of solid hypersequents into labelled sequents is given. (This is a minor restriction with respect to sequent calculi for intermediate logics, and an extension to address the case for all hypersequents is discussed in Appendix E on page 263.) The translation is shown to preserve truth and countermodels. An extension of the translation is given for translating schematic hypersequents, and a sound method is given for translating hypersequent rules into labelled rules. Finally, the method is applied to the hypersequent calculi in the framework **HG3ipm*** (Figure 4.4 on page 93) to obtain simply labelled calculi in the framework **LG3ipm***.

In Section 7.3, a reverse translation of labelled sequents to hypersequents is given. (A proof that the translations are the inverse of one another modulo equivalence of labels is given in Appendix E on page 263.) The translation is shown to preserve truth and countermodels. An extension of the translation is given for translating schematic labelled sequents, and a sound method is given for translating labelled sequent rules into hypersequent rules. The method is applied to obtain a hypersequent version of the simply labelled sequent calculus **O** (Figure 5.1 on page 112). We also show that hypersequents are mutually derivable in systems **O** and **HG1ip** (Figure 4.2 on page 84). (For brevity, we do not apply the translation from **FIL** in Figure 5.2 on page 114 to a hypersequent calculus.)

While this may be considered a fairly unsurprising result, we are unaware of previous work that makes this translation explicit.

7.2. Translating Hypersequents to Labelled Sequents

The procedure for translating solid hypersequents into simply labelled sequents is straightforward: for each component, assign a unique label. The result is the sequent union of all translated components. A translation function $\mathtt{sls}_3$ is given in Definition 7.1:

DEFINITION 7.1 (Translation of Solid Hypersequents to Labelled Sequents). We define a translation function $\mathtt{sls}_3$ from $\mathtt{Lab}_0^+ \times \mathtt{Seq}^+ \setminus (\Rightarrow)$ to $\mathtt{SLS}$:

$$\mathtt{sls}_\emptyset\ x\, A = A^x \qquad \text{where } A \in \mathtt{Prop}$$

$$\mathtt{sls}_1\ x\, \Gamma = (\mathtt{sls}_\emptyset\ x) \circledast \Gamma \qquad \text{where } \Gamma \in \mathtt{Prop}^*$$

$$\mathtt{sls}_2\ x\, S = (\mathtt{sls}_1\ x) \circledast S \qquad \text{where } S \in \mathtt{Seq}$$

where $x \in \mathtt{Lab}_0$.

$$\mathtt{sls}_3\ \bar{x}\, (S_1 \mid \ldots \mid S_n) = \begin{cases} \mathtt{sls}_2\ x_1\, S_1 & \text{if } n = 1 \\ (\mathtt{sls}_3\ \bar{x}\, (S_1 \mid \ldots \mid S_{n-1})) \sqcup (\mathtt{sls}_2\ x_n\, S_n) & \text{otherwise} \end{cases}$$

where $\bar{x} = x_1, \ldots, x_n \in \mathtt{Lab}_0^+$ is a list of labels, and $S_i \in \mathtt{Seq}$.

REMARK 7.2. Recall Remark 2.23 on page 26 that we can consider the components of hypersequents to be implicitly ordered by their representation. Had we chosen a different representation of the hypersequent (such as the order of components) or a different list of labels for the translation, we would have a different translation that is equivalent modulo permutation to $\mathtt{sls}_3$.

Solid hypersequents are required because without it $\mathtt{sls}_3$ would not be a 1-1 function, i.e. $\mathtt{sls}_3\ \bar{x}\, (\mathcal{H} \mid \emptyset \Rightarrow \emptyset) = \mathtt{sls}_3\ \bar{x}\, (\mathcal{H})$.

For proofs of hypersequents in **Int***/$\mathtt{Geo}$ this is not a problem, because $\emptyset \Rightarrow \emptyset$ is not derivable, and because the rule

$$\frac{\mathcal{H} \mid \Rightarrow}{\mathcal{H}}$$

is easily shown depth-preserving admissible in **HG3ipm** and similar systems. (It's also trivially shown by internal weakening and external contraction, though not necessarily as depth-preserving admissibly.) However, the empty component is still significant for other calculi (e.g. the system **GL** for Łukasiewicz logic in [MOG05]). This special case is handled by extending the language of simply labelled sequents with a placeholder for the empty antecedents or succedents and updating the definitions of translations, as well as equivalence modulo permutation. This is discussed in Appendix E on page 263.

THEOREM 7.3 (Truth Preservation). *Let* $\mathfrak{I}$ *be an interpretation, and let* $\mathcal{H} \in \mathtt{Seq}^+$ *such that* $\mathfrak{I} \vDash \mathcal{H}$. *Then for all non-empty lists* $\bar{x} \in \mathtt{Lab}_0^+$, $\mathfrak{I} \vDash \mathtt{sls}_3\ \bar{x}\ \mathcal{H}$.

Proof. There exists the kth component $S_k \in \mathcal{H}$, where $1 \leq k \leq |\bar{x}|$, such that $\mathfrak{I} \vDash S_k$ (Definition 4.10 on page 70). Let x_k be the kth label in a list $\bar{x}$ of labels. Let $S_k^{x_k} = \mathtt{sls}_2\ x_k\ S_k$. $\mathfrak{I} \vDash S_k^{x_k}$. Let $\underline{S} = \mathtt{sls}_3\ \bar{x}\ \mathcal{H}$. Since there exists a label $x_k \in \mathtt{lab}(\underline{S})$ such that $\mathfrak{I} \vDash \underline{S} \mathbin{/\!/} x_k$, $\mathfrak{I} \vDash \underline{S}$ (Definition 5.31 on page 109). So $\mathfrak{I} \vDash \mathtt{sls}_3\ \bar{x}\ \mathcal{H}$. □

COROLLARY 7.4 (Countermodel Preservation). *Let* $\mathfrak{I}$ *be an interpretation, and let* $\mathcal{H}$ *be a hypersequent such that* $\mathfrak{I} \nvDash \mathcal{H}$. *Then for all non-empty lists* $\bar{x} \in \mathtt{Lab}_0^+$, $\mathfrak{I} \nvDash \mathtt{sls}_3\ \bar{x}\ \mathcal{H}$.

Proof. The proof follows similarly to Theorem 7.3. □

The translation function can be extended to cover schematic hypersequents by adding cases for metavariables. We first define the sets of metavariables needed for the translation between schematic hypersequents and schematic simply labelled sequents.

Recall the definition of the set $(\mathtt{Seq}^{\mu} + \mathtt{HyperseqVar})^+$ of schematic hypersequents given in Definition 4.15 on page 71.

DEFINITION 7.5. Let `slsvar` be a function from `HyperseqVar` to a unique multilabelled sequent in `MultisetVar`, e.g. from $\mathcal{H}$ to $\underline{\Gamma}_{\dagger} \Rightarrow \underline{\Delta}_{\dagger}$. (The † subscript is used to indicate that these metavariables are in the denumerable set $\mathtt{MultisetVar}_{\dagger}$, and distinct from metavariables used to indicate unilabelled multisets.)

PROPOSITION 7.6. `slsvar` *is a* $1-1$ *function.*

Proof. The domain and codomain are denumerable. □

DEFINITION 7.7. We define a translation $\mathtt{sls}_3^\mu$ of schematic solid hypersequents to simply labelled sequents below:

$$\mathtt{sls}_{\mathtt{0}}^\mu\ x\ \alpha = \alpha^x \qquad \text{where } \alpha \in \mathtt{Prop} + \mathtt{MultisetVar}$$

$$\mathtt{sls}_1^\mu\ x\ \Gamma = (\mathtt{sls}_{\mathtt{0}}^\mu\ x) \circledast \Gamma \qquad \text{where } \Gamma \in (\mathtt{Prop} + \mathtt{MultisetVar})^*$$

where $x \in \mathtt{Lab}_0$.

$$\mathtt{sls}_2^\mu\ x\ S = \begin{cases} (\mathtt{sls}_1^\mu\ x) \circledast S & \text{where } S \in (\mathtt{Prop} + \mathtt{MultisetVar})^{*2} \\ (\mathtt{slsvar}\ S) & \text{where } S \in \mathtt{HyperseqVar} \end{cases}$$

where $x \in \mathtt{Lab}_0$.

$$\mathtt{sls}_3^\mu\ \bar{x}\ (\alpha_1 \mid \ldots \mid \alpha_n) = \begin{cases} \mathtt{sls}_2^\mu\ x_1\ \alpha_1 & \text{if } n = 1 \\ (\mathtt{sls}_3^\mu\ \bar{x}\ (\alpha_1 \mid \ldots \mid \alpha_{n-1})) \sqcup (\mathtt{sls}_2^\mu\ x_n\ \alpha_n) & \text{otherwise} \end{cases}$$

where $\bar{x} = x_1, \ldots, x_n \in \mathtt{Lab}_0^+$ is a list of labels.

REMARK 7.8. Note that the set $\mathtt{Prop}$ of formulae is identical to the set of metaformulae. The difference is one of interpretation: a propositional variable in a metaformula may represent an arbitrary formula rather than an atomic proposition.

Likewise schematic sequents may be treated as sequents where some of the formulae are metavariables that denote arbitrary multisets of formulae.

EXAMPLE. *The following schematic hypersequent*

$$\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2, A \vee B$$

would be translated by the $\mathtt{sls}_3^\mu$ *function and a list of labels* $x_1, \ldots, x_4$ *into the schematic simply labelled sequent*

$$\underline{\Gamma}_\dagger, \underline{\Gamma}'_\dagger \Gamma_1^{x_3}, \Gamma_2^{x_4} \Rightarrow \underline{\Delta}_\dagger, \underline{\Delta}'_\dagger, \Delta_1^{x_3}, \Delta_2^{x_4}, (A \vee B)^{x_4}$$

modulo the relationship between hypercontext variables and simply labelled sequent variables in $\mathtt{slsvar}$.

COROLLARY 7.9 (Translation of Rules). *Let ρ be a primitive n-premiss hypersequent rule,*

$$\frac{\mathcal{H}_1 \quad \ldots \quad \mathcal{H}_n}{\mathcal{H}_0}\ \rho$$

in a calculus **HGS** *for a logic* **S**. *Then ρ can be translated into a corresponding simply labelled sequent rule*

$$\frac{(\mathtt{sls}_3^{\mu}\ M_1\ \mathcal{H}_1) \quad \ldots \quad (\mathtt{sls}_3^{\mu}\ M_n\ \mathcal{H}_n)}{(\mathtt{sls}_3^{\mu}\ M_0\ \mathcal{H}_0)}\ \rho$$

where all $M_0, \ldots, M_n$ are mutually disjoint, for the corresponding simply labelled calculus **LGS** *for a logic* **S**.

Proof. Follows from Theorem 7.3 on page 153, when ρ is instantiated. □

REMARK 7.10. The use of disjoint lists of labels for each premiss and the conclusion is acceptable because of the semantics of hypersequents and simply labelled sequents: the actual label names play no rôle in the inference.

COROLLARY 7.11. *Let ρ be a (depth-preserving) admissible n-premiss hypersequent rule,*

$$\frac{\mathcal{H}_1 \quad \ldots \quad \mathcal{H}_n}{\mathcal{H}_0}\ \rho$$

in a calculus **HGS** *for a logic* **S**. *Then ρ can be translated into a corresponding (depth-preserving) admissible rule*

$$\frac{(\mathtt{sls}_3^{\mu}\ M_1\ \mathcal{H}_1) \quad \ldots \quad (\mathtt{sls}_3^{\mu}\ M_n\ \mathcal{H}_n)}{(\mathtt{sls}_3^{\mu}\ M_0\ \mathcal{H}_0)}\ \rho$$

where all $M_0, \ldots, M_n$ are mutually disjoint, for the corresponding simply labelled calculus **LGS** *for a logic* **S**.

Proof. Given a proof of admissibility of ρ in **HGS**, one can give a corresponding proof of admissibility of ρ in **LGS** by translating all rules in all (sub)cases appropriately. □

REMARK 7.12. Corollary 7.9 applies to admissible as well as primitive rules. One can translate admissibility proofs for a hypersequent calculus to admissibility proofs for the translated simply labelled sequent calculus by translating the rules in each case.

Remark 7.13. One can use the following "rules of thumb" for preserving labels between premisses and the conclusion without affecting soundness:

(1) Internal hypersequent rules will be translated to labelled sequent rules with only one explicit label in each premiss and the conclusion, so clearly these should all be the same label.
(2) The trivially invertible forms of hypersequent rules, or rules where a component is unchanged in a premiss and the conclusion, should be translated to that corresponding slices are equivalent modulo permutation of labels. Such slices should have the same label between premiss(es) and conclusion. However, multiple slices in the same premiss that are equivalent modulo permutation should not have the same label (e.g. a translation of the EC rule), as each active component corresponds to a different label.
(3) If the antecedents (resp. succedents) of a slice in a premiss (or premisses) and another slice in the conclusion are equivalent modulo permutation, then they can share a label.

Preserving labels between the premisses and conclusion will be needed for translating simply labelled rules into relational rules in Chapter 9.

Proposition 7.14. *The rules given in Remark 7.13 preserve soundness.*

Proof. By the disjunctive semantics of labelled sequents. □

Example. *The following hypersequent rule*

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow A \supset B, \Delta \mid \Gamma, A \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \supset B, \Delta}\ \mathsf{R}\supset_{\iota}$$

can translated to the simply labelled sequent rule

$$\frac{\underline{\Gamma}', \Gamma^x, \Gamma^y, A^y \Rightarrow \underline{\Delta}', (A \supset B)^x, \Delta^x, B^y}{\underline{\Gamma}', \Gamma^x \Rightarrow \underline{\Delta}', (A \supset B)^x, \Delta^x}\ \mathsf{R}\supset_{\iota}$$

where $x, y \# \underline{\Gamma}', \underline{\Delta}'$. (The † notation is omitted for simplicity, since it is clear that there is no overlap in names for the variables.) The x label is shared because the slice in the premiss and conclusion are equivalent modulo permutation.

To simplify the notation, the restriction that $x\#\underline{\Delta}'$ can be relaxed and Δ^x absorbed into $\underline{\Delta}'$, since it does not interact with any other variable in the rule.

EXAMPLE. *The following hypersequent rule*

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma, A \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \supset B, \Delta}\ \mathsf{R}{\supset}'$$

can translated to the simply labelled sequent rule

$$\frac{\underline{\Gamma}', \Gamma^x, \Gamma^y, A^y \Rightarrow \underline{\Delta}', \Delta^x, B^y}{\underline{\Gamma}', \Gamma^x \Rightarrow \underline{\Delta}', (A \supset B)^x, \Delta^x}\ \mathsf{R}{\supset}'$$

where $x, y\#\underline{\Gamma}', \underline{\Delta}'$. The x label is shared because the slice in the premiss and conclusion have antecedents that are equivalent modulo permutation.

7.2.1. The framework LG3ipm*. The system **LG3ipm** (Figure 7.1) can be obtained from the hypersequent calculus **HG3ipm** (Figure 4.4 on page 93) using the above procedure. **LG3ipm*** consists of rules of **LG3ipm** plus the rules from Figure 7.2 on page 159, which are obtained from the corresponding hypersequent rules.

$$\frac{}{P^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, P^x}\ \mathsf{Ax} \qquad \frac{}{\bot^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\bot$$

$$\frac{\underline{\Gamma}, A^x, B^x \Rightarrow \underline{\Delta}}{\underline{\Gamma}, A \wedge B^x \Rightarrow \underline{\Delta}}\ \mathsf{L}\wedge \qquad \frac{\underline{\Gamma} \Rightarrow A^x, \underline{\Delta} \quad \underline{\Gamma} \Rightarrow B^x, \underline{\Delta}}{\underline{\Gamma} \Rightarrow A \wedge B^x, \underline{\Delta}}\ \mathsf{R}\wedge$$

$$\frac{\underline{\Gamma}, A^x \Rightarrow \underline{\Delta} \quad \underline{\Gamma}, B^x \Rightarrow \underline{\Delta}}{\underline{\Gamma}, A \vee B^x \Rightarrow \underline{\Delta}}\ \mathsf{L}\vee \qquad \frac{\underline{\Gamma} \Rightarrow A^x, B^x, \underline{\Delta}}{\underline{\Gamma} \Rightarrow A \vee B^x, \underline{\Delta}}\ \mathsf{R}\vee$$

$$\frac{(A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x \quad B^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}{(A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}{\supset} \qquad \frac{A^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', B^x}{\underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, (A \supset B)^x}\ \mathsf{R}{\supset}$$

where $x\#\underline{\Delta}'$ in $\mathsf{R}{\supset}$.

$$\frac{\underline{\Gamma}' \Rightarrow \underline{\Delta}'}{\Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x}\ \mathsf{EW} \qquad \frac{\Gamma^x, \Gamma^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, \Delta^y}{\Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x}\ \mathsf{EC}$$

where $x, y\#\underline{\Gamma}', \underline{\Delta}'$ in EW and EC.

FIGURE 7.1. The simply labelled calculus **LG3ipm**.

THEOREM 7.15 (Soundness and Completeness). **LG3ipm** *is sound and complete.*

Proof. Soundness follows from the soundness and completeness of **HG3ipm**. □

PROPOSITION 7.16. *The structural rules*

$$\frac{\underline{\Gamma} \Rightarrow \underline{\Delta}}{\underline{\Gamma}, A^x \Rightarrow \underline{\Delta}}\ (\mathsf{LW}) \quad \frac{\underline{\Gamma} \Rightarrow \underline{\Delta}}{\underline{\Gamma} \Rightarrow A^x, \underline{\Delta}}\ (\mathsf{RW}) \quad \frac{\underline{\Gamma}, A^x, A^x \Rightarrow \underline{\Delta}}{\underline{\Gamma}, A^x \Rightarrow \underline{\Delta}}\ (\mathsf{LC}) \quad \frac{\underline{\Gamma} \Rightarrow A^x, A^x, \underline{\Delta}}{\underline{\Gamma} \Rightarrow A^x, \underline{\Delta}}\ (\mathsf{RC})$$

are depth-preserving admissible in **LG3ipm**.

Proof. Straightforward. Note that for weakening rules, we do not need to assume that the principal label occurs in the sequent. □

PROPOSITION 7.17. *The logical rules of* **LG3ipm** *are depth-preserving invertible.*

Proof. Semantically, by translation from **HG3ipm**. □

PROPOSITION 7.18 (Label substitution). *The following label substitution rule*

$$\frac{\underline{S}}{[y/x]\underline{S}}\ [\mathsf{y/x}]$$

is depth-preserving admissible in **LG3ipm**.

Proof. By induction on the derivation depth. □

LEMMA 7.19. *The **general weakening** and **general contraction** rules*

$$\frac{\underline{\Gamma}' \Rightarrow \underline{\Delta}'}{\Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x}\ (\mathsf{GW}) \qquad \frac{\Gamma^y, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, \Delta^y}{\Gamma^x, [y/x]\underline{\Gamma}' \Rightarrow [y/x]\underline{\Delta}', \Delta^x}\ (\mathsf{GC})$$

where are depth-preserving admissible in **LG3ipm**.

Proof. GW from multiple instances of LW and RW. GC from substitution (Proposition 7.18), then multiple instances of LC and RC. □

REMARK 7.20. These rules correspond to the GW and GC hypersequent rules (Lemmas 4.32 and 4.33 on page 74).

COROLLARY 7.21 (External Rules). *Instances of* EW *and* EC ***external weakening*** *and* ***external contraction*** *rules*

$$\frac{\underline{\Gamma}' \Rightarrow \underline{\Delta}'}{\Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x}\ (\mathsf{EW}) \qquad \frac{\Gamma^y, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, \Delta^y}{\Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x}\ (\mathsf{EC})$$

where $x, y \# \underline{\Gamma}', \underline{\Delta}'$*, can be eliminated from proofs* **LG3ipm**.

$$\frac{\Gamma_1^w, \Gamma_2^w, \underline{\Gamma}' \Rightarrow \underline{\Delta}}{\Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}}\ \mathsf{LQ}$$

$$\frac{\Gamma_1^x, \Gamma_1^y, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^x, \Delta_2^y \quad \Gamma_1^x, \Gamma_2^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_1^y, \Delta_2^y}{\Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}\ \mathsf{Com_m}$$

$$\frac{\Gamma_1^z, \Gamma_2^z, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^z, \Delta_2^z}{\Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}\ \mathsf{S}$$

where the labels $w, x, y \# \underline{\Gamma}', \underline{\Delta}'$ in LQ, $\mathsf{Com_m}$ and S.

FIGURE 7.2. Extension rules for **LG3ipm***.

Proof. Note that they are special cases of the admissible GW and GC rules. □

PROPOSITION 7.22 (Generalised axiom). **LG3ipm** $\vdash \underline{\Gamma}, A^x \Rightarrow A^x, \underline{\Delta}$.

Proof. Semantically, from the translation from **HG3ipm**. □

7.3. Translating Labelled Sequents to Hypersequents

We give the "inverse" translation from labelled sequents to hypersequents, and then obtain a hypersequent form of the simply labelled calculus **O** (Figure 5.1 on page 112):

DEFINITION 7.23 (Translation of Simply Labelled Sequents to Solid Hypersequents). We define a translation $\mathtt{seq}_3^+$ from SLS to $\mathtt{Seq}^+$:

$$\pi_{form}\ A^x = A \qquad \text{where } A^x \in \mathtt{Prop} \times \mathtt{Lab}_0$$

$$\mathtt{seq}_1^+\ \Gamma^x = \pi_{form} \circledast \Gamma^x \qquad \text{where } \Gamma^x \in (\mathtt{Prop} \times \mathtt{Lab}_0)^*$$

$$\mathtt{seq}_2^+\ S^x = \mathtt{seq}_1^+ \circledast S^x \qquad \text{where } S^x \in \mathrm{SLS} \mathbin{/\!\!/} x$$

where the domains of the $\mathtt{seq}_1^+$ and $\mathtt{seq}_2^+$ are unilabelled.

$$\mathtt{seq}_3^+\ (S_1^{x_1}, \ldots, S_n^{x_n}) = \begin{cases} \mathtt{seq}_2^+\ S_1^{x_1} & \text{if } n = 1 \\ (\mathtt{seq}_3^+\ S_1^{x_1}, \ldots, S_{n-1}^{x_{n-1}}) \mid (\mathtt{seq}_2^+\ S_n^{x_n}) & \text{otherwise} \end{cases}$$

where each $S_i^{x_i} \in \mathrm{SLS} \mathbin{/\!\!/} x_i$.

THEOREM 7.24 (Truth Preservation). *Let $\mathfrak{I}$ be an interpretation, and let $\underline{S} \in \mathrm{SLS}$ such that $\mathfrak{I} \vDash \underline{S}$. Then $\mathfrak{I} \vDash \mathtt{seq}_3^+\ \underline{S}$.*

Proof. There exists a label $x \in \mathtt{lab}(\underline{S})$ such that $\mathfrak{I} \vDash \underline{S} \mathbin{/\!/} x$ (Definition 5.31 on page 109). So $\mathfrak{I} \vDash \mathtt{seq}_2^+ (\underline{S} \mathbin{/\!/} x)$. Let $\mathcal{H} = \mathtt{seq}_3^+ \ \underline{S}$. Since $(\mathtt{seq}_2^+ (\underline{S} \mathbin{/\!/} x)) \in \mathcal{H}$, $\mathfrak{I} \vDash \mathcal{H}$ (Definition 4.10 on page 70). □

Corollary 7.25 (Countermodel Preservation). *Let $\mathfrak{I}$ be an interpretation, and let $\underline{S} \in$ SLS such that $\mathfrak{I} \nvDash \underline{S}$. Then $\mathfrak{I} \nvDash \mathtt{seq}_3^+ \ \underline{S}$.*

Proof. The proof follows similarly to Theorem 7.24. □

The above translation can be adapted to schematic labelled sequents:

Definition 7.26. Let $\mathtt{slsvar}^{-1}$ be a 1-1 function from members of the set of multilabelled sequents, e.g. $\underline{\Gamma}_\dagger \Rightarrow \underline{\Delta}_\dagger$ to the set of hypercontext variables $\mathtt{HyperseqVar}$. (Recall that there is a bijection between the sets.)

Definition 7.27. We define a translation $\mathtt{seq}_3^{+\mu}$ of labelled sequents that are not empty to hypersequents:

$$\pi^{\mu}_{form}\ \alpha^x =_{def} \alpha \qquad \text{where } \alpha^x \in (\mathtt{Prop} + \mathtt{MultisetVar}) \times \mathtt{Lab}_0$$

$$\mathtt{seq}_1^{+\mu}\ \Gamma^x =_{def} \pi^{\mu}_{form} \circledast \Gamma^x \qquad \text{where } \Gamma^x \in ((\mathtt{Prop} + \mathtt{MultisetVar}) \times \mathtt{Lab}_0)^*$$

where the domain of the $\mathtt{seq}_1^+$ is unilabelled.

$$\mathtt{seq}_2^{+\mu}\ \underline{S} = \begin{cases} \mathtt{seq}_1^{+\mu} \circledast \underline{S} & \text{where } \underline{S} \in \mathrm{SLS} \mathbin{/\!/} x \\ (\mathtt{slsvar}^{-1}\ \underline{S}) & \text{where } \underline{S} \in (\mathit{MultisetVar} \times \dagger)^2 \end{cases}$$

$$\mathtt{seq}_3^{+\mu}\ (\underline{S}_1, \ldots, \underline{S}_n) = \begin{cases} \mathtt{seq}_2^{+\mu}\ \underline{S}_1 & \text{if } n = 1 \\ (\mathtt{seq}_3^{+\mu}\ \underline{S}_1, \ldots, \underline{S}_{n-1}) \mid (\mathtt{seq}_2^{+\mu}\ \underline{S}_n) & \text{otherwise} \end{cases}$$

where each $\underline{S}_i \in \mathrm{SLS}^{\mu} \mathbin{/\!/} x_i + \langle \mathtt{MultisetVar}, \dagger \rangle^2$.

Corollary 7.28 (Translation of Rules). *Let ρ be a sound, context-sharing primitive n-premiss labelled rule,*

$$\frac{\underline{S}_1 \quad \cdots \quad \underline{S}_n}{\underline{S}_0}\ \rho$$

in a calculus **LGS** *for logic* **S**. *Then* ρ *can be translated into a corresponding hypersequent rule*

$$\frac{(\mathtt{seq}_3^{+\mu}\ \underline{S}_1) \quad \dots \quad (\mathtt{seq}_3^{+\mu}\ \underline{S}_n)}{\mathtt{seq}_3^{+\mu}\ \underline{S}_0}\ \rho$$

for a calculus **HGS** *for logic* **S**.

Proof. Follows from Theorem 7.24 on page 159, when ρ is instantiated. □

COROLLARY 7.29. *Let* ρ *be a sound, context-sharing (depth-preserving) admissible* n*-premiss labelled rule,*

$$\frac{\underline{S}_1 \quad \dots \quad \underline{S}_n}{\underline{S}_0}\ \rho$$

in a calculus **LGS** *for logic* **S**. *Then* ρ *can be translated into a corresponding (depth-preserving) admissible hypersequent rule*

$$\frac{(\mathtt{seq}_3^{+\mu}\ \underline{S}_1) \quad \dots \quad (\mathtt{seq}_3^{+\mu}\ \underline{S}_n)}{\mathtt{seq}_3^{+\mu}\ \underline{S}_0}\ \rho$$

for a calculus **HGS** *for logic* **S**.

Proof. Given a proof of admissibility of ρ in **LGS**, one can give a corresponding proof of admissibility of ρ in **HGS** by translating all rules in all (sub)cases appropriately. □

REMARK 7.30. Because context-sharing and context-splitting rules are interderivable with EW and EC, context-splitting rules can be translated as well.

7.3.1. The system HO. The labelled calculus **O** (Figure 5.1 on page 112) is translated into the single-succedent hypersequent calculus **HO** (Figure 7.3 on the following page), using a procedure based on Corollary 7.28 on the preceding page. (The translation method specified does not explicitly cover the translation of ranges of labels for the $\mathsf{L}\wedge_{\mathsf{O}}$ and $\mathsf{R}\supset_{\mathsf{O}}$ rules, however the unilabelled forms of the rules are translatable, and the multilabelled forms are derivable from those, as per Proposition 5.44 on page 113.)

PROPOSITION 7.31. *The rules of* **HO** *are invertible.*

Proof. Semantically from the translation of **O**. □

$$\frac{}{\mathcal{H} \mid A,\Gamma \Rightarrow A}\ \mathsf{Ax} \qquad \frac{}{\mathcal{H} \mid \bot,\Gamma \Rightarrow \Delta}\ \mathsf{L}\bot$$

$$\frac{\mathcal{H} \mid \Gamma_1, A, B \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, A, B \Rightarrow \Delta_n}{\mathcal{H} \mid \Gamma_1, A \wedge B \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, A \wedge B \Rightarrow \Delta_n}\ \mathsf{L}\wedge_{\mathsf{O}} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A \quad \mathcal{H} \mid \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \wedge B}\ \mathsf{R}\wedge$$

$$\frac{\mathcal{H} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{H} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \vee B \Rightarrow \Delta}\ \mathsf{L}\vee$$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow A \vee B \mid \Gamma \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow A \vee B}\ \mathsf{R}\vee_{\mathsf{O}_1} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A \vee B \mid \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \vee B}\ \mathsf{R}\vee_{\mathsf{O}_2}$$

$$\frac{\mathcal{H} \mid A \supset B, \Gamma \Rightarrow \Delta \mid A \supset B, \Gamma \Rightarrow A \quad \mathcal{H} \mid B, A \supset B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \supset B, \Gamma \Rightarrow \Delta}\ \mathsf{L}{\supset}_{\mathsf{O}}$$

$$\frac{\mathcal{H} \mid A, \Gamma_1 \Rightarrow B \mid \ldots \mid A, \Gamma_n \Rightarrow B}{\mathcal{H} \mid \Gamma_1 \Rightarrow A \supset B \mid \ldots \mid \Gamma_n \Rightarrow A \supset B}\ \mathsf{R}{\supset}_{\mathsf{O}}$$

FIGURE 7.3. The hypersequent calculus **HO**.

PROPOSITION 7.32. *The internal weakening rule*

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow C}{\mathcal{H} \mid \Gamma, A \Rightarrow C}\ \mathsf{LW}$$

is admissible in **HO**.

Proof. By induction on the derivation depth. □

PROPOSITION 7.33. *The external weakening rule*

$$\frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EW}$$

are admissible in **HO**.

Proof. Semantically, by translation from **O**. □

THEOREM 7.34. **HG1ip** $\vdash \mathcal{H}$ *iff* **HO** $\vdash \mathcal{H}$.

Proof. By induction on the derivation depths. (The proof is written out in Theorem E.4 on page 265.) □

7.4. Conclusion

Here we have shown that hypersequent calculi can be translated into labelled sequent calculi, and that labelled sequent calculi with the same disjunctive general semantics as hypersequents can be translated into hypersequent calculi. We have also given examples applying both translations to existing calculi, and thus introduced a new simply labelled framework of calculus **LG3ipm*** for some logics in **Int***/`Geo`, and a new single conclusioned hypersequent calculus **HO** for **Int**.

We have not explicitly defined a translation between hypersequents and labelled sequents where the hypersequent is a list of components rather than a multiset, as with the logics for $\mathbf{BD_k}$ (Section 4.4.3 on page 90) or **SIC** (Section 4.4.6 on page 97). We conjecture that the correspondence holds by similarly restricting permutation in simply labelled sequents between labels, i.e. that the rules

$$\frac{\underline{\Gamma}, B^y, A^x, \underline{\Gamma'} \Rightarrow \underline{\Delta}}{\underline{\Gamma}, A^x, B^y, \underline{\Gamma'} \Rightarrow \underline{\Delta}}\ \mathsf{LE} \qquad \frac{\underline{\Gamma} \Rightarrow \underline{\Delta}, B^y, A^x, \underline{\Delta'}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, B^y, \underline{\Delta'}}\ \mathsf{RE}$$

would not be allowed when $x \neq y$. Investigating the correspondence for systems with restrictions on exchange is an area for future work.

In the next chapter we will revisit the work on semantic relationship between formalisms and introduce methods for translating between labelled sequents and relational sequents.

CHAPTER 8

The Relationship between Labelled Sequents and Relational Sequents

8.1. Overview

In this chapter, we examine the semantic relationship between labelled sequents and relational sequents for **Int***/Geo. There are two aspects to this relationship. The first is that although both kinds of sequents have similar representations, they may have differing general semantics—some labelled sequents, like hypersequents, have a disjunctive semantics, whereas relational sequents have a conventional sequent semantics. (See Definition 5.34 on page 109 and Remark 5.35.) These were shown to be equivalent in Chapter 6 (Theorem 6.44 on page 148). That equivalence is trivially extended for relational sequents that have empty relational contexts.

The second aspect of this relationship is the set of methods for transforming sequents between formalisms in ways that preserve models. Transforming labelled sequents into relational sequents is straightforward: labelled sequents are treated as relational sequents with empty relational contexts, and transformed into relational sequents by adding relational formulae and applying **folding rules** to remove redundant formulae. The opposite operation, transforming relational sequents into labelled sequents by "unfolding" the relational formulae in them, requires some care in order to preserve the relationships between label, using a technique that we introduce, called **transitive unfolding**. The operations are not inverses of one another. Of note, the resulting labelled sequents from transitive unfolding are exponentially larger, and turn out to only have the same *linear* intuitionistic Kripke models that the relational sequents have.

This chapter is structured as follows: in Section 8.2 we extend the translations from Chapter 6 and introduce functions for translating between relational sequents $\mathtt{SPSF}_{\leq}$ sequents.

In Section 8.3, we show the admissibility of various rules for manipulating relations in the calculus **G3c/PSF** for PSF sequents. (These rules have corresponding relational rules, which we will use later in the chapters that follow.)

In Section 8.4, we examine the relationship between labelled and relational sequents. We show the "simple correspondence" of the former with a subset of the latter that have empty relational contexts. We also use the relational rules proven semantically in the previous section for deriving relational sequents from labelled sequents.

In Section 8.5, we introduce a function for translating acyclic relational sequents to labelled sequents. The resulting sequents are exponentially larger (based on the structure of the relational context). They also have the same *linear* models and countermodels as the corresponding relational sequents. That is, when the relational sequents are derivable in a logic that is weaker than **GD**, the labelled sequents are derivable in a logic at least as strong as **GD**. Later in the section we extend the function to cover cyclic sequents.

In Section 8.6, we introduce a simpler approach for eliminating relational formulae by adding converse relations and applying cycle elimination rules which merge labels by antisymmetry.

8.2. Translation of Relational Sequents into Partially Shielded Formulae

NOTATION 8.1. Although relational formulae have the type $\mathtt{Rel} \times \mathtt{Lab}_0^2$, we will use $\mathtt{Lab}_0^2$ to denote the type of relational formulae in RLS, because there is only one relational symbol used.

Recall the translation functions defined in Chapter 6, particularly the conventional translation of simply labelled sequents into a formulae in PSF from Definition 6.43 on page 147. Because the conventional translation of a simply labelled sequent is the same as a relational sequent with an empty relational context, the conventional translation from sequents in SLS into formulae and sequents in PSF can be extended to a translation of sequents in RLS into formulae and sequents in PSF. We add the function $\mathtt{psf}_\dagger$ from relational formulae in $\mathtt{Lab}_0^2$ to $\mathcal{R}$-formulae in PSF:

$$\mathtt{psf}_\leq\ x \leq y =_{def} \mathcal{R}\hat{x}\hat{y} \tag{119}$$

and use that to extend the conventional translation to cover relational sequents.

Definition 8.2. Recall the definition of $\mathtt{shield}$ from Definition 6.43 on page 147. We give a translation from RLS to PSF:

$$\mathtt{psf}_{\leq}\ (\underline{\Gamma} \Rightarrow \underline{\Delta}) =_{def} \forall \bar{y}.((\mathtt{shield}\ \bar{x}\ \bar{y}\ \bigwedge_1 [\bar{y}/\bar{x}]((\mathtt{psf}_{\leq}\ \circledast\ \Sigma)) \wedge (\mathtt{psf}_{\dagger}\ \circledast\ \underline{\Gamma})))$$
$$\supset \bigvee_1 [\bar{y}/\bar{x}](\mathtt{psf}_{\dagger}\ \circledast\ \underline{\Delta})) \tag{120}$$

where $\bar{x} \in \mathtt{Term}^{+}$ is a vector of parameters that corresponds to $\mathtt{lab}(\Sigma, \underline{\Gamma}, \underline{\Delta})$.

Using the root rule of **G3c/PSF***, a simpler translation of labelled sequents can be defined:

Definition 8.3. An alternative translation form RLS to $\mathsf{SPSF}^*{}_{\leq}$ is:

$$\mathtt{psf}'_{\leq}\ \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta} =_{def} (\mathtt{psf}_{\leq}\ \circledast\ \Sigma), (\mathtt{psf}_{\dagger}\ \circledast\ \underline{\Gamma}) \Rightarrow (\mathtt{psf}_{\dagger}\ \circledast\ \underline{\Delta}) \tag{121}$$

Lemma 8.4. **G3c/PSF*** $\vdash \mathtt{psf}_{\leq}\ \Sigma, \underline{\Gamma} \Rightarrow \underline{\Delta}$ *iff* **G3c/PSF*** $\vdash \mathtt{psf}'_{\leq}\ \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$.

Proof. Similar to Lemma 6.46 on page 149. □

We note that the inverse function $\mathtt{psf'}^{-1}_{\leq}$ is straightforward to define:

Definition 8.5. We define a function from $\mathcal{R}$-formulae in PSF to a relational formula in Lab^2_0:

$$\mathtt{psf}^{-1}_{\leq}\ \mathcal{R}\hat{x}\hat{y} =_{def} x \leq y \tag{122}$$

noting the bijection between a label x and the first-order parameter $\hat{x}$. We then define a function from PSF to simple relational sequents:

$$\mathtt{psf'}^{-1}_{\leq}\ \hat{\Sigma}, \Gamma \Rightarrow \Delta =_{def} (\mathtt{psf}^{-1}_{\leq}\ \circledast\ \hat{\Sigma}), (\mathtt{psf}^{-1}_{\dagger}\ \circledast\ \Gamma) \Rightarrow (\mathtt{psf}^{-1}_{\dagger}\ \circledast\ \Delta) \tag{123}$$

where $\hat{\Sigma} = \mathcal{R}\hat{x}_1\hat{y}_1, \ldots, \mathcal{R}\hat{x}_n\hat{y}_n$.

Lemma 8.6. $\mathtt{psf'}^{-1}_{\dagger}\ (\mathtt{psf'}_{\dagger}\ \underline{S}) = \underline{S}$.

Proof. Straightforward. □

We now show that labelled sequents correspond to relational sequents with empty relational contexts. Recall that although they have the same representation, the general semantics of labelled sequents are disjunctive, whereas the general semantics of relational sequents are conventional.

COROLLARY 8.7 (Simple Correspondence). *Let* $\Gamma_1^{x_1},\ldots,\Gamma_n^{x_n} \Rightarrow \Delta_1^{x_1},\ldots,\Delta_n^{x_n}$ *be a simply labelled sequent. Then* $\mathbf{G3c/PSF} \vdash \mathtt{psf}_\dagger\, \Gamma_1^{x_1},\ldots,\Gamma_n^{x_n} \Rightarrow \Delta_1^{x_1},\ldots,\Delta_n^{x_n}$ *iff* $\mathbf{G3c/PSF} \vdash \mathtt{psf}_\leq\, \Gamma_1^{x_1},\ldots,\Gamma_n^{x_n} \Rightarrow \Delta_1^{x_1},\ldots,\Delta_n^{x_n}$. *That is,* $\Gamma_1^{x_1},\ldots,\Gamma_n^{x_n} \Rightarrow \Delta_1^{x_1},\ldots,\Delta_n^{x_n}$ *is also a relational sequent with an empty relational context.*

Proof. Follows from Theorem 6.44 on page 148. □

THEOREM 8.8. $\mathbf{G3I}^* \vdash \Sigma;\underline{S}$ *iff* $\mathbf{G3c/PSF}^* \vdash \mathtt{psf}'_\leq\, \Sigma;\underline{S}$.

Proof. By induction on the derivation depths. We note that the translation of the axiom schema of $\mathbf{G3I}^*$ into $\mathsf{SPSF}^*{}_\leq$ is derivable in $\mathbf{G3c/PSF}^*$ by Lemma 3.68 on page 64, and that the $\mathsf{L}{\supset}_\leq$ and $\mathsf{R}{\supset}_\leq$ rules of $\mathbf{G3I}^*$ correspond to the $\mathsf{L}\forall{\supset}$ and $\mathsf{R}\forall{\supset}$ rules of $\mathbf{G3c/PSF}^*$. The cases for the other rules are straightforward. □

8.3. Rules for Manipulating Relations in G3c/PSF*

In order to extend the correspondence between simply labelled sequents and relational sequents to cover all relational sequents, we first need to show that various rules for manipulating relational formulae are sound. This is done semantically, by introducing rules for manipulating $\mathcal{R}$-formulae formulae in $\mathbf{G3c/PSF}^*$ (discussed in Section 3.5.3 on page 60). These rules have corresponding rules in relational sequent calculi (shown later in this chapter).

For the proofs below, it is assumed that the sequents are in $\mathsf{SPSF}^*{}_\leq$, that is, there are no $\mathcal{R}$-formulae in the succedents. (Recall the proofs of the persistence lemmas in Section 3.5.4 on page 64.)

The merging rules shown below allow for cycles to be eliminated, at the expense of combining parameters (which correspond to labels):

PROPOSITION 8.9 (Simple Cycle Merge). *The* cyc_2 *(simple cycle merging) rule*

$$\frac{\mathcal{R}xy, \mathcal{R}yx, \Gamma \Rightarrow \Delta}{[y/x]\Gamma \Rightarrow [y/x]\Delta}\ (\mathsf{cyc}_2)$$

where y *is free for* x *in* Γ, Δ, *is derivable in* $\mathbf{G3c/PSF}^*$.

Proof. Straightforward. (The proof is given in Proposition B.10 on page 246.) □

COROLLARY 8.10 (Cycle Merge). *The **cycle merging** rule*

$$\frac{\mathcal{R}x_1x_2,\ldots,\mathcal{R}x_{n-1}x_n,\mathcal{R}x_nx_1,\Gamma\Rightarrow\Delta}{[x_1/x_2]\ldots[x_1/x_n]\Gamma\Rightarrow[x_1/x_2]\ldots[x_1/x_n]\Delta}\ (\mathsf{cyc_n})$$

where x_1 is free for all x_i (for $1\le i\le n$), is admissible in **G3c/PSF***.

Proof. By induction on n. (The proof is given in Corollary B.12 on page 246.) □

PROPOSITION 8.11 (Extended Transitivity). *The extended transitivity rule*

$$\frac{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\ldots,\mathcal{R}y_nz,\mathcal{R}xz,\Gamma\Rightarrow\Delta}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\ldots,\mathcal{R}y_nz,\Gamma\Rightarrow\Delta}\ (\mathsf{trans_n})$$

where $n\ge 0$, is admissible in **G3c/PSF***.

Proof. Straightforward. (The proof is given in Proposition B.13 on page 247.) □

COROLLARY 8.12 (Extended Transitive Folding). *The **transitive folding** rules*

$$\frac{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\ldots,\mathcal{R}y_nz,A,[z/x]A,\Gamma\Rightarrow\Delta}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\ldots,\mathcal{R}y_nz,A,\Gamma\Rightarrow\Delta}\ (\mathsf{LF_\tau})$$

$$\frac{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\ldots,\mathcal{R}y_nz,\Gamma\Rightarrow A,[z/x]A,\Delta}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\ldots,\mathcal{R}y_nz,\Gamma\Rightarrow[z/x]A,\Delta}\ (\mathsf{RF_\tau})$$

where $n\ge 0$ and $A\in\mathsf{SPSF}$, are admissible in **G3c/PSF***.

Proof. Straightforward. (The proof is given in Corollary B.14 on page 247.) □

NOTATION 8.13 (Translation Pseudo-rules). We introduce the following "pseudo-rules" for translation between labelled or relational sequents and sequents in SPSF:

$$\frac{\underline{\Gamma}\Rightarrow\underline{\Delta}}{\hat{\Gamma}\Rightarrow\hat{\Delta}}\ \mathtt{psf}_\dagger \qquad \frac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta}}{\hat{\Sigma},\hat{\Gamma}\Rightarrow\hat{\Delta}}\ \mathtt{psf}_\preceq$$

where $\hat{\Gamma}=\mathtt{psf}_\dagger\circledast\underline{\Gamma}$, $\hat{\Delta}=\mathtt{psf}_\dagger\circledast\underline{\Delta}$ and $\hat{\Sigma}=\mathtt{psf}_\preceq\circledast\Sigma$.

Note that these are technically applications of the $\mathtt{psf}'_\dagger$ and $\mathtt{psf}'_\preceq$ rules.

TERMINOLOGY 8.14 (Semantic proof). A **semantic proof** of a relational sequent rule

$$\frac{\Sigma';\Gamma'\Rightarrow\Delta'}{\Sigma;\Gamma\Rightarrow\Delta}\ \rho_\preceq$$

in a calculus for **Int***/Geo denotes a proof using a corresponding rule ρ in **G3c/PSF***:

$$\dfrac{\dfrac{\dfrac{\Sigma';\Gamma' \Rightarrow \Delta'}{\hat{\Sigma},\hat{\Gamma}' \Rightarrow \hat{\Delta}'}\,\mathtt{psf}_{\leq}}{\hat{\Sigma},\hat{\Gamma} \Rightarrow \hat{\Delta}}\,\rho}{\Sigma;\Gamma \Rightarrow \Delta}\,\mathtt{psf}_{\leq}^{-1}$$

8.4. Sequent Folding

With the relational rules, the correspondence can be extended to cover relational sequents that do not have empty relational contexts. Here we present several techniques for translating between simply labelled sequents and relational sequents.

This first technique, called **sequent folding**, is straightforward: a labelled sequent is treated as a relational sequent with an empty relational context—allowable because of the simple correspondence (Corollary 8.7 on page 168). LW $\leq$ is is used to add relational formulae to the context, and the transitive folding rules are applied.

TERMINOLOGY 8.15 (Equivalence Class). Unless otherwise noted, equivalence classes of labelled or relational sequents will be the equivalence classes of those sequents determined by the equivalence relation $\approx$ (equivalence modulo permutation of labels).

We note that the rules proven semantically above have corresponding relational rules:

PROPOSITION 8.16 (Relational Formula Weakening). *The* LW $\leq$ *rule*

$$\frac{\Sigma;\underline{S}}{x \leq y, \Sigma;\underline{S}}\ \mathsf{LW}\leq$$

is sound for relational sequents in **Int***/Geo.

Proof. Semantically, using the LW rule in **G3c/PSF*** to add $\mathcal{R}$-formulae. □

PROPOSITION 8.17 (Extended Transitivity). *Extended transitivity*

$$\frac{x \leq y_1 \leq \ldots \leq y_n \leq z, x \leq z, \Sigma;\underline{S}}{x \leq y_1 \leq \ldots \leq y_n \leq z, \Sigma;\underline{S}}\ (\mathsf{trans_n})$$

where $n \geq 0$, *is sound for relational sequents in* **Int***/Geo.

Proof. Semantically from Proposition 8.11 on the preceding page. □

PROPOSITION 8.18. *Transitive folding*

$$\frac{x \leq y_1 \leq \ldots \leq y_n \leq z, \Sigma; A^x, A^z, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y_1 \leq \ldots \leq y_n \leq z, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}} \ (\mathsf{L} \leq_\tau) \qquad \frac{x \leq y_1 \leq \ldots \leq y_n \leq z, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^z}{x \leq y_1 \leq \ldots \leq y_n \leq z, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^z} \ (\mathsf{R} \leq_\tau)$$

where $n \geq 0$*, is sound for relational sequents in* **Int***/Geo.

Proof. Semantically from Corollary 8.12 on page 169. □

REMARK 8.19. Note that the $\mathsf{L} \leq_\tau$ rule for $n = 1$ corresponds with how the accessibility relation is defined for Hintikka frames in completeness proofs of calculi for **Int**, e.g. [Fit69].

REMARK 8.20. The proof of Proposition 8.17 can be directly translated to a relational calculus such as **G3I*** by using the $\mathsf{LW} \leq$ and trans rules. With that, the proof of Proposition 8.18 can be directly translated by using the $\mathsf{LW} \leq$ and $\mathsf{L} \leq$ or $\mathsf{R} \leq$ rules (the latter of which are admissible in **G3I*** by Lemma 5.100 on page 124).

LEMMA 8.21 (Sequent Folding Correspondence). *Let* $\underline{\Gamma}' \Rightarrow \underline{\Delta}' \in \mathsf{SLS}$ *Then there exists* $\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta} \in \mathsf{RLS}$ *such that* $\underline{\Gamma} \subset \underline{\Gamma}'$ *and* $\underline{\Delta} \subset \underline{\Delta}'$*, and if* $\vdash \underline{\Gamma}' \Rightarrow \underline{\Delta}'$*, then* $\vdash \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$.

Proof. $\underline{\Gamma}' \Rightarrow \underline{\Delta}' \in \mathsf{RLS}$, trivially. Also by derivation, using the $\mathsf{LW} \leq$, $\mathsf{L} \leq_\tau$ and $\mathsf{R} \leq_\tau$ rules. □

EXAMPLE 8.22.

$$\cfrac{\cfrac{\cfrac{\Gamma^x, \Gamma^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, \Delta^y}{x \leq y; \Gamma^x, \Gamma^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, \Delta^y} \ (\mathsf{LW} \leq)}{x \leq y; \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, \Delta^y} \ (\mathsf{L} \leq_\tau)^*}{x \leq y; \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^y} \ (\mathsf{R} \leq_\tau)^*$$

8.4.1. Grounded Sequents. Although the application of the transitive folding rules to a relational formula such as $x \leq y$ requires that slice y be a subset modulo permutation of slice x in the antecedent (or succedent), one can easily use weakening to reverse the subset relation between slices and apply the folding rules to the reverse relation $y \leq x$. In other words, given any two labels x, y in a labelled sequent, either folding can be applied.

One way of approaching the non-deterministic relationship between labelled sequents and relational sequents is to use a notion of a "normal form" of relational sequent, called **grounded (relational) sequent**, introduced below in Definition 8.23 on the next page.

Definition 8.23 (Grounded Sequent). Let $\Sigma;\underline{\Gamma} \Rightarrow \underline{\Delta}$ be a relational sequent. $\Sigma;\underline{\Gamma} \Rightarrow \underline{\Delta}$ is a **grounded sequent** iff all of the following hold:

(1) $\Sigma,\underline{\Gamma} \Rightarrow \underline{\Delta}$ is *connected*;
(2) $\underline{\Gamma} \cup \underline{\Delta} \neq \emptyset$;
(3) $x \leq x \notin \Sigma$;
(4) if $x \leq y, y \leq z \in \Sigma^+$, then $x \leq z \notin \Sigma$;
(5) if $x \leq y \in \Sigma^+$ and $A^x \in \underline{\Gamma}$, then $A^y \notin \underline{\Gamma}$;
(6) if $x \leq y \in \Sigma^+$ and $A^y \in \underline{\Delta}$, then $A^x \notin \underline{\Delta}$;
(7) if $\Sigma = x \leq y, \Sigma'$, then $x \leq y \notin \Sigma'$;
(8) if $\underline{\Gamma} = A^x, \underline{\Gamma}'$, then $A^x \notin \underline{\Gamma}'$;
(9) if $\underline{\Delta} = A^x, \underline{\Delta}'$, then $A^x \notin \underline{\Delta}'$.

We use the notation GRLS for the set of grounded sequents.

Remark 8.24. The definition of grounded sequent is a natural one. Conditions (3) through (9) require that there by no extraneous formulae in the sequent with respect to contraction, transitivity or folding. (This notion is applicable to logics that admit contraction, reflexivity and transitivity. Substructural or subintuitionistic logics would require a different definition.)

Proposition 8.25. $\mathsf{GRLS} \subset \mathsf{RLS}_a$.

Proof. By conditions 3 and 4 (where $x = z$) of Definition 8.23. □

Proposition 8.26. *Let* $\Sigma;\underline{S} \in \mathsf{GRLS}$. *Then either* $\mathtt{lab}(\underline{\Gamma},\underline{\Delta}) \subseteq \mathtt{lab}(\Sigma)$, *or* $\mathtt{lab}(\underline{\Gamma},\underline{\Delta})$ *is a singleton and* $\Sigma = \emptyset$.

Proof. Follows from condition 1 of Definition 8.23. □

Given a labelled sequent, we can obtain a grounded relational sequent:

Notation 8.27 (Sorted List). Let X be a finite poset. $\overrightarrow{X}$ denotes a sorted list of the elements in X. $\overleftarrow{X}$ denotes the reversed list of $\overrightarrow{X}$.

Remark 8.28. Any finite poset can be sorted by constructing a linear extension of it [Szp30]. One can also consider a poset as an acyclic digraph and use a topological sorting algorithm, e.g. [Kah62]. Note that cyclic digraphs may also be sorted, e.g. [Nuu94].

Notation 8.29. Let $\overrightarrow{X} = x_1, \ldots, x_n$ be a sorted list. $x_i \leqq x_j \in \overrightarrow{X}$ denotes that $x_i, x_j \in X$ and $i \leq j$. Informally, x_i occurs before x_j in $\overrightarrow{X}$ or is equal to x_j.

Definition 8.30 (Sorted List of Relations). Let Σ be an acyclic multiset of relational formulae, and let $\overrightarrow{V}$ be the sorted list of labels in Σ, ordered by the relational formulae in Σ.

Let $\overrightarrow{\Sigma}$ be the lexicographically sorted list of relational formulae in Σ, ordered by the labels in $\overrightarrow{V}$. That is, $(a \leq c) \leqq (b \leq d) \in \overrightarrow{\Sigma}$ iff $a \leqq b \in \overrightarrow{V}$, and $(a \leq c) \leqq (a \leq d) \in \overrightarrow{\Sigma}$ iff $c \leqq d \in \overrightarrow{V}$. $\overleftarrow{\Sigma}$ denotes the reverse of $\overrightarrow{\Sigma}$.

Lemma 8.31. *Let* $\underline{S}' \in \mathsf{SLS}$. *If* $\vdash \underline{S}'$, *then then there exists* $\Sigma; \underline{S} \in \mathsf{GRLS}$ *such that from* $\vdash \underline{S}'$, $\vdash \Sigma; \underline{S}$.

Proof. By derivation, using the following procedure:

(1) Apply the rules LC and RC until conditions 8 and 9 above are met.
(2) Apply the EC rule to remove equivalent slices (and non-determinism).
(3) Let the resulting sequent be $\underline{S}''$. For all labels $x_1, \ldots, x_n \in \mathtt{lab}(\underline{S}'')$, use $\mathsf{LW} \leq$ to add the context formula $w \leq x_1, \ldots, w \leq x_n$, where $w \# \underline{S}''$. This adds a common root label, and meets condition 1.
(4) Let $\underline{S}'' = \underline{\Gamma} \Rightarrow \underline{\Delta}$. For all pairs of labels $x, y \in \mathtt{lab}(\underline{S}'')$, whenever both $\underline{\Gamma} /\!/ x \subsetneqq \underline{\Gamma} /\!/ y$ and $\underline{\Delta} /\!/ y \subsetneqq \underline{\Delta} /\!/ x$, use $\mathsf{LW} \leq$ to add formulae $x \leq y$ to the relational context.
(5) Apply the $\mathsf{trans_n}$ rule, where applicable, to meet condition 4.
(6) For each relation $x \leq y$ in the list $\overleftarrow{\Sigma}$, if whenever $\underline{\Gamma} /\!/ x \subsetneqq \underline{\Gamma} /\!/ y$, apply the $\mathsf{L} \leq_\tau$ rule until condition 5 is met. (Note that folding is applied in the order that relational formulae occur in $\overleftarrow{\Sigma}$.)
(7) For each relation $x \leq y$ in the list $\overrightarrow{\Sigma}$, if whenever $\underline{\Delta} /\!/ y \subsetneqq \underline{\Delta} /\!/ x$, apply the $\mathsf{R} \leq_\tau$ rule until condition 6 is met. (Note that folding is applied in the order that relational formulae occur in $\overrightarrow{\Sigma}$.)

□

Proposition 8.32. *The procedure given in Lemma 8.31 results in a grounded sequent.*

Proof. Straightforward. □

PROPOSITION 8.33. *The procedure given in Lemma 8.31 on the previous page terminates.*

Proof. By induction on the size of the sequent. □

REMARK 8.34. Note that the procedure given in Lemma 8.31 on the preceding page assumes that the labelled sequent and its resulting grounded sequent are derivable. So the notion them having the same model or countermodel is irrelevant.

EXAMPLE 8.35. *From the labelled sequent* $(A \vee B)^x, (A \vee B)^y \Rightarrow A^x, B^x, B^y$*, we can derive a grounded sequent:*

$$\dfrac{\dfrac{\dfrac{(A \vee B)^x, (A \vee B)^y \Rightarrow A^x, B^x, B^y}{w \leq x, w \leq y, x \leq y; (A \vee B)^x, (A \vee B)^y \Rightarrow A^x, B^x, B^y}\ (\mathsf{LW} \leq)^+}{w \leq x, w \leq y, x \leq y; (A \vee B)^x \Rightarrow A^x, B^x, B^y}\ (\mathsf{L} \leq_\tau)}{w \leq x, w \leq y, x \leq y; (A \vee B)^x \Rightarrow A^x, B^y}\ (\mathsf{R} \leq_\tau)$$

COROLLARY 8.36 (Unique Grounding). *Let* $\underline{S}' \in \mathsf{SLS}$*. If* $\vdash \underline{S}'$*, then then there exists a unique equivalence class of sequents modulo permutation of labels* $\Sigma; \underline{S} \in \mathsf{GRLS}$ *such that* $\vdash \Sigma; \underline{S}$.

Proof. From Lemma 8.31 on the previous page, there exists a sequent $\Sigma; \underline{S} \in \mathsf{GRLS}$ such that $\vdash \Sigma; \underline{S}$. Suppose that there are two such sequents, $\Sigma_1; \underline{S}_1$ and $\Sigma_2; \underline{S}_2$.

Let $\underline{S}_1'' \in \mathsf{SLS}$ and $\underline{S}_2'' \in \mathsf{SLS}$ be the resulting sequents after applying contractions (steps 1 and 2). The application of the LC and RC rules yield the same result, whereas the application of EC allows for a choice of principal labels. However, EC can only be applied to labels x and y iff $\underline{S}' /\!/ x \approx \underline{S}' /\!/ y$, so by induction on the number of instances of EC, $\underline{S}_1'' \approx \underline{S}_2''$.

The choice of a root label is not fixed, but by virtue of the sequents being equivalent modulo permutation of labels, the relational contexts resulting from adding root labels in step 3 will be equivalent modulo permutation.

The set of relational formulae added in step 4 will also be equivalent modulo permutation of labels, by virtue of both sequents having the an equivalent modulo permutation of labels set of slices.

The application of trans_n and transitive folding rules will not affect the equivalence of sequents, because for each formula removed from one sequent, the corresponding formula (modulo permutation of labels) will be removed from the other.

□

From Corollary 8.36 on the facing page, given an equivalence class of labelled sequents, there is a unique equivalence class of grounded relational sequents. However, due to contraction, there are multiple equivalence classes of labelled sequents that map to the same equivalence class of grounded sequents, so the relationship between equivalence classes is not a bijection.

An area of future work is to define a notion of equivalence for sequents modulo the duplication of formulae (and slices). We conjecture that the composition of that equivalence relation with equivalence modulo permutation of labels allows for a bijection between members of that equivalence class and the equivalence class of their grounded relational sequents.

8.5. Transitive Unfolding

Here we introduce a technique called "transitive unfolding", for translating arbitrary relational sequents into linearly equivalent simply labelled sequents—that is, the simply labelled sequents obtained by this technique have the same *linear* models and countermodels as the relational sequents. What this means is that while the relational sequents may derivable be in a weaker logic such as $\mathbf{Int}_{\leq}$, for some special cases the corresponding labelled sequents are will be derivable in **GD**. (Those cases will be discussed with regards to the translation of proofs later in Chapter 9 on page 191.)

Definition 8.37 (Transitive Unfolding of Acyclic Sequents). Let $\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta}\in\mathtt{RLS}_a$. The **left and right transitive unfolding functions** from $\mathtt{Lab}_0^2\times(\mathtt{Prop}\times\mathtt{Lab}_0)^*$ to $\mathtt{Prop}^*$ are defined below:

$$\mathtt{LU}\,\overrightarrow{\Sigma}\,\underline{\Gamma} =_{def} \begin{cases} \underline{\Gamma} & \text{if } \overrightarrow{\Sigma} = \emptyset \\ \mathtt{LU}\,\overrightarrow{\Sigma}'\,(\underline{\Gamma}\cup[y/x](\underline{\Gamma}\,/\!\!/\,x)) & \text{if } \overrightarrow{\Sigma} = x\leq y, \overrightarrow{\Sigma}' \end{cases} \tag{124}$$

$$\mathtt{RU}\ \overleftarrow{\Sigma}\ \underline{\Delta} =_{def} \begin{cases} \underline{\Delta} & \text{if } \overleftarrow{\Sigma} = \emptyset \\ \mathtt{RU}\ \overleftarrow{\Sigma'}\ (\underline{\Delta} \cup [x/y](\underline{\Delta} \mathbin{/\!\!/} y)) & \text{if } \overleftarrow{\Sigma} = \overleftarrow{\Sigma'}, x \preceq y \end{cases} \tag{125}$$

from these functions, the **transitive unfolding of an acyclic sequent** from RLS_a to SLS is defined as:

$$(\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta})^{\bullet} =_{def} (\mathtt{LU}\ \overrightarrow{\Sigma}\ \underline{\Gamma}) \Rightarrow (\mathtt{RU}\ \overleftarrow{\Sigma}\ \underline{\Delta}) \tag{126}$$

Note that the $^{\bullet}$ operator is not defined for multisets of labelled formulae. However, (in a slight abuse of notation) it is sometimes used to indicate the side formulae resulting from the transitive unfolding of a sequent, e.g. $A^{x_1}, \ldots, A^{x_n}, \underline{\Gamma}^{\bullet}$.

We note that there are many ways of sorting a relational context. For example, $x \preceq y, y \preceq z, y \preceq w$ can be sorted as $xyzw$ or $xywz$. However, the different permutations of sorting do not affect transitive unfolding:

Proposition 8.38. *Let Σ be an acyclic multiset of relational formulae, where $\overrightarrow{\Sigma}$ has n permutations of sorting, $\overrightarrow{\Sigma}_1, \ldots, \overrightarrow{\Sigma}_n$, and let $\underline{\Gamma}, \underline{\Delta}$ be multisets of labelled formulae. Then* $\mathtt{LU}\ \overrightarrow{\Sigma}_1\ \underline{\Gamma} = \ldots = \mathtt{LU}\ \overrightarrow{\Sigma}_n\ \underline{\Gamma}$ *and* $\mathtt{RU}\ \overleftarrow{\Sigma}_1\ \underline{\Delta} = \ldots = \mathtt{RU}\ \overleftarrow{\Sigma}_n\ \underline{\Delta}$.

Proof. By induction on the size of $\mathtt{lab}(\Sigma, \underline{\Gamma}, \underline{\Delta})$. Note that any two permutations $\overrightarrow{\Sigma}_i$ and $\overrightarrow{\Sigma}_j$, of sorting Σ, will differ by the following cases for the $\mathtt{LU}$ function, where $\underline{\Gamma}^{\bullet}_i = \mathtt{LU}\ \overrightarrow{\Sigma}_i\ \underline{\Gamma}$ and $\underline{\Gamma}^{\bullet}_j = \mathtt{LU}\ \overrightarrow{\Sigma}_j\ \underline{\Gamma}$.

(1) In $\overrightarrow{\Sigma}_i$, $(w \preceq y) \leqq (x \preceq y)$ whereas in $\overrightarrow{\Sigma}_j$, $(x \preceq y) \leqq (w \preceq y)$. By Definition 8.37 on the preceding page, $\underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} w \underset{\sim}{\subsetneq} \underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} y$ and $\underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} x \underset{\sim}{\subsetneq} \underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} y$; and $\underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} w \underset{\sim}{\subsetneq} \underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} y$ and $\underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} x \underset{\sim}{\subsetneq} \underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} y$.

So $\underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} y = \underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} y$, as both slices share the same subsets.

(2) In $\overrightarrow{\Sigma}_i$, $(w \preceq x) \leqq (w \preceq y)$ whereas in $\overrightarrow{\Sigma}_j$, $(w \preceq y) \leqq (w \preceq x)$. By Definition 8.37, $\underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} w \underset{\sim}{\subsetneq} \underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} x$ and $\underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} w \underset{\sim}{\subsetneq} \underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} y$; and $\underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} w \underset{\sim}{\subsetneq} \underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} x$ and $\underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} w \underset{\sim}{\subsetneq} \underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} y$.

So $\underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} y = \underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} y$, as both slices share the same subsets. (Similarly, $\underline{\Gamma}^{\bullet}_i \mathbin{/\!\!/} x = \underline{\Gamma}^{\bullet}_j \mathbin{/\!\!/} x$.)

The cases for the $\mathtt{RU}$ function are the reverse of the above cases.

□

Proposition 8.39. *Let* $\underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet = (\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta})^\bullet$, *and* $x \in \mathtt{lab}(\Sigma, \underline{\Gamma}, \underline{\Delta})$. *Then* $\underline{\Gamma} \mathbin{/\!/} x \subseteq \underline{\Gamma}^\bullet \mathbin{/\!/} x$ *and* $\underline{\Delta} \mathbin{/\!/} x \subseteq \underline{\Delta}^\bullet \mathbin{/\!/} x$.

Proof. From Definition 8.37. □

Corollary 8.40. *Let* $\underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet = (\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta})^\bullet$. *Then* $\underline{\Gamma} \subseteq \underline{\Gamma}^\bullet$ *and* $\underline{\Delta} \subseteq \underline{\Delta}^\bullet$.

Proof. From Proposition 8.39. □

Theorem 8.41 (Semantic Correspondence). *Let* $\mathfrak{M} \in \mathcal{K}_{\mathbf{GD}}$, *and let* $\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta} \in \mathsf{RLS}_a$. $\mathfrak{M} \models \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$ *iff* $\mathfrak{M} \models (\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta})^\bullet$.

Proof. Recall that $\mathfrak{M} = \langle W, R, v \rangle$, and that $\mathfrak{M} \models \bigwedge\!\!\!\bigwedge \underline{\Gamma}$ iff for all $A^x \in \underline{\Gamma}$, $x \in W$ and $x \Vdash A$; and $\mathfrak{M} \not\models \bigvee\!\!\!\bigvee \underline{\Gamma}$ iff for all $A^x \in \underline{\Gamma}$, $x \in W$ and $x \nVdash A$. Let $\underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet = (\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta})^\bullet$.

First note that accessibility relations are preserved. Suppose that $\mathfrak{M} \models \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet$. Then (by Definition 5.31 on page 109) for some label $x \in \mathtt{lab}(\underline{\Gamma}^\bullet, \underline{\Delta}^\bullet)$, $\mathfrak{M} \models \underline{\Gamma}^\bullet \mathbin{/\!/} x \Rightarrow \underline{\Delta}^\bullet \mathbin{/\!/} x$, which is equivalent to $\mathfrak{M} \not\models \bigvee\!\!\!\bigvee \underline{\Delta}^\bullet \mathbin{/\!/} x$ implies $\mathfrak{M} \not\models \bigwedge\!\!\!\bigwedge \underline{\Gamma}^\bullet \mathbin{/\!/} x$. Suppose $x \leq y \in \Sigma^+$. $\mathfrak{M} \not\models \bigvee\!\!\!\bigvee \underline{\Delta}^\bullet \mathbin{/\!/} x$ implies $\mathfrak{M} \not\models \bigvee\!\!\!\bigvee \underline{\Delta}^\bullet \mathbin{/\!/} y$, since $\underline{\Delta}^\bullet \mathbin{/\!/} y \precsim \underline{\Delta}^\bullet \mathbin{/\!/} x$, and $\mathfrak{M} \not\models \bigwedge\!\!\!\bigwedge \underline{\Gamma}^\bullet \mathbin{/\!/} x$, which implies $\mathfrak{M} \not\models \bigwedge\!\!\!\bigwedge \underline{\Gamma}^\bullet \mathbin{/\!/} y$, since $\underline{\Gamma}^\bullet \mathbin{/\!/} x \precsim \underline{\Gamma}^\bullet \mathbin{/\!/} y$. So $\mathfrak{M} \models \underline{\Gamma}^\bullet \mathbin{/\!/} y \Rightarrow \underline{\Delta}^\bullet \mathbin{/\!/} y$.

From left-to-right (by contraposition): suppose $\mathfrak{M} \not\models \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet$. Then $\mathfrak{M} \models \bigwedge\!\!\!\bigwedge \underline{\Gamma}^\bullet$ and $\mathfrak{M} \not\models \bigvee\!\!\!\bigvee \underline{\Delta}^\bullet$. From Corollary 8.40, $\mathfrak{M} \models \bigwedge\!\!\!\bigwedge \underline{\Gamma}$ and $\mathfrak{M} \not\models \bigvee\!\!\!\bigvee \underline{\Delta}$. So $\mathfrak{M} \models \underline{\Gamma} \Rightarrow \underline{\Delta}$, and since relations are preserved $\mathfrak{M} \models \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$.

From right-to-left (again by contraposition): suppose $\mathfrak{M} \not\models \underline{\Gamma} \Rightarrow \underline{\Delta}$. Then $\mathfrak{M} \models \bigwedge\!\!\!\bigwedge \underline{\Gamma}$ and $\mathfrak{M} \not\models \bigvee\!\!\!\bigvee \underline{\Delta}$. But by persistence, $\mathfrak{M} \models \bigwedge\!\!\!\bigwedge \underline{\Gamma}^\bullet$ and $\mathfrak{M} \not\models \bigvee\!\!\!\bigvee \underline{\Delta}^\bullet$. That is, if every $A^x \in \underline{\Gamma}$ is true in $\mathfrak{M}$, then for every y such that $x \leq y$, A^y (which is in $\underline{\Gamma}^\bullet$) is true in $\mathfrak{M}$. Likewise, if every $B^y \in \underline{\Delta}$ is false in $\mathfrak{M}$, then for every x such that $x \leq y$, A^x (which is in $\underline{\Delta}^\bullet$) is false in $\mathfrak{M}$.

Therefore, $\mathfrak{M} \models \hat{\Sigma}, \underline{\Gamma} \Rightarrow \underline{\Delta}$, iff $\mathfrak{M} \models \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet$.

However, $\mathfrak{M}$ is stricter than a standard intuitionistic Kripke model (Definition 3.10 on page 45), as it is also linear, i.e. $\forall x, y \in W$, either $(x, y) \in R$ or $(y, x) \in R$. Suppose $\mathfrak{M} \not\models \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet$. Then for all $x \in \mathtt{lab}(\underline{\Gamma}^\bullet, \underline{\Delta}^\bullet)$, $\mathfrak{M} \not\models \underline{\Gamma}^\bullet \mathbin{/\!/} x \Rightarrow \underline{\Delta}^\bullet \mathbin{/\!/} x$, that is, $\mathfrak{M} \models \bigwedge\!\!\!\bigwedge \underline{\Gamma}^\bullet \mathbin{/\!/} x$ and $\mathfrak{M} \not\models \bigvee\!\!\!\bigvee \underline{\Delta}^\bullet \mathbin{/\!/} x$. Let $x, y \in \mathtt{lab}(\underline{\Gamma}^\bullet, \underline{\Delta}^\bullet)$ be labels such that $x \leq y \in \Sigma^+$. Recall that $\underline{\Gamma} \mathbin{/\!/} x \precsim \underline{\Gamma}^\bullet \mathbin{/\!/} y$ and $\underline{\Gamma} \mathbin{/\!/} y \precsim \underline{\Gamma}^\bullet \mathbin{/\!/} y$, so $\mathfrak{M} \models \bigwedge\!\!\!\bigwedge (\underline{\Gamma} \mathbin{/\!/} x, \underline{\Gamma} \mathbin{/\!/} y)$. Similarly, recall that $\underline{\Delta} \mathbin{/\!/} x \precsim \underline{\Delta}^\bullet \mathbin{/\!/} x$ and $\underline{\Delta} \mathbin{/\!/} y \precsim \underline{\Delta}^\bullet \mathbin{/\!/} x$, so $\mathfrak{M} \not\models \bigvee\!\!\!\bigvee (\underline{\Delta} \mathbin{/\!/} x, \underline{\Delta} \mathbin{/\!/} y)$. Note that the same would be the case if $y \leq x \in \Sigma^+$ instead. □

EXAMPLE 8.42. *Note that the derivation of the relational sequent in* **G3I** *below does not use superintuitionistic relational rules:*

$$\dfrac{x \le y; A^x, (A \supset C)^y \Rightarrow B^x, C^y, A^y \qquad \dfrac{y \le y, x \le y; C^y, A^x, (A \supset C)^y \Rightarrow B^x, C^y}{x \le y; C^y, A^x, (A \supset C)^y \Rightarrow B^x, C^y}\,\mathsf{refl}}{x \le y; A^x, (A \supset C)^y \Rightarrow B^x, C^y}\,\mathsf{L{\supset}_{\le}} \tag{127}$$

$$\dfrac{\vdots\,(127) \qquad \dfrac{x \le x, x \le y; B^x, (A \supset C)^y \Rightarrow B^x, C^y}{x \le y; B^x, (A \supset C)^y \Rightarrow B^x, C^y}\,\mathsf{refl}}{x \le y; (A \vee B)^x, (A \supset C)^y \Rightarrow B^x, C^y}\,\mathsf{L{\vee}}$$

However, the transitive unfolding of that sequent is only derivable in **LG3ipm**+$\mathsf{Com_m}$*:*

$$\dfrac{\dfrac{\dfrac{A^x, A^y, B^y \Rightarrow A^x, B^x, A^y \qquad A^x, B^x, B^y \Rightarrow B^x, A^y, B^y}{A^x, B^y \Rightarrow B^x, A^y}\,\mathsf{Com}}{A^x, B^y, (A \supset C)^y \Rightarrow B^x, C^x, C^y, A^y}\,\mathsf{(GW)} \qquad A^x, B^y, C^y \Rightarrow B^x, C^x, C^y}{A^x, B^y, (A \supset C)^y \Rightarrow B^x, C^x, C^y}\,\mathsf{L{\supset}} \tag{128}$$

$$\dfrac{A^x, A^y, (A \supset C)^y \Rightarrow B^x, C^x, C^y \qquad \overset{\vdots\,(128)}{B^x, (A \vee B)^y, (A \supset C)^y \Rightarrow B^x, C^x, C^y}}{(A \vee B)^x, (A \vee B)^y, (A \supset C)^y \Rightarrow B^x, C^x, C^y}\,\mathsf{L{\vee}^+}$$

Note that the $\mathsf{Com_m}$ *is only (but not always) required for proving sequents that have disjunctions in negative positions that are labelled with non-maximal labels. This will be discussed further in Chapter 9 on page 191.*

EXAMPLE 8.43. *The transitive unfolding of the relational sequent* $x \le y; (A \vee B)^x, C^y \Rightarrow A^x, B^y$ *is the labelled sequent* $(A \vee B)^x, (A \vee B)^y, C^y \Rightarrow A^x, B^x, B^y$*. (Note that* Com *not required to derive this sequent.)*

EXAMPLE 8.44. *Take the sequent* $w \le x, x \le z, w \le y, y \le z; A^x, \neg A^y \Rightarrow \bot^x, \bot^y$*. A possible topological sorting of the set of labels is wyxz,so a sorted list of the relational formula is* $w \le y, w \le x, y \le z, x \le z$*. The transitive unfolding is* $A^x, A^z, \neg A^y, \neg A^z \Rightarrow \bot^w, \bot^w, \bot^x, \bot^y$*. (Similarly note that* Com *not required to derive this sequent.)*

8.5.1. Transitive Unfolding and Cyclic sequents. Transitive unfolding can be extended to cyclic relational sequents by application of cycle merging rules. We first note some properties of cycle merging rules and the relationship between RLS_c and RLS_a.

PROPOSITION 8.45. *Cycle merging*

$$\frac{x_1 \le x_2, \ldots, x_{n-1} \le x_n, x_n \le x_1, \Sigma; \underline{S}}{[x_1/x_2] \ldots [x_1/x_n]\Sigma; [x_1/x_2] \ldots [x_1/x_n]\underline{S}} \ (\mathsf{cyc_n})$$

*where x_1 is called the **principal label** of the rule, is sound for relational sequents in* **Int***/Geo.

Proof. Semantically from Corollary 8.10 on page 168. □

TERMINOLOGY 8.46 (Cycle elimination). Let $\Sigma; \underline{S} \in \mathrm{RLS}_c$. The application of the $\mathsf{cyc_n}$ rule 1 or more times to derive the sequent $\Sigma'; \underline{S}' \in \mathrm{RLS}_a$ is called **cycle elimination**.

Note that the choice of the principal label to substitute for other labels in a cycle using the $\mathsf{cyc_n}$ rule is non-deterministic. However, the results of cycle elimination are in the same equivalence class formed by the $\approx$ relation:

LEMMA 8.47 (Cycle elimination equivalence). *Let $\Sigma'; \underline{S}' \in \mathrm{RLS}_c$, and let $\Sigma_1; \underline{S}_1 \in \mathrm{RLS}_a$ and $\Sigma_2; \underline{S}_2 \in \mathrm{RLS}_a$, derived from $\Sigma'; \underline{S}'$ by cycle elimination. Then $\Sigma_1; \underline{S}_1 \approx \Sigma_2; \underline{S}_2$.*

Proof. By observing that the $\mathsf{cyc_n}$ rule only eliminates the relational formulae that are in the cycle formed by the active formulae, and that it substitutes all labels in the rest of the sequent that are part of that cycle by a single label (i.e., it merges the labels). □

LEMMA 8.48. *Let $\Sigma_1; \underline{S}_1 \in \mathrm{RLS}_a$ and $\Sigma_2; \underline{S}_2 \in \mathrm{RLS}_a$.*
If $\Sigma_1; \underline{S}_1 \approx \Sigma_2; \underline{S}_2$, then $(\Sigma_1; \underline{S}_1)^\bullet \approx (\Sigma_2; \underline{S}_2)^\bullet$.

Proof. By induction on the size of Σ_1, using Lemma 5.118 on page 132. □

COROLLARY 8.49. *Let $\Sigma'; \underline{S}' \in \mathrm{RLS}_c$, and let $\Sigma_1; \underline{S}_1 \in \mathrm{RLS}_a$ and $\Sigma_2; \underline{S}_2 \in \mathrm{RLS}_a$, derived from applying the $\mathsf{cyc_n}$ rule one or more times to $\Sigma'; \underline{S}'$. Then $(\Sigma_1; \underline{S}_1)^\bullet \approx (\Sigma_2; \underline{S}_2)^\bullet$.*

Proof. From Lemmas 8.47 and 8.48. □

LEMMA 8.50. *Cycle merging is invertible:*

$$\frac{[x_1/x_2] \ldots [x_1/x_n]\Sigma; [x_1/x_2] \ldots [x_1/x_n]\underline{S}}{x_1 \le x_2, \ldots, x_{n-1} \le x_n, x_n \le x_1, \Sigma; \underline{S}} \ (\mathsf{cyc_n^{-1}})$$

Proof. Straightforward. (The proof is written out in Lemma B.15 on page 248.) □

Alternatively, we could extend the unfolding functions to cover cyclic sequents:

Definition 8.51 (Transitive Unfolding of Cyclic Relations). We extend the functions from Definition 8.37 on page 175 to cover cyclic relational contexts:

$$\begin{aligned}\mathtt{LU}\; x_1 \leq \ldots \leq x_n \leq x_1, \overrightarrow{\Sigma}\; \underline{\Gamma} =_{def} \mathtt{LU}\; \overrightarrow{\Sigma}\; \Gamma \cup ([x_1/x_2]\underline{\Gamma} \,/\!\!/\, x_2) \cup \ldots \cup ([x_1/x_n]\underline{\Gamma} \,/\!\!/\, x_n) \cup \ldots \\ \cup ([x_n/x_1]\underline{\Gamma} \,/\!\!/\, x_1) \cup \ldots \cup ([x_n/x_{n-1}]\underline{\Gamma} \,/\!\!/\, x_{n-1}) \\ \mathtt{RU}\; \overleftarrow{\Sigma}, x_1 \leq \ldots \leq x_n \leq x_1\; \underline{\Delta} =_{def} \mathtt{RU}\; \overleftarrow{\Sigma}\; \Delta \cup ([x_1/x_2]\underline{\Delta} \,/\!\!/\, x_2) \cup \ldots \cup ([x_1/x_n]\underline{\Delta} \,/\!\!/\, x_n) \cup \ldots \\ \cup ([x_n/x_1]\underline{\Delta} \,/\!\!/\, x_1) \cup \ldots \cup ([x_n/x_{n-1}]\underline{\Delta} \,/\!\!/\, x_{n-1})\end{aligned}$$

We similar extend the definition of the $^\bullet$ operator to cover cyclic sequents.

Lemma 8.52. *Let* $\Sigma;\underline{S} \in \mathrm{RLS}_c$*, and let* $\Sigma';\underline{S}' \in \mathrm{RLS}_a$ *be the result of applying cycle elimination to* $\Sigma;\underline{S}$*. Then* $(\Sigma;\underline{S})^\bullet$ *and* $(\Sigma';\underline{S}')^\bullet$ *are interderivable.*

Proof. From left-to-right, by applications of the EC rule. From right-to-left by applications of the EW rule. □

8.5.2. Size of Transitive Unfolded Sequents. Transitively unfolded sequents can be significantly larger. Let $\mathcal{G}$ be the graph that corresponds to a relational context Σ, such that $\mathcal{G}$ is a tree with a root x (corresponding to the minimum label), a maximum depth of n (corresponding to the longest chain of relations $x \leq y_1, y_1 \leq y_2, \ldots, y_{n-1} \leq y_n$ in Σ), and a maximum branching width of w (corresponding to the maximum number of relations with the same left label, $x \leq y_1, \ldots, x \leq y_w$). Now let the maximum size of any slice of the antecedent $\underline{\Gamma} \,/\!\!/\, x_i$ be m. The size of the unfolding antecedent has an upper bound of mw^n formulae. The upper bounds of unfolded succedents are similar.

8.5.3. On the Relationship between Formalisms. Below we discuss the relationship between the sets SLS and RLS_a.

Proposition 8.53. *Let* $\underline{S} \in \mathrm{SLS}$. $\underline{S}^\bullet = \underline{S}$.

Proof. Trivial. Recall that $\mathrm{SLS} \subset \mathrm{RLS}_a$. □

Corollary 8.54. *Let the following set be defined:*

$$\mathrm{RLS}_a^\bullet =_{def} \{\, (\Sigma;\underline{S})^\bullet : \Sigma;\underline{S} \in \mathrm{RLS}_a \,\} \tag{129}$$

The relationship between RLS_a *and* $\mathrm{RLS}_a^\bullet$ *determined by transitive unfolding is surjective and non-injective.*

Proof. $\mathrm{RLS}_a^\bullet \subseteq \mathrm{SLS}$, so it follows trivially from Proposition 8.53. However, suppose $(\Sigma;\underline{S}')^\bullet = \underline{S}$, where $\Sigma;\underline{S}' \neq \underline{S}$. But $\underline{S}^\bullet = \underline{S}$ as well. So the relationship is not injective. □

Corollary 8.55. $\mathrm{RLS}_a^\bullet = \mathrm{SLS}$.

Proof. Follows from Corollary 8.54. □

The proof of non-injectivity in Corollary 8.54 is trivial. We give a stronger proof that eliminates the trivial cases:

Lemma 8.56. *The relationship between* $\mathrm{RLS}_a \setminus \mathrm{SLS}$ *and* $\mathrm{RLS}_a^\bullet$ *determined by transitive unfolding is non-injective.*

Proof. By counterexample. Let $x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta} \in \mathrm{RLS}_a$, where y is maximal (i.e. $y \le y' \notin \Sigma$) and $z \# \Sigma$. Then $(x \le y \le z, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta})^\bullet = (x \le y, x \le z, \Sigma; \underline{\Gamma}, [x/z](\underline{\Gamma} /\!\!/ y) \Rightarrow \underline{\Delta})^\bullet$. Note that a dual counterexample can be constructed where y is minimal and the sequents differ similarly by their succedents. □

Example 8.57. *The transitive unfolding of the sequents* $x \le y \le z; A^x \Rightarrow A^y$ *and* $x \le y, x \le z; A^x \Rightarrow A^y$ *are both* $A^x, A^y, A^z \Rightarrow A^y$.

By Corollary 8.54 and Lemma 8.56, there are cases where two different relational sequents have the same transitive unfolding. It follows from this that there is no deterministic function for deriving the original relational sequent from its transitive unfolding. While subset relations between slices (modulo permutation of labels) can indicate accessibility relations, e.g. $\underline{\Gamma} /\!\!/ x \subsetsim \underline{\Gamma} /\!\!/ y$ corresponding to $x \le y$, the relationship is not one-to-one. For example, the sequent $A^x, A^y, A^z \Rightarrow A^z$ can be folded into two different relational sequents:

$$\dfrac{\dfrac{A^x, A^y, A^z \Rightarrow A^z}{x \le y \le z; A^x, A^y, A^z \Rightarrow A^z}\,(\mathsf{LW} \le)^+}{x \le y \le z; A^x \Rightarrow A^z}\,(\mathsf{L} \le_\tau)^+ \qquad \dfrac{\dfrac{A^x, A^y, A^z \Rightarrow A^z}{x \le y, x \le z; A^x, A^y, A^z \Rightarrow A^z}\,(\mathsf{LW} \le)^+}{x \le y, x \le z; A^x \Rightarrow A^z}\,(\mathsf{L} \le_\tau)^+$$

both of which have as their transitive unfolding $A^x, A^y, A^z \Rightarrow A^z$. This is not unexpected, since relational sequents are more expressive than labelled sequents. The ordering imposed on labels in a relational sequent $\Sigma; \underline{S}$ by Σ is only implicit in the transitive unfolding of that sequent, so slices that are equivalent modulo permutation are interchangeable.

Some of the equivalence of slices can be eliminated by modifying the definition of transitive unfolding, so that slices double in size by each unfolding, e.g.

$$\mathtt{LU}' \; \overrightarrow{\Sigma} \; \underline{\Gamma} =_{def} \begin{cases} \underline{\Gamma} & \text{if } \overrightarrow{\Sigma} = \emptyset \\ \mathtt{LU}' \; \overrightarrow{\Sigma}' \; (\underline{\Gamma} \cup [y/x](2 \cdot \underline{\Gamma} /\!\!/ x)) & \text{if } \overrightarrow{\Sigma} = x \leq y, \overrightarrow{\Sigma}' \end{cases}$$

$$\mathtt{RU}' \; \overleftarrow{\Sigma} \; \underline{\Delta} =_{def} \begin{cases} \underline{\Delta} & \text{if } \overleftarrow{\Sigma} = \emptyset \\ \mathtt{RU}' \; \overleftarrow{\Sigma}' \; (\underline{\Delta} \cup [x/y](2 \cdot \underline{\Delta} /\!\!/ y)) & \text{if } \overleftarrow{\Sigma} = x \leq y, \overleftarrow{\Sigma}' \end{cases}$$

This is allowable because contraction is admissible for **Int***/$\mathtt{Geo}$. However, this will not affect cases where the labels of slices have the same distance from a common label, e.g. $x \leq y \leq z, x \leq y' \leq z', y \leq w, A^x \Rightarrow A^x$ and $x \leq y \leq z, x \leq y' \leq z', y' \leq w, A^x \Rightarrow A^x$. One aspect about these sequents is that they differ by formulae which do not contribute to the validity of the sequent.

The notion of a grounded sequent (Definition 8.23 on page 172) is not helpful here. The grounded sequent of $A^x, A^y, A^z \Rightarrow A^z$ is $w \leq z, z \leq x, z \leq y, A^z \Rightarrow A^z$. But the transitive unfolding of that is $A^z, A^x, A^y \Rightarrow A^w, A^z$. The grounded sequent of that sequent is $w' \leq w, w \leq z, z \leq x, z \leq y, A^z \Rightarrow A^z$, and the transitive unfolding of that grounded sequent is $A^z, A^x, A^y \Rightarrow A^{w'}, A^w, A^z$. So there is no bijection between transitive unfoldings and grounded sequents.

This suggests that a stronger notion than grounding is needed, which eliminates formulae that not needed for the proof. However, it is not clear how to mechanically determine which formulae in a sequent are irrelevant, without first having a proof of the sequent. Even then, this is problematic: another counterexample to bijection between relational sequents and their transitive unfolding is with the sequents $x \leq y \leq z; A^y, (B \wedge A)^z \Rightarrow A^x$ and $x \leq y, x \leq z; A^y, (B \wedge A)^z \Rightarrow A^x$, both of which are derivable in **G3I*** for **Cl** using the sym rule—the formula $(B \wedge A)^z$ could be used in the proof instead of A^y:

$$\dfrac{\dfrac{\dfrac{\dfrac{x \le y, x \le z, z \le x; (B \wedge A)^z, B^x, A^x \Rightarrow A^x}{x \le y, x \le z, z \le x; (B \wedge A)^z, (B \wedge A)^x \Rightarrow A^x}\,\mathsf{L}\wedge}{x \le y, x \le z, z \le x; (B \wedge A)^z \Rightarrow A^x}\,(\mathsf{L} \le_\tau)}{x \le y, x \le z; (B \wedge A)^z \Rightarrow A^x}\,\mathsf{sym}}{x \le y, x \le z; A^y, (B \wedge A)^z \Rightarrow A^x}\,(\mathsf{LW})$$

Indeed, the problem of eliminating extraneous formulae changes the notion of a "normal" sequent into one that resembles the notion of a "normal" proof, which is a larger problem that is beyond the scope of this thesis.

What is also noteworthy about the counterexamples is that they have different corresponding graphs. One is linear, and the other is a tree with two branches. Defining an equivalence between such graphs appears to be counterintuitive.

Although the notion of a grounded sequent does not solve the above problems, we give some interesting results about the relationship between RLS_a and GRLS below:

THEOREM 8.58 (Grounding). *Let* $\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta} \in \mathsf{RLS}$ *Then there exists a sequent* $\Sigma'; \underline{\Gamma}' \Rightarrow \underline{\Delta}' \in \mathsf{GRLS}$ *such that* $\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$ *and* $\Sigma'; \underline{\Gamma}' \Rightarrow \underline{\Delta}'$ *are interderivable.*

Proof. By derivation. From left-to-right, suppose $\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$ is non-grounded.

(1) For all occurrences of $x \le x \in \Sigma$, apply the refl rule,

$$\dfrac{x \le x, \Sigma; \underline{S}}{\Sigma; \underline{S}}\,\mathsf{refl}$$

which is admissible semantically from the refl in **G3c/PSF***.

(2) If the sequent is not connected, then for all minimal labels $x_1, \dots, x_n$ use LW $\le$ to add relations $w \le x_1, \dots, w \le x_n$ to the relational context, where the label $w \notin \mathtt{lab}(\Sigma, \underline{\Gamma}, \underline{\Delta})$.

(3) If the sequent is acyclic, apply the $\mathsf{trans_n}$ rule where needed; otherwise apply instances of the $\mathsf{cyc_n}$ rule to eliminate cycles.

(4) Apply the LC $\le$, LC and RC rules as needed.

(5) Apply the L $\le_\tau$ and R $\le_\tau$ rules as needed.

For each of the above cases, the size of the conclusion of the rule is less than the size of the premiss. So this process terminates.

From right-to-left, use the root rule to eliminate relational formulae added to make the sequent connected. Otherwise use instances of weakening. To obtain a cyclic sequent, use the $\mathsf{cyc}_{\mathsf{n}}^{-1}$ rule. □

REMARK 8.59. Clearly grounding is not functional, because the root label added to make the sequent connected is not fixed.

COROLLARY 8.60. *Let* $\Sigma;\underline{\Gamma} \Rightarrow \underline{\Delta} \in \mathrm{RLS}_a$. *Then then there exists a unique equivalence class of sequents modulo permutation of labels* $\Sigma';\underline{\Gamma}' \Rightarrow \underline{\Delta}' \in \mathtt{GRLS}$ *such that* $\Sigma;\underline{\Gamma} \Rightarrow \underline{\Delta}$ *and* $\Sigma';\underline{\Gamma}' \Rightarrow \underline{\Delta}'$ *are interderivable.*

Proof. Assume that there are two different grounded sequents, $\Sigma_1',\underline{\Gamma}_1' \Rightarrow \underline{\Delta}_1'$ and $\Sigma_2',\underline{\Gamma}_2' \Rightarrow \underline{\Delta}_2'$ derivable using the procedure given in Theorem 8.58. There are the following cases:

(1) $\Sigma_1' \not\approx \Sigma_2'$. First note the following observations about the procedure for deriving a grounded sequent from Theorem 8.58:
   (a) Since Σ is acyclic, Σ_1' and Σ_2' are acyclic;
   (b) The only relational formulae that are added by the procedure are for adding a fresh root label w_1 or w_2, such that $\mathtt{lab}(\Sigma') = \{x_1, \ldots, x_n\}$, with $\Sigma_1' = w_1 \leq x_1, \ldots, w_1 \leq x_n, \Sigma_1''$ and $\Sigma_2' = w_2 \leq x_2, \ldots, w_2 \leq x_n, \Sigma_2''$, so $w_1 \leq x_1, \ldots, w_1 \leq x_n \approx w_2 \leq x_2, \ldots, w_2 \leq x_n$;
   (c) $\Sigma_1'' \subseteq \Sigma$ and $\Sigma_2'' \subseteq \Sigma$,
   (d) $\mathtt{lab}(\Sigma_1'') = \mathtt{lab}(\Sigma_2'') = \mathtt{lab}(\Sigma)$, with
   (e) the order of labels preserved—note that because Σ is acyclic, the refl rule is not applicable.

   Without loss of generality, assume that $x \leq y \in \Sigma_1'$ but $x \leq y \notin \Sigma_2'$. So $x \leq y$ was removed during the derivation of Σ_2' by an application of the $\mathsf{trans}_{\mathsf{n}}$ rule. Since labels and their order are preserved, $x \leq w_1 \leq \ldots \leq w_n \leq y \in \Sigma_2'$. But label and order preservation requires that each $x \leq^* w_i \in \Sigma_1'$, which is only possible if $y \leq^* w_i \in \Sigma_1'$—i.e., if Σ_1' is cyclic, which is a contradiction.
(2) $\underline{\Gamma}_1' \neq \underline{\Gamma}_2'$. Without loss of generality, assume that $A^x \in \underline{\Gamma}_1'$ but $A^x \notin \underline{\Gamma}_2'$. From case 1, $\Sigma_1' = \Sigma_2'$.But if both sequents have the same relational context, and thus the same minimum labels for each chain in the context, then the transitive folding rules

cannot produce different conclusions, unless the relational contexts are cyclic, which is a contradiction.

(3) $\underline{\Delta}_1' \neq \underline{\Delta}_2'$. Similar to case 2.

□

REMARK 8.61. By altering condition (4) of Definition 8.23 to disallow cases where $x = z$, some cyclic sequents would be considered grounded. However, the grounding procedure would not be functional for cyclic sequents. For example, the sequent $x \leq y, y \leq x; A^x, A^y \Rightarrow A^x$ could then be transformed into either of two different grounded sequents, $x \leq y, y \leq x; A^x \Rightarrow A^x$ and $x \leq y, y \leq x; A^y \Rightarrow A^x$. It's noteworthy that the composition of grounding with cycle elimination (and vice versa) would derive sequents that were in the same equivalence class determined by the $\approx$ relation.

LEMMA 8.62. *The relationship between* RLS *and* GRLS *determined by grounding procedure in Theorem 8.58 on page 183 is surjective and non-injective.*

Proof. Trivial, as $\mathsf{GRLS} \subset \mathsf{RLS}_a$. □

LEMMA 8.63. *The relationship between* GRLS *and* $\mathsf{RLS}_a^\bullet$ *determined by transitive unfolding is non-injective.*

Proof. Follows from Corollary 8.54 on page 181. Note that $(\mathsf{RLS}_a^\bullet \subseteq \mathsf{SLS}) \subset \mathsf{GRLS}$. The counterexample given in Lemma 8.56 on page 181 also applies. □

LEMMA 8.64. *The transitive unfolding of a sequent in* GRLS *is not necessarily in* GRLS.

Proof. Let $x \leq z, y \leq z, \Sigma, \underline{\Gamma} \Rightarrow \underline{\Delta} \in \mathsf{GRLS}$, where x, y are minimal, and $(\underline{\Gamma} \mathbin{/\!/} x) \cap [x/y](\underline{\Gamma} \mathbin{/\!/} y) \neq \emptyset$. Then $(x \leq z, y \leq z, \Sigma, \underline{\Gamma} \Rightarrow \underline{\Delta})^\bullet$ would violate condition (8) of Definition 8.23. □

8.6. Sequent Flattening

A simpler "shotgun approach" to deriving labelled sequents from relational sequents is to add symmetric relational formulae to the sequent (by weakening) and apply instances of the $\mathsf{cyc_n}$ rule to "flatten" the sequent by merging the labels in relational formulae.

PROPOSITION 8.65. *The* root *rule*

$$\frac{x \leq y_1, \ldots, x \leq y_n, \Sigma; \underline{S}}{\Sigma; \underline{S}}\ (\mathsf{root})$$

is sound for relational sequents in $\mathbf{Int}^*/\mathsf{Geo}$.

Proof. Semantically from root rule in $\mathbf{G3c/PSF}^*$. □

LEMMA 8.66. *The flattening rule*

$$\frac{x_1 \le y_1, \ldots, x_n \le y_n; \underline{S}}{[x_1/y_1] \ldots [x_n/y_n]\underline{S}} \text{ (flat)}$$

is sound for relational sequents in $\mathbf{Int}^*/\mathsf{Geo}$.

Proof. By derivation. Use instances of refl, root or $\mathsf{cyc_n}$ rules, where applicable. Otherwise, for each relational formula $x_i \le y_i \in \Sigma$, use $\mathsf{LW} \le$ to add $y_i \le x_i$ and then apply the $\mathsf{cyc_2}$ rule. (For the refl rule the substitution is trivial. For the root rule, this corresponds to substitution.) □

EXAMPLE 8.67. *The relational sequent* $x \le y; (A \vee B)^x, C^y \Rightarrow A^x, B^y$ *can be translated into a valid labelled sequent:*

$$\frac{x \le y; (A \vee B)^x, C^y \Rightarrow A^x, B^y}{(A \vee B)^x, C^x \Rightarrow A^x, B^x} \text{ (flat)}$$

Although the choice of labels in flattening is non-deterministic, the results are always in the same equivalence class:

COROLLARY 8.68. *Let* $\Sigma'; \underline{S}'$ *be a relational sequent, and let* $\underline{S}_1$ *and* $\underline{S}_2$ *be labelled sequents derived from applying the flattening rule to* $\Sigma'; \underline{S}'$. *Then* $\underline{S}_1 \approx \underline{S}_2$.

Proof. From Lemma 8.47. □

LEMMA 8.69. *Flattening is surjective and non-injective.*

Proof. Let $\underline{S} \in \mathrm{SLS}$, and let labels $x, z \# \underline{S}$ and $y \in \mathtt{lab}(\underline{S})$. Then $\underline{S}$ is a result of flattening $x \le y; \underline{S}$ by applying the substitution $[y/x]$. So flattening is surjective.

Then $\underline{S}$ is a result of flattening $x \le y; \underline{S}$ or $x \le z; \underline{S}$ by applying the substitutions $[y/x]$ or $[z/x]$, respectively, so flattening is not injective. □

From Lemma 8.69, flattening is not invertible. While weakening can be used to re-add the relational contexts, the labels in both sides of the sequent cannot be "split" into their original form for **Int**. It is possible to use weakening and the folding rules to split either the antecedent or succedent. A general schema for splitting the antecedent is and succedent is shown here:

$$\dfrac{\dfrac{\dfrac{\Sigma;\Gamma^x,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta_1^x,\Delta_2^x}{x \le y,\Sigma;\Gamma^x,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta_1^x,\Delta_2^x}\,(\mathsf{LW}\le)}{x \le y,\Sigma;\Gamma^x,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta_1^x,\Delta_2^x,\Delta_2^y}\,(\mathsf{RW})^*}{x \le y,\Sigma;\Gamma^x,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta_1^x,\Delta_2^y}\,(\mathsf{R}\le)^* \qquad \dfrac{\dfrac{\dfrac{\Sigma;\Gamma_1^x,\Gamma_2^x,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta^x}{y \le x,\Sigma;\Gamma_1^x,\Gamma_2^x,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta^x}\,(\mathsf{LW}\le)}{y \le x,\Sigma;\Gamma_1^x,\Gamma_1^y,\Gamma_2^x,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta^x}\,(\mathsf{LW})^*}{y \le x,\Sigma;\Gamma_1^y,\Gamma_2^x,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta^x}\,(\mathsf{L}\le)^*$$

where $x\#\underline{\Gamma'},\underline{\Delta'}$. (When $y\#\Sigma,\underline{\Gamma'},\underline{\Delta'}$, the labels x,y can be transposed to obtain the relation $x \le y$.) Note that **Cl** does admit the splitting rule, (see case 5 on page 287), so that the original sequent can be recovered if they are classical.

These methods for deriving "unflattened" sequents are similar to the one used for translating labelled sequent rules into relational rules given in Chapter 9.

EXAMPLE 8.70. *Translating the labelled sequent from the previous example back into a relational sequent, by derivation. Note that the original antecedent (or succedent) cannot be recovered by derivations within* **Int***:*

$$\dfrac{\dfrac{\dfrac{(A\vee B)^x,C^x \Rightarrow A^x,B^x}{x \le y;(A\vee B)^x,C^x \Rightarrow A^x,B^x}\,(\mathsf{LW}\le)}{x \le y;(A\vee B)^x,C^x \Rightarrow A^x,B^x,B^y}\,(\mathsf{RW})}{x \le y;(A\vee B)^x,C^x \Rightarrow A^x,B^y}\,(\mathsf{R}\le) \qquad \dfrac{\dfrac{\dfrac{\dfrac{(A\vee B)^x,C^x \Rightarrow A^x,B^x}{y \le x;(A\vee B)^x,C^x \Rightarrow A^x,B^x}\,(\mathsf{LW}\le)}{y \le x;(A\vee B)^y,(A\vee B)^x,C^x \Rightarrow A^x,B^x}\,(\mathsf{LW})}{y \le x;(A\vee B)^y,C^x \Rightarrow A^x,B^x}\,(\mathsf{L}\le)}{x \le y;(A\vee B)^x,C^y \Rightarrow A^y,B^y}\,[\mathsf{xy/yx}]$$

In **Cl***, the original sequent from the previous example can be derived:*

$$\dfrac{\dfrac{\dfrac{(A\vee B)^x,C^x \Rightarrow A^x,B^x}{(A\vee B)^{x'},C^y \Rightarrow A^{x'},B^y}\,(\mathsf{S})}{(A\vee B)^x,C^y \Rightarrow A^x,B^y}\,[\mathsf{x/x'}]}{x \le y;(A\vee B)^x,C^y \Rightarrow A^x,B^y}\,(\mathsf{LW}\le)$$

A property of flattening relational sequents is that **connected chains** (that is, chains with an element in common) that do not contain reflexive relations are merged into a single label:

LEMMA 8.71. *Let $\Sigma;\underline{S} \in \mathsf{RLS}$, and let $\underline{S'}$ be the result of sequent flattening. Let $x_1 \le \ldots \le x_n$ be a chain in Σ, such that $x_i \le x_i$ does not occur in that chain. Then for each label x_i in that chain, $[x_j/x_i](\underline{S} \mathbin{/\!\!/} x_i) \subseteq \underline{S'} \mathbin{/\!\!/} x_j$. That is, all labels on that chain are merged into a single label $x_j \in \{x_1,\ldots,x_n\}$.*

Proof. By induction on the size of the chain.

For the base case for $n = 2$, $x_1 \leq x_2$. If $x_1 \# \underline{S}$, then the property holds trivially, and the relation can be eliminated by the root rule. Otherwise we use LW$\leq$ to add $x_2 \leq x_1$ to the relational context, and apply the cyc_2 rule, which substitutes one for the other.

For the induction step on chains $x_1 \leq \ldots \leq x_n$, apply the rule for chain $x_1 \leq \ldots \leq x_{n-1} \leq x_n$, resulting in a chain $x_j \leq n_n$ for $1 \leq j \leq n-1$. Apply the same procedure as the base case.

□

For example, the sequent $x \leq y \leq z; A^x \Rightarrow A^z$ becomes simply $A^x \Rightarrow A^x$. However, this structure can be restored (minus relational formulae) using the intermediate structural rules from the corresponding labelled calculus **LG3ipm** (Figure 7.1 on page 157).

EXAMPLE 8.72. *Suppose we have a derivation in* **Jan** *(using* **G3I** *plus* dir*) of the sequent $w \leq x, w \leq y; A^x, \neg A^y \Rightarrow$. The flattened form of this sequent is $A^w, \neg A^w \Rightarrow$, which appears to have lost structure. However, we can apply the rule* LQ *in the corresponding labelled framework* **LG3ipm*** *to derive $A^x, \neg A^y \Rightarrow$.*

Note that flattening assumes that the sequent is derivable. It makes no sense to examine whether it preserves countermodels. Indeed, applying flattening to an underivable sequent may produce a derivable sequent: For example, given the relational sequent $x \leq y, x \leq z; A^y, B^z \Rightarrow B^y, A^z$ (which is derivable in **GD**, but not in **Int**: the countermodel has a rooted, non-linear Kripke frame with two branches, where A is true on one branch and B is true in the other). The flattened sequent is $A^x, B^x \Rightarrow B^x, A^x$, which has no countermodel at all.

An alternative approach to sequent flattening for a subset of acyclic sequents is called **implicational flattening**.

$$\cfrac{\cfrac{\cfrac{x \leq y, \Sigma; \Gamma^y, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, \Delta^y}{x \leq y, \Sigma; \bigwedge\Gamma^y, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, \Delta^y}\ \mathsf{L}\wedge^*}{x \leq y, \Sigma; \bigwedge\Gamma^y, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, \bigvee\Delta^y}\ \mathsf{R}\vee^*}{\Sigma; \underline{\Gamma'} \Rightarrow \underline{\Delta'}, (\bigwedge\Gamma \supset \bigvee\Delta)^x}\ \mathsf{R}\supset$$

where $y\#\Sigma$, where Γ^y is assumed to be $\top$ when empty and Δ^y is assumed to be $\bot$ when empty. This technique is only for eliminating relational contexts that are linear or non-directed.

8.7. Conclusion

We've extended the conventional translation of labelled sequents to PSF so that it covers relational sequents, and later used it to show a "simple correspondence" between labelled sequents and relational sequents with empty relational contexts.

We've also proven the admissibility of rules in the calculus **G3c/PSF** for PSF that manipulating relations. These rules correspond to relational rules in relational calculi for **Int***/Geo, including cycle merging, transitivity and transitive folding rules. These rules were used to show the relationship between labelled sequents and relational sequents.

From a labelled sequent one can derive relational sequents with non-empty relational contexts. The relationship is not injective, because there are many relational sequents than can be derived from an arbitrary labelled sequent.

We have introduced the notion of a grounded relational sequent as a kind of normal form of relational sequent, and that from an equivalence class (modulo permutation of labels) of labelled sequents one can derive a corresponding equivalence class of grounded relational sequents. Because of the presence of contraction, this is not a bijection. This suggests that a larger equivalence class that includes labelled sequents with duplicate formulae from which there may be a bijection with a a corresponding equivalence class of grounded sequents. This is an area for future research.

From an acyclic relational sequent one can obtain, using a method called transitive unfolding, a labelled sequent, that shares the same model and countermodel for linear intuitionistic Kripke frames (which correspond to the intermediate logic **GD**).

We have shown that this method is applicable to cyclic relational sequents by composition with cycle elimination, and that composition of cycle elimination with transitive unfolding yields results that are in the same equivalence class (modulo permutation of labels). We also introduce an extension to transitive unfolding that covers cyclic sequents, and show that the resulting sequents by transitive unfolding or by cycle elimination composed with transitive folding are interderivable.

We also examined the relationship between labelled sequents, relational sequents and their transitive unfolding and shown that that the relationship is not bijective. In particular, transitive unfolding obscures the branching relationships between labels (which is consistent with linear models).

We also revisited the notion of grounded sequents and shown for every equivalence class of relational sequents there is an equivalence class of grounded sequents. However, the relationship is not a bijection.

We've also introduced simpler approaches to obtaining labelled sequents from relational sequents called flattening, which involve using rules such as root, "implicational flattening" (using $\mathsf{L}\wedge$, $\mathsf{R}\vee$ and $\mathsf{R}\supset$) to eliminate relational formulae, or weakening to transform all relations into cycles and applying cycle elimination rules. Despite the non-determinism, we've shown that the results of flattening produce sequents which are in the same equivalence class. However, the resulting sequents lose much of the original structure that different labels provided.

Although we give multiple methods describing how relational sequents can be obtained from labelled sequents and vice versa, none of the methods gives a bijection between formalisms. However, all of them preserve derivability (and in the case transitive unfolding, preserve stricter models).

In the next chapter, we will investigate the translation of labelled calculi into relational calculi, and the translation of relational proofs into labelled proofs.

CHAPTER 9

Translating between Labelled Sequents and Relational Sequents

9.1. Overview

In this chapter we show how to translate a labelled sequent calculus to a relational sequent calculus, and how translate the proofs in a relational sequent calculus into proofs in a labelled sequent calculus. The outline of this chapter is as follows:

In Section 9.2, we will introduce methods for translating labelled sequent calculi to relational sequent calculi. This requires extending not only the language of the rules but adding new rules to ensure that the calculus is complete with respect to a so-called logic extended with relations, e.g. $\mathbf{Int}_{\leq}$ (see Definition 5.94 on page 123), so that sequents such as $x \leq y; (A \vee B)^x \Rightarrow A^y, B^x$ can be proven in that logic. We will also address issues of preserving the admissibility of cut, weakening and contraction and present a method that can be automated. We introduce two relational calculi **RG3ipm′** and **RG3ipm** (and a framework of calculi **RG3ipm***) obtained by these methods, and show that these calculi are equivalent to the **G3i**. We also show that proofs in the original framework of labelled calculi **LG3ipm*** can be translated into proofs in the corresponding calculi in the relational framework.

In Section 9.3, we show how to translate *proofs* in the relational calculi of **RG3ipm*** into proofs in the corresponding labelled calculi of **LG3ipm***+$\mathsf{Com_m}$. Recall that in Theorem 8.41 on page 177 we showed that the a relational sequent and its transitive unfolding share the same *linear* models, which corresponds to the logic **GD**. We outline cases where the translation requires the $\mathsf{Com_m}$ (which is the structural rule for **GD**). Note that a method of translating relational *calculi* to labelled calculi is not given here. The translation assumes that one has both kinds of calculi. Some work on an alternative notation to extend the translation to weaker logics (unsuccessfully) is given. We also present some initial work and conjectures relating to translating pure relational formulae into structural labelled rules.

Corollaries to the work here is that hypersequent calculi (and proofs) can be translated into relational calculi (and proofs), and that relational proofs can be translated into hypersequent proofs. We do not explicitly apply the translations here to hypersequents for brevity.

9.2. Translating Labelled Proofs to Relational Proofs

In this section we show how to translate a labelled calculus into a relational calculus for logics in **Int***/Geo. We discuss the issues involved with translating calculi so that they are *complete* for the language of their corresponding relational logic, and give a simple method of translating labelled calculi into relational calculi by adding relational contexts to the rules of a labelled calculus along with a set of relational rules. We present a simple translation of the calculus **LG3ipm** into a relational calculus **RG3ipm′**. We note that this method is not suitable for an automated translation, and provide a second method which modifies the labelled rules, and adds a smaller set of rules (which absorb some of the rules proposed in the simple translation) which are known to preserve the admissibility of cut, weakening and contraction. We then use this method and present a framework of cut-free calculi **RG3ipm***. (Clearly by translating calculi, one is able to translate proofs.)

9.2.1. Relational Extensions of Labelled Calculi. The problem of extending a labelled calculus such as **LG3ipm** (Figure 7.1 on page 157) to a relational calculus is similar to that of extending a sequent calculus to a hypersequent calculus. Hyperextending a sequent calculus **GS** for a logic **S** (Section 4.3.2 on page 75) with hypercontexts does not necessarily guarantee that the calculus is *complete* with respect to the language of hypersequents derivable in **HGS** without the addition of new rules for manipulating the hypercontexts, such as EW and EC.

Unlike hypersequents, relational sequents may contain relational formulae, and do not have a natural translation into formula. Recall the extended notion of a logic, e.g. $\mathbf{Int}_{\leq}$ from Definition 5.94 on page 123. We need to ensure that the **relational extension** of a labelled calculus for a logic **S** is complete for $\mathbf{S}_{\leq}$. Merely adding relational contexts to the rules (shown sound in Lemma 9.1 on the facing page below) is not enough. Appropriate rules for manipulating relational formulae are needed. $\mathsf{LW} \leq$ and $\mathsf{LC} \leq$ are obvious candidates for such rules:

$$\frac{\Sigma;\underline{S}}{x \leq y, \Sigma;\underline{S}}\ \mathsf{LW}\leq \qquad \frac{x \leq y, x \leq y, \Sigma;\underline{S}}{x \leq y, \Sigma;\underline{S}}\ \mathsf{LC}\leq$$

By themselves, the $\mathsf{LW}\leq$ and $\mathsf{LC}\leq$ rules do not make the new system complete for relational sequents in $\mathbf{S}_{\leq}$, and they do not show how relational formulae interact with the logical formula. So the folding rules are also needed:

$$\frac{\Sigma; A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\leq' \qquad \frac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ \mathsf{R}\leq'$$

Semantically, variants of the rules have already been shown sound in Lemma 3.71 on page 64.

The folding rules introduce relational formulae and define them as indicating the persistence relationship between labels (or the dual anti-persistence on the right side). Note that the folding rules have the **separation**, and **symmetric** and **explicitness** properties (as given by Wansing [Wan94]), but not the **uniqueness** property, since given a relational formula, it is not obvious whether it was introduced by $\mathsf{L}\leq'$ and $\mathsf{R}\leq'$.

LEMMA 9.1 (Relational Extension). *If ρ is a sound labelled sequent rule for a calculus in* $\mathbf{Int}^*/\mathsf{Geo}$,

$$\frac{\underline{S}'_1 \quad \ldots \quad \underline{S}'_n}{\underline{S}}\ \rho$$

then the extension of that rule with relational contexts, (called the ***relational extension of*** *ρ),*

$$\frac{\Sigma;\underline{S}'_1 \quad \ldots \quad \Sigma;\underline{S}'_n}{\Sigma;\underline{S}}\ \rho_{\leq}$$

is sound for $\mathbf{Int}^*/\mathsf{Geo}$.

Proof. Semantically. Let $\underline{S}'_i = \underline{\Gamma}'_i \Rightarrow \underline{\Delta}'_i$ and $\underline{S} = \underline{\Gamma} \Rightarrow \underline{\Delta}$. If ρ is sound, then the rule

$$\frac{\mathtt{psf} \circledast \underline{\Gamma}'_1 \Rightarrow \mathtt{psf} \circledast \underline{\Delta}'_1 \quad \ldots \quad \mathtt{psf} \circledast \underline{\Gamma}'_n \Rightarrow \mathtt{psf} \circledast \underline{\Delta}'_n}{\mathtt{psf} \circledast \underline{\Gamma} \Rightarrow \mathtt{psf} \circledast \underline{\Delta}}\ \rho'$$

is admissible in **G3c/PSF*** by the correctness of the translation (Theorem 6.42). From the simple correspondence (Corollary 8.7 on page 168), the rule

$$\frac{\mathtt{psf}_{\leq} \circledast \underline{\Gamma}'_1 \Rightarrow \mathtt{psf} \circledast \underline{\Delta}'_1 \quad \ldots \quad \mathtt{psf}_{\leq} \circledast \underline{\Gamma}'_n \Rightarrow \mathtt{psf} \circledast \underline{\Delta}'_n}{\mathtt{psf}_{\leq} \circledast \underline{\Gamma} \Rightarrow \mathtt{psf} \circledast \underline{\Delta}}$$

is also admissible. By weakening the premisses, we get the rule

$$\frac{\mathtt{psf}_{\leq} \circledast \Sigma, \mathtt{psf}_{\leq} \circledast \underline{\Gamma}'_1 \Rightarrow \mathtt{psf} \circledast \underline{\Delta}'_1 \quad \ldots \quad \mathtt{psf}_{\leq} \circledast \Sigma, \mathtt{psf}_{\leq} \circledast \underline{\Gamma}'_n \Rightarrow \mathtt{psf} \circledast \underline{\Delta}'_n}{\mathtt{psf}_{\leq} \circledast \Sigma, \mathtt{psf}_{\leq} \circledast \underline{\Gamma} \Rightarrow \mathtt{psf} \circledast \underline{\Delta}}$$

provided that Σ meets any restrictions imposed on the premiss(es) or conclusion by ρ (e.g. freshness conditions on labels). By Proposition 2.49 on page 31, the above rule is equivalent to the rule

$$\frac{\mathtt{psf}_{\leq} \circledast \Sigma, \underline{\Gamma}'_1 \Rightarrow \mathtt{psf} \circledast \underline{\Delta}'_1 \quad \ldots \quad \mathtt{psf}_{\leq} \circledast \Sigma, \underline{\Gamma}'_n \Rightarrow \mathtt{psf} \circledast \underline{\Delta}'_n}{\mathtt{psf}_{\leq} \circledast \Sigma, \underline{\Gamma} \Rightarrow \mathtt{psf} \circledast \underline{\Delta}}$$

We translate these rules in **G3c/PSF*** back to relational rules:

$$\frac{\Sigma; \underline{S}'_1 \quad \ldots \quad \Sigma; \underline{S}'_n}{\Sigma; \underline{S}} \; \rho_{\leq}$$

□

Definition 9.2 (Simple Relational Extension). Let **LGS** be a labelled calculus. A **simple relational extension RGS** is obtained by adding (shared) relational context variables to the axioms and all premisses and the conclusion of all of the rules of **LGS**, along with the relational rules:

$$\frac{\Sigma; \underline{S}}{x \leq y, \Sigma; \underline{S}} \; \mathsf{LW}\leq \qquad \frac{x \leq y, x \leq y, \Sigma; \underline{S}}{x \leq y, \Sigma; \underline{S}} \; \mathsf{LC}\leq$$

$$\frac{\Sigma; A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}} \; \mathsf{L}\leq' \qquad \frac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y} \; \mathsf{R}\leq'$$

Lemma 9.3. *Let* **RGS** *be a relational extension of a calculus* **LGS** *with contraction and weakening. Then the reflexivity and transitivity rules*

$$\frac{x \leq x, \Sigma; \underline{S}}{\Sigma; \underline{S}} \; (\mathsf{refl}) \qquad \frac{x \leq y \leq z, x \leq z, \Sigma; \underline{S}}{x \leq y \leq z, \Sigma; \underline{S}} \; (\mathsf{trans})$$

can be shown admissible in **RGS**.

Proof. By induction on the derivation depth. (The proof is written out in Lemma G.1 on page 273.) □

REMARK 9.4. There are subintuitionistic logics, e.g. [Res94], where reflexivity and transitivity are considered separable from persistence. But the admissibility of reflexivity (with contraction) and transitivity (with weakening) in the presence of the folding rules suggests a relationship between substructural logics and subintuitionistic logics. This is an area for future investigation. .

The rules for **RG3ipm′**, a relational extension of **LG3ipm**, are given in Figure 9.1.

$$\frac{}{\Sigma;\underline{\Gamma},P^x \Rightarrow P^x,\underline{\Delta}}\ \mathsf{Ax} \qquad \frac{}{\Sigma;\bot^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\ \mathsf{L}\bot$$

$$\frac{\Sigma;\underline{\Gamma},A^x,B^x\Rightarrow\underline{\Delta}}{\Sigma;\underline{\Gamma},(A\wedge B)^x\Rightarrow\underline{\Delta}}\ \mathsf{L}\wedge \qquad \frac{\Sigma;\underline{\Gamma}\Rightarrow A^x,\underline{\Delta} \quad \Sigma;\underline{\Gamma}\Rightarrow B^x,\underline{\Delta}}{\Sigma;\underline{\Gamma}\Rightarrow(A\wedge B)^x,\underline{\Delta}}\ \mathsf{R}\wedge$$

$$\frac{\Sigma;\underline{\Gamma},A^x\Rightarrow\underline{\Delta} \quad \Sigma;\underline{\Gamma},B^x\Rightarrow\underline{\Delta}}{\Sigma;\underline{\Gamma},(A\vee B)^x\Rightarrow\underline{\Delta}}\ \mathsf{L}\vee \qquad \frac{\Sigma;\underline{\Gamma}\Rightarrow A^x,B^x,\underline{\Delta}}{\Sigma;\underline{\Gamma}\Rightarrow(A\vee B)^x,\underline{\Delta}}\ \mathsf{R}\vee$$

$$\frac{\Sigma;(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^x \quad \Sigma;B^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}{\Sigma;(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\ \mathsf{L}\supset \qquad \frac{\Sigma;A^x,\underline{\Gamma}'\Rightarrow\underline{\Delta}',B^x}{\Sigma;\underline{\Gamma}'\Rightarrow\underline{\Delta}',\Delta^x,(A\supset B)^x}\ \mathsf{R}\supset$$

where $x\#\underline{\Delta}'$ in $\mathsf{R}\supset$.

$$\frac{\Sigma;\underline{S}}{x\le y,\Sigma;\underline{S}}\ \mathsf{LW}\le \qquad \frac{x\le y,x\le y,\Sigma;\underline{S}}{x\le y,\Sigma;\underline{S}}\ \mathsf{LC}\le$$

$$\frac{\Sigma;A^x,A^y,\underline{\Gamma}\Rightarrow\underline{\Delta}}{x\le y,\Sigma;A^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\ \mathsf{L}\le' \qquad \frac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^x,A^y}{x\le y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^y}\ \mathsf{R}\le'$$

FIGURE 9.1. The simple relational calculus **RG3ipm′**.

COROLLARY 9.5. *The* refl *and* trans *rules are admissible in* **RG3ipm′**.

Proof. Follows from Lemma 9.3 on the facing page. □

PROPOSITION 9.6. *Instances of the* $\mathsf{LW}\le$ *in a proof in* **RG3ipm′** *rule can be eliminated.*

Proof. By induction on the derivation depth. □

PROPOSITION 9.7. *The following rules*

$$\frac{x \le y, \Sigma; A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \le y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\le \qquad \frac{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ \mathsf{R}\le$$

are admissible in **RG3ipm′**.

Proof. By the LC ≤ rule. □

LEMMA 9.8. *The weakening and contraction rules*

$$\frac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma}, A^x \Rightarrow \underline{\Delta}}\ (\mathsf{LW}) \quad \frac{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow A^x, \underline{\Delta}}\ (\mathsf{RW}) \quad \frac{\Sigma; \underline{\Gamma}, A^x, A^x \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma}, A^x \Rightarrow \underline{\Delta}}\ (\mathsf{LC}) \quad \frac{\Sigma; \underline{\Gamma} \Rightarrow A^x, A^x, \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow A^x, \underline{\Delta}}\ (\mathsf{RC})$$

are depth-preserving admissible in RG3ipm′.

Proof. Straightforward, by simultaneous (in the case of contraction) induction on the derivation height. (The folding rules have no effect on the permutability of weakening or contraction.) □

REMARK 9.9. We note that the contraction rules can be derived using the folding rules (as is done in **L** [PU09]), but they would not be considered depth-preserving admissible.

THEOREM 9.10 (Interderivability). **RG3ipm′** $\vdash \Sigma; \underline{S}$ *iff* **G3I** $\vdash \Sigma; \underline{S}$.

Proof. By showing that the rules of **RG3ipm′** are admissible in **G3I** and viz. that the rules of **G3I** are admissible in **RG3ipm′**. (The proof is written out in Theorem G.2 on page 274.) □

COROLLARY 9.11. *The system* **RG3ipm′** *is complete for* $\mathbf{Int}_{\le}$.

Proof. Follows from the Interderivability Theorem above and Definition 5.94 on page 123. □

It's not altogether clear that a relational extension preserves cut admissibility (although in the case of **RG3ipm′**, we do have interderivability with a cut-free system **G3I**.) As noted in the section on the logic **L** [PU09] (Subsection 5.3.4 on page 124), cut admissibility may be problematic to prove with non-atomic folding rules.

One idea is to use a labelled form of Gentzen-style multicut that allows multiple cut formulae with multiple labels, but which accounts for the formulae in the relational context. We show a form of cut is admissible in Theorem 9.15 on the facing page.

PROPOSITION 9.12. *The following weakening and contraction rules*

$$\frac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta}}{\Sigma;\underline{\Gamma},A^x\Rightarrow\underline{\Delta}}\ (\mathsf{LW})\qquad \frac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta}}{\Sigma;\underline{\Gamma}\Rightarrow A^x,\underline{\Delta}}\ (\mathsf{RW})\qquad \frac{\Sigma;\underline{\Gamma},A^x,A^x\Rightarrow\underline{\Delta}}{\Sigma;\underline{\Gamma},A^x\Rightarrow\underline{\Delta}}\ (\mathsf{LC})\qquad \frac{\Sigma;\underline{\Gamma}\Rightarrow A^x,A^x,\underline{\Delta}}{\Sigma;\underline{\Gamma}\Rightarrow A^x,\underline{\Delta}}\ (\mathsf{RC})$$

are depth-preserving admissible in **RG3ipm′**.

Proof. By induction on the derivation height. □

PROPOSITION 9.13. *The* L∧, R∧, L∨, R∨ *and* L⊃ *rules are depth-preserving invertible in* **RG3ipm′**.

Proof. By induction on the derivation depth. □

PROPOSITION 9.14. *Let δ be a proof in* **RG3ipm′**. *Then δ can be rewritten so that all instances of* $\mathsf{L}\leq'$ *and* $\mathsf{R}\leq'$ *that occur in δ are replaced by instances of* $\mathsf{L}\leq$ *and* $\mathsf{R}\leq$.

Proof. By induction on the derivation depth. □

THEOREM 9.15 (Polycut Admissibility).

$$\frac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},n_1\cdot A^{x_1},\ldots,n_k\cdot A^{x_k}\qquad \Sigma';m_1\cdot A^{y_1},\ldots,m_{k'}\cdot A^{y_{k'}},\underline{\Gamma'}\Rightarrow\underline{\Delta'}}{\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma'}\Rightarrow\underline{\Delta},\underline{\Delta'}}\ (\mathsf{pcut})$$

where for all $1\leq i\leq k$ and $1\leq j\leq k'$, $x_i\leq y_j\in\Sigma,\Sigma'$, is admissible in **RG3ipm′**.

Recall the notation on page 21, that $n\cdot A^x$ denotes n copies of A^x.

TERMINOLOGY 9.16. Each x_i is a **left cut label** and each y_i is a **right cut label**. A is called the **cut formula**, and each A^{x_i} and A^{y_i} (for $1\leq i\leq k$) are called the **left cut formulae** and **right cut formula**, respectively.

REMARK 9.17. Without the restriction on the relational context, pcut would be unsound, e.g.

$$\frac{A^y\Rightarrow A^y,A^x\qquad A^y,A^x\Rightarrow A^x}{A^y\Rightarrow A^x}\ (\mathsf{pcut})$$

Proof. By induction on the rank defined by (a) the size of the cut formula; and (b) the sum of the derivation depths of the premisses. (The proof is written out in Theorem G.3 on page 275.) □

COROLLARY 9.18. *The* cut *rule*

$$\frac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^x\qquad \Sigma';A^x\underline{\Gamma'}\Rightarrow\underline{\Delta'}}{\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma'}\Rightarrow\underline{\Delta},\underline{\Delta'}}\ (\mathsf{cut})$$

is admissible in **RG3ipm′**.

Proof. Using weakening and refl. □

9.2.2. Mechanically Extending Labelled Calculi. The simple relational extension is problematic because it requires a manual proof of cut admissibility, as in Theorem 9.15 above. In particular, the following properties are derivable in a mechanical translation τ from a labelled calculus **LGS** to a relational calculus **RGS**:

(1) **Preservation of admissible rules.** If ρ is (depth-preserving) admissible in **LGS**, then $\tau\rho$ is (depth-preserving) admissible in **RGS**. The includes the preserving the admissibility of the standard structural rules, substitution of labels, and cut, as well as preserving the invertibility of rules.
(2) **Presence of standard relational rules.** The rules LW $\leq$, LC $\leq$, L $\leq'$, R $\leq'$, refl and trans should be admissible or primitive in **RGS**.
(3) **Soundness.** The calculus should be sound in the logic $\mathbf{S}_{\leq}$ that corresponds to the relational extension of $\mathbf{S} \in \mathbf{Int}^*/\mathtt{Geo}$.
(4) **Completeness.** The calculus should be complete for the logic $\mathbf{S}_{\leq}$ that corresponds to the relational extension of $\mathbf{S} \in \mathbf{Int}^*/\mathtt{Geo}$.

Assuming that the translation τ on individual rules has no effect on the admissibility of the standard structural rules (weakening, contraction, substitution of labels and cut) as part of (1), then we can insure that these are preserved by adding only geometric rules to the system, as per [Neg03].

We can add the atomic form of the folding rule,

$$\frac{x \leq y, \Sigma; P^x, P^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; P^x, \underline{\Gamma} \Rightarrow \underline{\Delta}} \ \mathsf{L} \leq_0$$

that conforms to the geometric rule schema discussed in Section 3.5.3 on page 60. The corresponding atomic R $\leq_0$ rule

$$\frac{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, P^x, P^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, P^y} \ \mathsf{R} \leq_0$$

is trivially shown admissible (Proposition G.9 on page 280), and the non-atomic L $\leq$ and R $\leq$ rules follow from cut.

Also note that $\mathsf{L}\leq_0$ corresponds to the semantic axiom of persistence for intuitionistic Kripke frames, which is defined in terms of atomic formulae.

If relational contexts are added to the axioms of **RGS** obtained by τ, then the rule $\mathsf{LW}\leq$ is straightforwardly admissible. Likewise, if the multipremiss rules of **RGS** obtained by τ are relational-context-sharing, then $\mathsf{LC}\leq$ is admissible. (Note that folding rules also include the principal relational formula in the premisses, absorbing the $\mathsf{LC}\leq$ rule.) Note this is similar to how the EW and EC rules can be made admissible in a hyperextension by adding hypercontexts to axioms.

Because we cannot be sure that refl and trans admissible in **RGS**, but because they are geometric rules, they too can be added without affecting (1).

Clearly, item (2) would be satisfied.

Soundness (3) w.r.t. $\mathbf{S}_{\leq}$ for $\mathbf{S}\in\mathbf{Int}^*/\mathsf{Geo}$ are straightforward, so long as the translation τ obtains sound rules. If the rules of **LGS** are complete w.r.t. **S**, then the translated rules of **RGS** with folding, refl and trans rules should be complete (4) for $\mathbf{Int}_{\leq}$.

The main concern, then, is in using a method τ for translating the rules of a calculus **LGS** that is sound, and translates structural rules so as to be complete for $\mathbf{S}_{\leq}$ for $\mathbf{S}\in\mathbf{Int}^*/\mathsf{Geo}$.

We conjecture that the procedure outlined in Definition 9.19 below meets satisfies these requirements. We show later that it does for the translation of **LG3ipm** to **RG3ipm**.

Definition 9.19 (Cut-Free Relational Extension). Let **LGS** be a cut-free labelled calculus with weakening and contraction. The **cut-free relational extension RGS** is obtained by adding the rules

$$\frac{x\leq y,\Sigma;P^x,P^y,\underline{\Gamma}\Rightarrow\underline{\Delta}}{x\leq y,\Sigma;P^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\ \mathsf{L}\leq_0 \qquad \frac{x\leq x,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta}}{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta}}\ \mathsf{refl} \qquad \frac{x\leq z,x\leq y\leq z,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta}}{x\leq y\leq z,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta}}\ \mathsf{trans}$$

where P is atomic, and translating the rules of **LGS** using the following steps:

(1) **Relational contexts.** A shared relational context metavariable is added to all premisses and the conclusions of all rules and axioms of **LGS**, similar to Definition 9.2 on page 194. This is necessary for the $\mathsf{LW}\leq$ and $\mathsf{LC}\leq$ rules to be admissible.

(2) **Non-invertible rules.** Non-invertible rules can be identified syntactically using procedures adapted from [Cha08]. Such rules can be transformed into relational rules by using the trivial invertible forms (Definition 5.38 on page 110. By virtue of missing meta-variables in some of the premisses of the original rules, these rules will impose an *eigenlabel condition* on the conclusions of the trivially invertible forms.

For example, the trivially invertible form of the $\mathsf{R}\supset$ rule from **LG3ipm** is:

$$\frac{\Sigma; \Gamma^{xy}, A^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', B^y, \Delta^x, (A \supset B)^x}{[x/y]\Sigma; \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, (A \supset B)^x} \ \mathsf{R}\supset_\iota$$

where $x, y \# \underline{\Gamma}', \underline{\Delta}'$. (Recall that the substitution comes from the EC rule.) Because y is a fresh label that is added (with respect to root-first application of the rules), we can conclude that $y\#\Sigma$, despite the substitution $[x/y]$ in the conclusion of $\mathsf{R}\supset_\iota$. (It is implicit because the use of EC in the derivation of the trivially invertible rule uses substitution.)

Suppose we had a calculus with a non-invertible form of the $\mathsf{L}\supset$ rule:

$$\frac{\Sigma; (A \supset B)^x, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta} \setminus x, A^x \quad \Sigma; B^x, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}}{\Sigma; (A \supset B)^x, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}} \ \mathsf{L}\supset'$$

Then we can obtain the trivially invertible form, noting Remark 5.40 on page 111:

$$\frac{\Sigma; (A \supset B)^{xy}, \Gamma^{xy}, \underline{\Gamma}' \Rightarrow \underline{\Delta} A^y \quad \Sigma; B^x, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}}{[x/y]\Sigma; (A \supset B)^x, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}} \ \mathsf{L}\supset'_\iota$$

Note that a side effect of this is to make such rules invertible in **RGS**.

(3) **Structural rules.** The procedure in step (2) is also be applied to structural rules (even invertible ones), to make the relationship between labels in the premisses and conclusion explicit. (This is done for structural rules of **RG3ipm*** later in Section 9.2.3.)

(4) **Premiss Unfolding.** In each premiss, for each schematic multiset variable in the antecedent that is shared by multiple labels x, y, add $x \leq y$ to the schematic relational context where the relation $\underline{\Gamma} /\!/ x \underset{\sim}{\subseteq} \underline{\Gamma} /\!/ y$ holds. (If the relation is symmetric, then add both possibilities.) Do the same for the succedent.

For example, the $\mathsf{R}{\supset_\iota}$ rule becomes

$$\frac{x \leq y, \Sigma; \Gamma^x, \Gamma^y, A^y, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, B^y, \Delta^x, (A \supset B)^x}{\Sigma; \Gamma^x, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, \Delta^x, (A \supset B)^x}\ \mathsf{R}{\supset_\iota}$$

(Recall that y is fresh for the conclusion.)

Then implicitly apply the unfolding rules (weakening) to the premisses and folding rules to the conclusion, to eliminate redundant variables from the rule. Using the $\mathsf{R}{\supset_\iota}$ rule as an example:

$$\cfrac{\cfrac{x \leq y, \Sigma; \Gamma^x, A^y, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, B^y, \Delta^x, (A \supset B)^x}{x \leq y, \Sigma; \Gamma^x, \Gamma^y, A^y, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, B^y, \Delta^x, (A \supset B)^x}\ \mathsf{LW}^*}{\Sigma; \Gamma^x, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, \Delta^x, (A \supset B)^x}\ \mathsf{R}{\supset_\iota}$$

Another example, applying the same technique to the LQ rule:

$$\cfrac{\cfrac{x \leq z, y \leq z, \Sigma; \Gamma_1^x, \Gamma_2^y, \underline{\Gamma'} \Rightarrow \underline{\Delta'}}{x \leq z, y \leq z, \Sigma; \Gamma_1^x, \Gamma_2^y, \Gamma_1^z, \Gamma_2^z, \underline{\Gamma'} \Rightarrow \underline{\Delta'}}\ \mathsf{LW}^*}{\Sigma; \Gamma_1^x, \Gamma_2^y, \underline{\Gamma'} \Rightarrow \underline{\Delta'}}\ \mathsf{LQ}_\iota$$

where $x, y, z \# \underline{\Gamma'}, \underline{\Delta'}$. The resulting rule conforms to the geometric rule schema, so does not affect the admissibility of cut, contraction or weakening. (Note that no further justification is needed for the soundness of eliminating these relational formulae in this rule, because that is part of the rule itself.)

(5) **Conclusion folding.** We apply a similar technique to step (4) to the conclusions of rules, with the main difference that each relational formula added to the conclusion is also added to every premiss. This is done to preserve the admissibility of $\mathsf{LC}\leq$, and (for structural rules), to insure that they are geometric rules.

(6) **Preserving symmetry.** A final step is with regards to dual rules of logical rules that have have active relational formulae in the premiss, as per step (2), as well as active logical formulae with different labels than the principal formulae. For example, the translated $\mathsf{R}{\supset_{\iota\leq}}$ above requires the dual rule $\mathsf{L}{\supset_\leq}$:

$$\frac{x \leq y, \Sigma; (A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y \quad x \leq y, \Sigma; (A \supset B)^x, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; (A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}{\supset_\leq}$$

Without this rule, cut admissibility may problematic (as would be the generalised axiom, Lemma 9.34 on page 207), since the cuts could not be simply permuted to lower derivation depths without accounting for the different labels.

Note that this rule is interderivable with $\mathsf{L}\supset$. A derivation of $\mathsf{L}\supset_{\leq}$ from $\mathsf{L}\supset$:

$$\dfrac{\dfrac{\dfrac{x \leq y, \Sigma; (A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}{x \leq y, \Sigma; (A \supset B)^x, (A \supset B)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ (\mathsf{LW}) \quad x \leq y, \Sigma; (A \supset B)^x, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; (A \supset B)^x, (A \supset B)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\supset}{x \leq y, \Sigma; (A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ (\mathsf{L} \leq)$$

The $\mathsf{L}\supset$ rule is derivable from $\mathsf{L}\supset_{\leq}$ using refl (again, this means that refl must be a primitive rather than admissible rule). Note that the $\mathsf{L}\supset_{\leq}$ rule absorbs the non-atomic $\mathsf{L} \leq$ rule, and this allows the system to be complete with a primitive $\mathsf{L} \leq_0$ rule.

When both rules have been transformed as per step (2), then this step can be omitted.

Remark 9.20. We note for a calculus **LGS** with the non-invertible $\mathsf{L}\supset'$ (Remark 5.40 on page 111) and $\mathsf{R}\supset$ rules that we are unsure if this method produces a complete calculus for RGS w.r.t. $\mathbf{S}_{\leq}$. We do note that the admissibility of $\mathsf{L} \leq$ may make the calculus complete, e.g.

$$\dfrac{\dfrac{\dfrac{\dfrac{x \leq y \leq z; (A \supset B)^{xy}, A^{yz} \Rightarrow A^z}{x \leq y \leq z; (A \supset B)^{xy}, A^y \Rightarrow A^z}\ (\mathsf{L} \leq) \quad x \leq y; (A \supset B)^x, A^y, B^y \Rightarrow B^y}{x \leq y; (A \supset B)^{xy}, A^y \Rightarrow B^y}\ \mathsf{L}\supset'_{\leq}}{x \leq y; (A \supset B)^x, A^y \Rightarrow B^y}\ (\mathsf{L} \leq)}{(A \supset B)^x \Rightarrow (A \supset B)^x}\ \mathsf{R}\supset_{\leq}$$

This is an area for future work.

Remark 9.21. A question that we have not examined is how the translation from labelled to relational calculus affects the invertibility of rules. We conjecture that labelled rules which are invertible, remain invertible when translated to relational rules, by virtue of the added rules ($\mathsf{L} \leq_0$ and relational rules) having only atomic active and principal formulae. A related question is whether the translation procedure can generate invertible rules from non-invertible rules. We note the $\mathsf{R}\supset_{\leq}$ rule from **G3I**:

$$\dfrac{x \leq y, \Sigma; \Gamma^x, A^y, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, B^y, \Delta^x}{\Sigma; \Gamma^x, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, \Delta^x, (A \supset B)^x}\ \mathsf{R}\supset_{\leq}$$

This rule is derivable from $\mathsf{R}{\supset}_{\iota\leq}$ by RW. But it is also invertible (see [DN10]), not only because the side formulae Δ^x occur in the premisses as well as conclusion, but because the *relationship* between the formulae labelled with y and the formulae labelled with x is preserved, by virtue of the relational formula $x \leq y$, even when permuting the rule with other rules. Contrast this with the simply labelled sequent rule

$$\frac{\Sigma; A^y, \Gamma^y, \Gamma^x, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, B^y, \Delta^x}{\Sigma; \Gamma^x, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, \Delta^x, (A \supset B)^x}\ \mathsf{R}{\supset}'$$

which is not invertible. For example, in the sequent $(A \vee B)^x \Rightarrow (C \supset A)^x, B^x$, analysing the implication before the disjunction using the $\mathsf{R}{\supset}'$ rule will not result in a proof. That is because the formulae in Γ^x may be analysed independently of the formulae in Γ^y in a simply labelled sequent calculus.

Remark 9.22. It is worth noting that it appears that a calculus with the atomic $\mathsf{L}\leq_0$ rule requires the relational $\mathsf{L}{\supset}_{\leq}$ rule rather than the "classical" $\mathsf{L}{\supset}$ rule. Suppose that the calculus included the dir rule (Figure 5.4 on page 122). Then the sequent,

$$x \leq y, x \leq z, \neg(A \wedge B)^x, A^y, B^z$$

which is valid in **Jan**, cannot be derived.

9.2.3. The Framework RG3ipm*. Applying the procedure outlined above, we obtain the calculus **RG3ipm** in Figure 9.2 by translating the rules from **LG3ipm** (Figure 7.1), and how various relational analogues to labelled rules, as well as the non-atomic folding rules are admissible. Afterwards we apply the procedure outlined above to syntactically obtain the relational rules for intermediate logics from **LG3ipm*** (Figure Figure 7.2 on page 159), which are the same as the rules from **G3I*** given in Figure 5.4 on page 122.

Lemma 9.23 (Label substitution). *The following label substitution rule*

$$\frac{\Sigma; \underline{S}, S^x}{[y/x]\Sigma; \underline{S}, S^y}\ [\mathsf{y/x}]$$

where $x\#\underline{S}$, is admissible in **RG3ipm**.

Proof. By induction on the derivation height. □

$$\frac{}{\Sigma; P^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, P^x}\ \mathsf{Ax} \qquad \frac{}{\Sigma; \underline{\Gamma}, \bot^x \Rightarrow \underline{\Delta}}\ \mathsf{L}\bot$$

$$\frac{\Sigma; \underline{\Gamma}, A^x, B^x \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma}, A \wedge B^x \Rightarrow \underline{\Delta}}\ \mathsf{L}\wedge \qquad \frac{\Sigma; \underline{\Gamma} \Rightarrow A^x, \underline{\Delta} \quad \Sigma; \underline{\Gamma} \Rightarrow B^x, \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow A \wedge B^x, \underline{\Delta}}\ \mathsf{R}\wedge$$

$$\frac{\Sigma; \underline{\Gamma}, A^x \Rightarrow \underline{\Delta} \quad \Sigma; \underline{\Gamma}, B^x \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma}, A \vee B^x \Rightarrow \underline{\Delta}}\ \mathsf{L}\vee \qquad \frac{\Sigma; \underline{\Gamma} \Rightarrow A^x, B^x, \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow A \vee B^x, \underline{\Delta}}\ \mathsf{R}\vee$$

$$\frac{x \leq y, \Sigma; (A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y \quad x \leq y, \Sigma; (A \supset B)^x, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; (A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}{\supset_{\leq}}$$

$$\frac{x \leq y, \Sigma; \Gamma^x, A^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', B^y, \Delta^x, (A \supset B)^x}{\Sigma; \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, (A \supset B)^x}\ \mathsf{R}{\supset_{\iota\leq}}$$

where $y \# \Sigma$ for $\mathsf{R}{\supset_{\iota\leq}}$ rule.

$$\frac{x \leq y, \Sigma; P^x, P^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; P^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\leq_0 \qquad \frac{x \leq x, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{refl} \qquad \frac{x \leq z, x \leq y \leq z, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y \leq z, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{trans}$$

Note that P^x, P^y are atomic in Ax and $\mathsf{L}\leq_0$.

FIGURE 9.2. The simple relational calculus **RG3ipm**.

LEMMA 9.24. *The corresponding structural rules*

$$\frac{\Sigma; \underline{\Gamma}' \Rightarrow \underline{\Delta}'}{\Sigma; \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x}\ (\mathsf{GW}) \qquad \frac{\Sigma; \Gamma^y, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, \Delta^y}{[x/y]\Sigma; \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x}\ (\mathsf{GC}_{\leq})$$

where $y \# \underline{\Gamma}', \underline{\Delta}'$ *are depth-preserving admissible in* **RG3ipm**.

Proof. Similar to Lemma 7.19 on page 158. □

REMARK 9.25. Note that the substitution used in the derivation of the labelled form of GC must also be applied to the relational context in $\mathsf{GC}_{\leq}$.

COROLLARY 9.26 (External Rules). *The* ***external weakening*** *and* ***external contraction*** *rules*

$$\frac{\Sigma; \underline{\Gamma}' \Rightarrow \underline{\Delta}'}{\Sigma; \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x}\ (\mathsf{EW}) \qquad \frac{\Sigma; \Gamma^y, \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x, \Delta^y}{[x/y]\Sigma; \Gamma^x, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^x}\ (\mathsf{EC}_{\leq})$$

where $y \# \Sigma$ *for* EW *and* $x, y \# \underline{\Gamma}', \underline{\Delta}'$*, are depth-preserving admissible in* **LG3ipm**.

Proof. They are special cases of GW and $\mathsf{GC}_{\leq}$. □

LEMMA 9.27. *The* $\mathsf{LW}\leq$ *and* $\mathsf{LC}\leq$ *rules*

$$\frac{\Sigma;\underline{S}}{x\leq y,\Sigma;\underline{S}}\ (\mathsf{LW}\leq) \qquad \frac{x\leq y,x\leq y,\Sigma;\underline{S}}{x\leq y,\Sigma;\underline{S}}\ (\mathsf{LC}\leq)$$

are depth-preserving admissible in **RG3ipm**.

Proof. By induction on the derivation depth. The proof for $\mathsf{LW}\leq$ is straightforward.

□

REMARK 9.28. The rules can be shown admissible (without the depth-preserving feature) semantically, using LW and LC in **G3c/PSF***.

LEMMA 9.29 (Folding Rules). *The* ***folding rules****:*

$$\frac{x\leq y,\Sigma;A^x,A^y,\underline{\Gamma}\Rightarrow\underline{\Delta}}{\overline{\overline{x\leq y,\Sigma;A^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}}}\ (\mathsf{L}\leq) \qquad \frac{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^x,A^y}{\overline{\overline{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^y}}}\ (\mathsf{R}\leq)$$

are admissible in **RG3ipm***.

Proof. By cut, similar to proof for Lemma 5.100 on page 124. □

An alternative proof that the folding rules are admissible in **RG3ipm** by induction on the derivation depth is given in Lemma G.10 on page 281.

LEMMA 9.30 (Root). *The* ***root*** *rule*

$$\frac{x\leq y_1,\ldots,x\leq y_n,\Sigma;\underline{S}}{\Sigma;\underline{S}}\ (\mathsf{root})$$

where $x\#\Sigma,\underline{S}$*, is admissible in* **RG3ipm**.

Proof. Semantically from root rule in **G3c/PSF***. (The rule is shown also depth-preserving admissible by induction on the derivation depth in Lemma G.11 on page 283.) □

REMARK 9.31. The root rule has no analogous rule in **LG3ipm**. It roughly corresponds to the elimination of empty components in a hypersequent calculus, i.e.

$$\frac{\mathcal{H}\mid\ \Rightarrow}{\mathcal{H}}\ (\mathsf{E}\emptyset)$$

from Definition 4.57 on page 79.

REMARK 9.32. Although the root rule is not necessary for proving formulae in the logics in **Int***/Geo, it is useful for proving meta-properties.

We apply the technique for translating the base calculus to the extension rules of the framework **LG3ipm*** (Figure 7.2 on page 159) to obtaining the additional relational rules for **RG3ipm*** (Figure 9.3):

$$\frac{x \le z, y \le z, \Sigma; \underline{S}}{\Sigma; \underline{S}}\ \mathsf{wk\ dir}$$

where $z \# \Sigma, \underline{S}$.

$$\frac{x \le y, \Sigma; \underline{S} \quad y \le x, \Sigma; \underline{S}}{\Sigma; \underline{S}}\ \mathsf{lin}$$

$$\frac{y \le x, x \le y, \Sigma; \underline{S}}{\Sigma; \underline{S}}\ \mathsf{sym}'$$

FIGURE 9.3. Extension rules for **RG3ipm***.

(1) From the LQ rule, we obtain the trivially invertible form:

$$\frac{\Sigma; \Gamma_1^x, \Gamma_2^y, \Gamma_1^z, \Gamma_2^z, \underline{\Gamma}' \Rightarrow \underline{\Delta}'}{\Sigma; \Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}'}\ \mathsf{LQ}_\iota$$

where $x, y, z \# \underline{\Gamma}', \underline{\Delta}'$ and $z \# \Sigma$. We add relational formulae and folding rules, and obtain the derivation in item 4 on page 201. The resulting rule is a variant of wk dir where the active labels do not occur in the succedent. Note that wk dir is interderivable with dir (Proposition 3.66 on page 63).

(2) From the $\mathsf{Com_m}$ rule,

$$\frac{\Gamma_1^x, \Gamma_2^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_1^y, \Delta_2^y \quad \Gamma_1^x, \Gamma_1^y, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^x, \Delta_2^y}{\Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}\ \mathsf{Com_m}$$

where $x, y \# \underline{\Gamma}', \underline{\Delta}'$, we do not need the trivially invertible form, as the relationship between labels is already apparent. We add relational contexts and formulae and apply folding/unfolding rules, to obtain the lin rule:

$$\frac{\dfrac{y \le x, \Sigma; \Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}{x \le y, \Sigma; \Gamma_1^x, \Gamma_2^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_1^y, \Delta_2^y}\ (\mathsf{GW}) \quad \dfrac{y \le x, \Sigma; \Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}{y \le x, \Sigma; \Gamma_1^x, \Gamma_1^y, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^x, \Delta_2^y}\ (\mathsf{GW})}{\Sigma; \Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}\ \mathsf{Com_m}$$

(3) From the S rule, we use the trivially invertible form

$$\dfrac{\Gamma_1^x, \Gamma_2^y, \Gamma_1^z, \Gamma_2^z, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^z, \Delta_2^z, \Delta_1^x, \Delta_2^y}{\Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}\ \mathsf{S}_\iota$$

where $x, y, z \# \underline{\Gamma}', \underline{\Delta}'$, and apply the technique to obtain the sym' rule:

$$\dfrac{\dfrac{\dfrac{x \le y, y \le x, \Sigma; \Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}{x \le z \le x, y \le z \le y, \Sigma; \Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}\ (\mathsf{LW} \le)}{x \le z \le x, y \le z \le y, \Sigma; \Gamma_1^x, \Gamma_2^y, \Gamma_1^z, \Gamma_2^z, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^z, \Delta_2^z, \Delta_1^x, \Delta_2^y}\ (\mathsf{GW})}{\Sigma; \Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}\ \mathsf{S}_\iota$$

We note the extra $\mathsf{LW} \le$ inference in the premiss is not part of the procedure. But because z is part of a cycle and is fresh for the logical context, it is acceptable to remove it.

Using $\mathsf{LW} \le$, we can derive the syn rule.

Remark 9.33. For the structural rules, the translations result in purely relational rules. The logical contexts remain unchanged between premisses and conclusion.

Proposition 9.34 (Generalised Axiom). *The generalised axiom* $\Sigma; A^x, \underline{\Gamma} \Rightarrow A^x, \underline{\Delta}$ *is derivable in* **RG3ipm***.

Proof. By induction on the derivation depth. □

Theorem 9.35 (Interderivability). **RG3ipm**$^* \vdash \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$ *iff* **G3I**$^* \vdash \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}$.

Proof. By induction on the derivation depth. (A proof is given in Theorem G.14 on page 284.) □

Corollary 9.36. *The calculus* **RG3ipm** *is sound and complete for* **Int**$_\le$.

Proof. From Theorem 9.35 Theorem and 5.92 on page 121. □

Remark 9.37. The calculi in **RG3ipm** are sound and complete for their respective logics given in Definition 5.94 on page 123.

Theorem 9.38. *If* **LG3ipm**$^* \vdash \underline{\Gamma} \Rightarrow \underline{\Delta}$, *then* **RG3ipm**$^* \vdash \underline{\Gamma} \Rightarrow \underline{\Delta}$.

Proof. By induction on the derivation depth. (A proof is given in Theorem G.15 on page 285.) □

COROLLARY 9.39. *All proofs in* **LG3ipm*** *can be translated into proofs in* **G3I***.

Proof. From Theorems 9.38 and 9.35 on the preceding page. □

9.3. Translating Relational Proofs to Labelled Proofs

Here we show how to translate relational proofs into simply labelled proofs. Specifically, we will show that a proof of a relational sequent $\Sigma;\underline{S}$ in a calculus in the **RG3ipm*** framework (Figures 9.2 on page 204 and 9.3 on page 206) for **Int***/Geo can be translated into a proof of $(\Sigma;\underline{S})^\bullet$ in the corresponding calculus in the **LG3ipm*** framework (Figures 7.1 on page 157 and 7.2 on page 159), although in some cases the proofs will require the $\mathsf{Com_m}$ rule for **GD**. (Recall the definition of transitive unfolding in Section 8.5 on page 175, and the proof that relational sequents and simply labelled sequents share the same *linear* models in Theorem 8.41 on page 177.)

LEMMA 9.40. *The rule*

$$\frac{A^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta} \qquad B^x, B^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}}{(A \vee B)^x, (A \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}} \ (\mathsf{L}\vee^*_\bullet)$$

is admissible in **LG3ipm**+$\mathsf{Com_m}$.

Proof. By derivation, using GW. We use Γ^x_{12} as shorthand for Γ^x_1, Γ^x_2 below:

$$\frac{\dfrac{A^x, A^y, \Gamma^x_1, \Gamma^y_2, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x_1, \Delta^y_2}{A^x, A^y, B^y, \Gamma^x_1, \Gamma^y_{12}, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x_{12}, \Delta^y_2}\ (\mathsf{GW}) \qquad \dfrac{B^x, B^y, \Gamma^x_1, \Gamma^y_2, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x_1, \Delta^y_2}{A^x, B^x, B^y, \Gamma^x_{12}, \Gamma^y_2, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x_1, \Delta^y_{12}}\ (\mathsf{GW})}{A^x, B^y, \Gamma^x_1, \Gamma^y_2, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x_1, \Delta^y_2}\ \mathsf{Com_m} \tag{130}$$

$$\frac{\dfrac{B^x, B^y, \Gamma^x_1, \Gamma^y_2, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x_1, \Delta^y_2}{B^x, B^y, A^y, \Gamma^x_1, \Gamma^y_{12}, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x_{12}, \Delta^y_2}\ (\mathsf{GW}) \qquad \dfrac{A^x, A^y, \Gamma^x_1, \Gamma^y_2, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x_1, \Delta^y_2}{B^x, A^x, A^y, \Gamma^x_{12}, \Gamma^y_2, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x_1, \Delta^y_{12}}\ (\mathsf{GW})}{B^x, A^y, \Gamma^x_1, \Gamma^y_2, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x_1, \Delta^y_2}\ \mathsf{Com_m} \tag{131}$$

where $x, y \# \underline{\Gamma}', \underline{\Delta}'$ in (130) and (131).

$$\dfrac{\dfrac{A^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta} \quad \begin{array}{c}\vdots\ (130)\\ A^x, B^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}\end{array}}{A^x, (A \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ \mathsf{L}\vee \qquad \dfrac{\begin{array}{c}\vdots\ (131)\\ B^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}\end{array} \quad B^x, B^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{B^x, (A \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ \mathsf{L}\vee}{(A \vee B)^x, (A \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ \mathsf{L}\vee$$

□

LEMMA 9.41. *The following rule*

$$\dfrac{(A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x \quad (A \supset B)^x, B^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}{(A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\supset_\iota$$

is admissible in **LG3ipm**.

Proof. Straightforward. (The proof is written out in Lemma G.16 on page 287.) □

PROPOSITION 9.42. *The rule*

$$\dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, \perp^x}{\underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{R}\perp$$

is admissible in **LG3ipm**.

Proof. By induction on the derivation depth. □

LEMMA 9.43. *The rule*

$$\dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ \mathsf{R}\underset{\sim}{\subseteq}$$

where $\underline{\Gamma} /\!\!/ x \underset{\sim}{\subseteq} \underline{\Gamma} /\!\!/ y$, *is admissible in* **LG3ipm**+$\mathsf{Com_m}$.

Proof. By induction on the structure of A, and the derivation depth of the premiss.

(1) For the base case, A is atomic, and the premiss is an axiom (with derivation depth of 0). There are two subcases:
   (a) Suppose A^x is the principal formula. $A^x \in \underline{\Gamma}$, but by the constraints on side formulae, so is $A^y \in \underline{\Gamma}$. Therefore the conclusion of the rule is also an axiom.
   (b) Otherwise, the conclusion is also an axiom.

   Note that this case applies to generalised axioms of greater derivation depth.

(2) Suppose A is atomic, but at a derivation depth greater than 0. Then there are two subcases:

(a) If $A = \bot$, then the conclusion is derivable by Proposition 9.42 on the preceding page.

(b) Otherwise the $\mathsf{R}\precsim$ rule can be permuted to lower derivation depth.

(3) Suppose the premiss is the conclusion of an instance of $\mathsf{L}\wedge$. We have the following subcases:

(a) Suppose the principal label is x:

$$\cfrac{\cfrac{C^x, D^x, (C \wedge D)^y, \underline{\Gamma'} \Rightarrow \underline{\Delta}, A^x, A^y}{(C \wedge D)^x, (C \wedge D)^y, \underline{\Gamma'} \Rightarrow \underline{\Delta}, A^x, A^y}\ \mathsf{L}\wedge}{(C \wedge D)^x, (C \wedge D)^y, \underline{\Gamma'} \Rightarrow \underline{\Delta}, A^y}\ (\mathsf{R}\precsim)$$

Then the following can be derived, where $\mathsf{R}\precsim$ is permuted to a lower depth:

$$\cfrac{\cfrac{\cfrac{C^x, D^x, (C \wedge D)^y, \underline{\Gamma'} \Rightarrow \underline{\Delta}, A^x, A^y}{C^x, D^x, C^y, D^y, \underline{\Gamma'} \Rightarrow \underline{\Delta}, A^x, A^y}\ (\mathsf{L}\wedge^{-1})}{C^x, D^x, C^y, D^y, \underline{\Gamma'} \Rightarrow \underline{\Delta}, A^x}\ (\mathsf{R}\precsim)}{(C \wedge D)^x, (C \wedge D)^y, \underline{\Gamma'} \Rightarrow \underline{\Delta}, A^y}\ \mathsf{L}\wedge^2$$

(b) For all other cases, the constraint on the antecedent is not affected, so the $\mathsf{R}\precsim$ instance can be permuted upwards.

(4) Suppose the premiss is derived by an instance of $\mathsf{R}\wedge$. We have two subcases:

(a) The principal formula of $\mathsf{R}\wedge$ is one of the active formulae of $\mathsf{R}\precsim$, such that $A = C \wedge D$. Without loss of generality, we assume that A^x is the principal formula. The following is derivable, by permuting the $\mathsf{R}\precsim$ rule to smaller formulae and a lower derivation depth:

$$\cfrac{\cfrac{\cfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, C^x, (C \wedge D)^y}{\underline{\Gamma} \Rightarrow \underline{\Delta}, C^x, C^y}\ (\mathsf{R}\wedge_1^{-1})}{\underline{\Gamma} \Rightarrow \underline{\Delta}, C^y}\ (\mathsf{R}\precsim) \qquad \cfrac{\cfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, D^x, (C \wedge D)^y}{\underline{\Gamma} \Rightarrow \underline{\Delta}, D^x, D^y}\ (\mathsf{R}\wedge_2^{-1})}{\underline{\Gamma} \Rightarrow \underline{\Delta}, D^y}\ (\mathsf{R}\precsim)}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (C \wedge D)^y}\ \mathsf{R}\wedge$$

(b) Otherwise the $\mathsf{R}\precsim$ rule can be permuted to lower derivation depth.

(5) Suppose the premiss is the conclusion of an instance of $\mathsf{L}\vee$. We have the following subcases:

(a) Suppose the principal label is x:

$$\dfrac{\dfrac{C^x,(C\lor D)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y \qquad D^x,(C\lor D)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}{(C\lor D)^x,(C\lor D)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}\ \mathsf{L}\lor}{(C\lor D)^x,(C\lor D)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^y}\ (\mathsf{R}\underset{\sim}{\subset})$$

Then the following can be derived, where $\mathsf{R}\underset{\sim}{\subset}$ is permuted to a lower depth:

$$\dfrac{\dfrac{\dfrac{C^x,(C\lor D)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}{C^x,C^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}\ (\mathsf{L}\lor_1^{-1})}{C^x,C^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^y}\ (\mathsf{R}\underset{\sim}{\subset}) \qquad \dfrac{\dfrac{D^x,(C\lor D)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}{D^x,D^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}\ (\mathsf{L}\lor_2^{-1})}{D^x,D^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^y}\ (\mathsf{R}\underset{\sim}{\subset})}{(C\lor D)^x,(C\lor D)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^y}\ (\mathsf{L}\lor_\bullet)$$

Recall that $\mathsf{L}\lor_\bullet$ requires the $\mathsf{Com_m}$ rule.

(b) For all other cases, the constraint on the antecedent is not affected, so the $\mathsf{R}\underset{\sim}{\subset}$ instance can be permuted upwards.

(6) Suppose the premiss is derived by an instance of $\mathsf{R}\lor$. We have two subcases:

(a) The principal formula of $\mathsf{R}\lor$ is one of the active formulae of $\mathsf{R}\underset{\sim}{\subset}$, such that $A = C \lor B$. Without loss of generality, we assume that A^x is the principal formula. The following is derivable, by permuting the $\mathsf{R}\underset{\sim}{\subset}$ rule to smaller formulae and a lower derivation depth:

$$\dfrac{\dfrac{\dfrac{\underline{\Gamma}\Rightarrow\underline{\Delta},C^x,D^x,(C\lor D)^y}{\underline{\Gamma}\Rightarrow\underline{\Delta},C^x,D^x,C^y,D^y}\ (\mathsf{R}\lor^{-1})}{\underline{\Gamma}\Rightarrow\underline{\Delta},C^y,D^y}\ (\mathsf{R}\underset{\sim}{\subset})^+}{\underline{\Gamma}\Rightarrow\underline{\Delta},(C\lor D)^y}\ \mathsf{R}\lor$$

(b) Otherwise the $\mathsf{R}\underset{\sim}{\subset}$ rule can be permuted to lower derivation depth.

(7) Suppose the premiss is the conclusion of an instance of $\mathsf{L}\supset$. We have the following subcases:

(a) Suppose the principal label is x:

$$\dfrac{\dfrac{(C\supset D)^x,(C\supset D)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y,C^x \qquad (C\supset D)^y,D^x,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}{(C\supset D)^x,(C\supset D)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}\ \mathsf{L}\supset}{(C\supset D)^x,(C\supset D)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^y}\ (\mathsf{R}\underset{\sim}{\subset})$$

Then the following can be derived, where $\mathsf{R}\underset{\sim}{\subseteq}$ is permuted to a lower depth, abbreviating $C \supset D$ as CD:

$$\dfrac{\dfrac{\dfrac{(CD)^x,(CD)^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^x,A^y,C^x}{(CD)^x,(CD)^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^y,C^x}\,(\mathsf{R}\underset{\sim}{\subseteq})}{(CD)^x,(CD)^y,D^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^y,C^x}\,(\mathsf{LW}) \quad \dfrac{\dfrac{(CD)^y,D^x,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^x,A^y}{(CD)^x,(CD)^y,D^x,D^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^x,A^y}\,(\mathsf{LW})^+}{(CD)^x,(CD)^y,D^x,D^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^y}\,(\mathsf{R}\underset{\sim}{\subseteq})}{(CD)^x,(CD)^y,D^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^y}\,(\mathsf{L}{\supset_\iota}) \tag{132}$$

$$\dfrac{\dfrac{\dfrac{(CD)^x,(CD)^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^x,A^y,C^x}{(CD)^x,(CD)^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^x,A^y,C^x,C^y}\,(\mathsf{RW})}{(CD)^x,(CD)^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^y,C^y}\,(\mathsf{R}\underset{\sim}{\subseteq})^+ \quad \begin{array}{c}\vdots\ (132)\\ (CD)^x,(CD)^y,D^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^y\end{array}}{(CD)^x,(CD)^y,\underline{\Gamma}' \Rightarrow \underline{\Delta},A^y}\,(\mathsf{L}{\supset_\iota})$$

(b) For all other cases, the constraint on the antecedent is not affected, so the $\mathsf{R}\underset{\sim}{\subseteq}$ instance can be permuted upwards.

(8) Suppose the premiss is derived by an instance of $\mathsf{R}{\supset}$. We have two subcases:

(a) The principal formula of $\mathsf{R}{\supset}$ is one of the active formulae of $\mathsf{R}\underset{\sim}{\subseteq}$, such that $A = C \supset B$. Without loss of generality, we assume that A^x is the principal formula. The following is derivable, by permuting the $\mathsf{R}\underset{\sim}{\subseteq}$ rule to smaller formulae and a lower derivation depth:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C^x,\underline{\Gamma} \Rightarrow \underline{\Delta}',D^x,(C\supset D)^y}{\underline{\Gamma} /\!\!/ x, C^x, [x'/x]\underline{\Gamma} \Rightarrow [x'/x]\underline{\Delta}, D^x, (C\supset D)^{x'}, (C\supset D)^y}\,(\mathsf{GW})}{\underline{\Gamma} /\!\!/ x, C^x, [x'/x][y'/y]\underline{\Gamma} \Rightarrow [y'/y][x'/x]\underline{\Delta}, D^x, (C\supset D)^{x'}, (C\supset D)^{y'}}\,[\mathsf{y'/y}]}{\underline{\Gamma} /\!\!/ x, \underline{\Gamma} /\!\!/ y, C^x, C^y, [x'/x][y'/y]\underline{\Gamma} \Rightarrow [y'/y][x'/x]\underline{\Delta}, D^x, D^y, (C\supset D)^{x'}, (C\supset D)^{y'}}\,(\mathsf{R}{\supset_\iota^{-1}})}{\underline{\Gamma} /\!\!/ x, \underline{\Gamma} /\!\!/ y, C^x, C^y, [x'/x][y'/y]\underline{\Gamma} \Rightarrow [y'/y][x'/x]\underline{\Delta}, D^y, (C\supset D)^{y'}}\,(\mathsf{R}\underset{\sim}{\subseteq})^+}{[y/x]\underline{\Gamma} /\!\!/ x, \underline{\Gamma} /\!\!/ y, C^y, C^y, [x'/x][y'/y]\underline{\Gamma} \Rightarrow [y'/y][x'/x]\underline{\Delta}, D^y, (C\supset D)^{y'}}\,[\mathsf{y/x}]}{\underline{\Gamma} /\!\!/ y, C^y, [x'/x][y'/y]\underline{\Gamma} \Rightarrow [y'/y][x'/x]\underline{\Delta}, D^y, (C\supset D)^{y'}}\,(\mathsf{LC})^+}{[x'/x][y'/y]\underline{\Gamma} \Rightarrow [y'/y][x'/x]\underline{\Delta}, (C\supset D)^{y'}}\,(\mathsf{R}{\supset_\iota})}{[y'/y]\underline{\Gamma} \Rightarrow [y'/y]\underline{\Delta}, (C\supset D)^{y'}}\,[\mathsf{x/x'}]}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (C\supset D)^y}\,[\mathsf{y/y'}]$$

where $\underline{\Delta}' = \underline{\Delta} \setminus (\underline{\Delta} /\!\!/ x)$.

(b) Otherwise the $\mathsf{R}\underset{\sim}{\subseteq}$ rule can be permuted to lower derivation depth.

The inverted form of $\mathsf{R}\underset{\sim}{\subseteq}$ is admissible using RW. □

REMARK 9.44. Note that the $\mathsf{R}\subsetsim$ rule corresponds to the $\mathsf{R}\leq$ rule in **RG3ipm**. However, the dual $\mathsf{L}\subsetsim$ rule

$$\frac{A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\subsetsim$$

where $\underline{\Delta} /\!\!/ y \subsetsim \underline{\Delta} /\!\!/ x$ (that would correspond to $\mathsf{L}\leq$ in **RG3ipm**) is not admissible in **LG3ipm**+$\mathsf{Com_m}$. Suppose the premiss is $A^x, A^y, (A \supset B)^y \Rightarrow B^x, B^y$. The conclusion $A^x, (A \supset B)^y \Rightarrow B^x, B^y$ is not derivable.

However, the rule is admissible in **LG3ipm**+S, which is adequate for **Cl**. (A proof is given in Lemma G.17 on page 288.)

LEMMA 9.45. *The following rule*

$$\frac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y \quad \underline{\Gamma} \Rightarrow \underline{\Delta}, B^x, B^y}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^x, A^y}\ \mathsf{R}\wedge_\bullet$$

where $\underline{\Gamma} /\!\!/ x \subsetsim \underline{\Gamma} /\!\!/ y$*, is admissible in* **LG3ipm**.

Proof. Straightforward. (The proof is written out in Lemma G.18 on page 288.) □

COROLLARY 9.46. *The following rule*

$$\frac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^{x_1}, \ldots, A^{x_n} \quad \underline{\Gamma} \Rightarrow \underline{\Delta}, B^{x_1}, \ldots, B^{x_n}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^{x_1}, \ldots, (A \wedge B)^{x_n}}\ \mathsf{R}\wedge_\bullet^*$$

where $\underline{\Gamma} /\!\!/ x_i \subsetsim \underline{\Gamma} /\!\!/ x_n$ *(for* $1 \leq i \leq n$*), is admissible in* **LG3ipm**.

Proof. Straightforward. (The proof is written out in Corollary G.19 on page 288.) □

LEMMA 9.47. *The following rule*

$$\frac{\underline{\Gamma} \Rightarrow B^x, (A \supset B)^x, \underline{\Delta}}{\underline{\Gamma} \Rightarrow (A \supset B)^x, \underline{\Delta}}\ \mathsf{RC}\supset$$

is admissible in **LG3ipm***.

Proof. By induction on the derivation depth. (The proof is written out in Lemma G.20 on page 289.) □

THEOREM 9.48. *Let* $\Sigma; \underline{S} \in \mathsf{RLS}_a$*. If* **RG3ipm**$^* \vdash \Sigma; \underline{S}$*, then* **LG3ipm**$^* + \mathsf{Com_m} \vdash (\Sigma; \underline{S})^\bullet$.

Proof. By induction on the derivation depth, with the following cases:

(1) Suppose $\Sigma; \underline{S}$ is an axiom. Then $(\Sigma; \underline{S})^\bullet$ is also an axiom.

(2) Suppose $\Sigma;\underline{S}$ is the conclusion of an instance of refl. We apply an instance of contraction to remove duplicate formulae labelled with x in the antecedent and succedent.

(3) Suppose $\Sigma;\underline{S}$ is the conclusion of an instance of trans. We apply an instances of contraction to remove duplicate formulae labelled with z (unfolded from x) in the antecedent, and to remove duplicate formulae labelled with x (unfolded from z) in the succedent.

(4) Suppose $\Sigma;\underline{S}$ is the conclusion of an instance of $\mathsf{L}\leq_0$:

$$\frac{x_1 \leq x_2, \Sigma; \underline{\Gamma}, A^{x_1}, A^{x_2} \Rightarrow \underline{\Delta}}{x_1 \leq x_2, \Sigma; \underline{\Gamma}, A^{x_1} \Rightarrow \underline{\Delta}} \ \mathsf{L}\leq_0$$

Let $(x_1 \leq x_2, \Sigma; \underline{\Gamma}, A^{x_1}, A^{x_2} \Rightarrow \underline{\Delta})^\bullet = \underline{\Gamma}^\bullet, A^{x_1}, A^{x_2}, A^{x_2}, \dots, A^{x_n} \Rightarrow \underline{\Delta}^\bullet$, where for each $x_2 \leq x_i \in \Sigma^+$ $(3 \leq i \leq n)$, there are multiple occurrences of A^{x_i} in the antecedent. The corresponding proof in **LG3ipm*** is derived using LC:

$$\frac{\underline{\Gamma}^\bullet, A^{x_1}, A^{x_2}, A^{x_2}, \dots, A^{x_n} \Rightarrow \underline{\Delta}^\bullet}{\underline{\Gamma}^\bullet, A^{x_1}, A^{x_2}, \dots, A^{x_n} \Rightarrow \underline{\Delta}^\bullet} \ (\mathsf{LC})^+$$

(5) Suppose $\Sigma;\underline{S}$ is the conclusion of an instance of $\mathsf{L}\wedge$:

$$\frac{\Sigma; \underline{\Gamma}, A^{x_1}, B^{x_1} \Rightarrow \underline{\Delta}}{\Sigma; \underline{\Gamma}, (A \wedge B)^{x_1} \Rightarrow \underline{\Delta}} \ \mathsf{L}\wedge$$

Let $(\Sigma; \underline{\Gamma}, A^{x_1}, B^{x_1} \Rightarrow \underline{\Delta})^\bullet = \underline{\Gamma}^\bullet, A^{x_1}, B^{x_1}, \dots, A^{x_n}, B^{x_n} \Rightarrow \underline{\Delta}^\bullet$, where $x_1 \leq x_i \in \Sigma^+$ $(2 \leq i \leq n)$. The corresponding proof in **LG3ipm*** is derived using n instances of $\mathsf{L}\wedge$:

$$\frac{\underline{\Gamma}^\bullet, A^{x_1}, B^{x_1}, \dots, A^{x_n}, B^{x_n} \Rightarrow \underline{\Delta}^\bullet}{\underline{\Gamma}^\bullet, (A \wedge B)^{x_1}, \dots, (A \wedge B)^{x_n} \Rightarrow \underline{\Delta}^\bullet} \ \mathsf{L}\wedge^{\mathsf{n}}$$

(6) Suppose $\Sigma;\underline{S}$ is the conclusion of an instance of $\mathsf{R}\wedge$:

$$\frac{\Sigma; \underline{\Gamma} \Rightarrow A^{x_1}, \underline{\Delta} \quad \Sigma; \underline{\Gamma} \Rightarrow B^{x_1}, \underline{\Delta}}{\Sigma; \underline{\Gamma} \Rightarrow (A \wedge B)^{x_1}, \underline{\Delta}} \ \mathsf{R}\wedge$$

where $x_2 \le x_1, \ldots, x_n \le x_1 \in \Sigma^+$ for $n \ge 1$. Let

$$(\Sigma; \underline{\Gamma} \Rightarrow A^{x_1}, \underline{\Delta})^\bullet = \underline{\Gamma}^\bullet \Rightarrow A^{x_1}, \ldots, A^{x_n}, \underline{\Delta}^\bullet$$

$$(\Sigma; \underline{\Gamma} \Rightarrow B^{x_1}, \underline{\Delta})^\bullet = \underline{\Gamma}^\bullet \Rightarrow B^{x_1}, \ldots, B^{x_n}, \underline{\Delta}^\bullet$$

where $\underline{\Gamma} \mathbin{/\!/} x_1 \subsetsim \underline{\Gamma} \mathbin{/\!/} x_i$ and $\underline{\Delta} \mathbin{/\!/} x_i \subsetsim \underline{\Delta} \mathbin{/\!/} x_1$ for $1 \le i \le n$. The corresponding proof in **LG3ipm*** is derived using the $\mathsf{R}\wedge^\bullet$ rule from Corollary 9.46 on page 213:

$$\frac{\underline{\Gamma}^\bullet \Rightarrow A^{x_1}, \ldots, A^{x_n}, \underline{\Delta}^\bullet \qquad \underline{\Gamma}^\bullet \Rightarrow B^{x_1}, \ldots, B^{x_n}, \underline{\Delta}^\bullet}{\underline{\Gamma}^\bullet \Rightarrow (A \wedge B)^{x_1}, \ldots, (A \wedge B)^{x_n}, \underline{\Delta}^\bullet}\ (\mathsf{R}\wedge^\bullet)$$

Note that $(\Sigma; \underline{\Gamma} \Rightarrow (A \wedge B)^{x_1}, \underline{\Delta})^\bullet = \underline{\Gamma}^\bullet \Rightarrow (A \wedge B)^{x_1}, \ldots, (A \wedge B)^{x_n}, \underline{\Delta}^\bullet$.

(7) Suppose $\Sigma; \underline{S}$ is the conclusion of an instance of $\mathsf{L}\vee$. The case is the dual of case 6 above, using the $\mathsf{L}\vee_\bullet$ rule from Lemma 9.40 on page 208.

(8) Suppose $\Sigma; \underline{S}$ is the conclusion of an instance of $\mathsf{R}\vee$. The case is the dual of case 5 above.

(9) Suppose $\Sigma; \underline{S}$ is the conclusion of an instance of $\mathsf{L}{\supset_\le}$:

$$\frac{x_1 \le y_1 \Sigma; (A \supset B)^{x_1}, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^{y_1} \qquad x_1 \le y_1 \Sigma; (A \supset B)^{x_1}, B^{y_1}, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x_1 \le y_1 \Sigma; (A \supset B)^{x_1}, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}{\supset_\le}$$

where $x_1 \le y_1, \ldots, x_m \le y_1, y_1 \le y_2, \ldots, y_1 \le y_n$ for $m, n \ge 1$. Let

$$(x_1 \le y_1 \Sigma; (A \supset B)^{x_1}, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^{y_1})^\bullet = \overline{(A \supset B)}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet, A^{x_1}, \ldots, A^{x_m}, A^{y_1} \tag{133}$$

$$(x_1 \le y_1 \Sigma; (A \supset B)^{x_1}, B^{y_1}, \underline{\Gamma} \Rightarrow \underline{\Delta})^\bullet = \overline{(A \supset B)}, B^{y_1}, \ldots, B^{y_n}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet \tag{134}$$

where $\overline{(A \supset B)} = (A \supset B)^{x_1}, (A \supset B)^{y_1}, (A \supset B)^{y_n}$. We can derive the following from (133), for $1 \le i \le n$:

$$\dfrac{\dfrac{\dfrac{\overline{(A \supset B)}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet, A^{x_1}, \ldots, A^{x_m}, A^{y_1}}{\overline{(A \supset B)}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet, A^{y_1}}\ (\mathsf{R}\subsetsim)^+}{\overline{(A \supset B)}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet, A^{y_1}, A^{y_i}}\ (\mathsf{RW})}{\overline{(A \supset B)}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet, A^{y_i}}\ (\mathsf{R}\subsetsim)$$

(Clearly the last two inference steps are omitted for $i = 1$.) We first derive the following:

$$\dfrac{\dfrac{\overline{(A \supset B)}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet, A^{y_1}}{\overline{(A \supset B)},, B^{y_2}, \ldots, B^{y_n}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet, A^{y_1}}\ \mathsf{LW}^+ \qquad \vdots\,(134)}{\overline{(A \supset B)}, B^{y_2}, \ldots, B^{y_n}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet}\ (\mathsf{L}\supset_\iota) \tag{135}$$

For $n \geq 2$, we apply the result of (135) to

$$\dfrac{\dfrac{\overline{(A \supset B)}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet, A^{y_2}}{\overline{(A \supset B)},, B^{y_{i+1}}, \ldots, B^{y_n}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet, A^{y_i}}\ \mathsf{LW}^+ \qquad \begin{matrix}\vdots \\ \overline{(A \supset B)}, B^{y_i}, \ldots, B^{y_n}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet\end{matrix}}{\overline{(A \supset B)}, B^{y_{i+1}}, \ldots, B^{y_n}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet}\ (\mathsf{L}\supset_\iota)$$

and apply repeatedly until we have derived $\overline{(A \supset B)}, \underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet$.

(10) Suppose $\Sigma; \underline{S}$ is the conclusion of an instance of $\mathsf{R}\supset_{\iota\leq}$:

$$\dfrac{x_1 \leq y, \Sigma; \Gamma^{x_1}, A^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', B^y, \Delta^{x_1}, (A \supset B)^{x_1}}{\Sigma; \Gamma^{x_1}, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta^{x_1}, (A \supset B)^{x_1}}\ \mathsf{R}\supset_{\iota\leq}$$

where $x_2 \leq x_1, \ldots, x_n \leq x_1 \in \Sigma^+$ for $n \geq 1$. Let

$$(x_1 \leq y, \Sigma; \Gamma^{x_1}, A^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', B^y, \Delta^{x_1}, (A \supset B)^{x_1})^\bullet =$$

$$\underline{\Gamma}^\bullet, \Gamma^y, A^y \Rightarrow \underline{\Delta}^\bullet, B^y, B^{x_1}, \ldots, B^{x_n}, (A \supset B)^{x_1}, \ldots, (A \supset B)^{x_n}$$

where $\Gamma^y \approx \underline{\Gamma}^\bullet /\!\!/ x_1$. The corresponding proof in **LG3ipm*** is derived:

$$\dfrac{\dfrac{\underline{\Gamma}^\bullet, \Gamma^y, A^y \Rightarrow \underline{\Delta}^\bullet, B^y, B^{x_1}, \ldots, B^{x_n}, (A \supset B)^{x_1}, \ldots, (A \supset B)^{x_n}}{\underline{\Gamma}^\bullet, \Gamma^y, A^y \Rightarrow \underline{\Delta}^\bullet, B^y, (A \supset B)^{x_1}, \ldots, (A \supset B)^{x_n}}\ (\mathsf{RC}\supset)^n}{\underline{\Gamma}^\bullet \Rightarrow \underline{\Delta}^\bullet, (A \supset B)^{x_1}, \ldots, (A \supset B)^{x_n}}\ (\mathsf{R}\supset_\iota)$$

(11) Suppose $\Sigma; \underline{S}$ is the conclusion of an instance of wk dir:

$$\dfrac{x \leq z, y \leq z, \Sigma; \underline{S}}{\Sigma; \underline{S}}\ \text{wk dir}$$

where $z\#\Sigma, \underline{S}$. Let $(x \leq z, y \leq z, \Sigma; \underline{S})^\bullet = \underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^y, \Gamma_1^z, \Gamma_2^z \Rightarrow \Delta_1^x, \Delta_2^y, \underline{\Delta}^\bullet$. The corresponding proof in **LG3ipm*** is derived:

$$\dfrac{\dfrac{\dfrac{\underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^y, \Gamma_1^z, \Gamma_2^z \Rightarrow \Delta_1^x, \Delta_2^y, \underline{\Delta}^\bullet}{\underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^y, \Gamma_1^{x'}, \Gamma_2^{y'} \Rightarrow \Delta_1^x, \Delta_2^y, \underline{\Delta}^\bullet}\ \mathsf{LQ}}{\underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^y, \Gamma_1^{x'}, \Gamma_2^{y'} \Rightarrow \Delta_1^x, \Delta_2^y, \Delta_1^{x'}, \Delta_2^{y'}, \underline{\Delta}^\bullet}\ (\mathsf{RW})^*}{\underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^y \Rightarrow \Delta_1^x, \Delta_2^y, \underline{\Delta}^\bullet}\ (\mathsf{GC})^+$$

Note that dir is a special case of wk dir.

(12) Suppose $\Sigma; \underline{S}$ is the conclusion of an instance of lin:

$$\dfrac{x \leq y, \Sigma; \underline{S} \quad y \leq x, \Sigma; \underline{S}}{\Sigma; \underline{S}}\ \mathsf{lin}$$

Let

$$(x \leq y, \Sigma; \underline{S})^\bullet = \underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_1^y, \Gamma_2^y \Rightarrow \Delta_1^x, \Delta_2^x, \Delta_2^y, \underline{\Delta}^\bullet$$

$$(y \leq x, \Sigma; \underline{S})^\bullet = \underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^x, \Gamma_2^y \Rightarrow \Delta_1^x, \Delta_1^y, \Delta_2^y, \underline{\Delta}^\bullet$$

Then the corresponding proof in **LG3ipm*** is derived:

$$\dfrac{\underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_1^y, \Gamma_2^y \Rightarrow \Delta_1^x, \Delta_2^x, \Delta_2^y, \underline{\Delta}^\bullet \quad \underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^x, \Gamma_2^y \Rightarrow \Delta_1^x, \Delta_1^y, \Delta_2^y, \underline{\Delta}^\bullet}{\underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^y \Rightarrow \Delta_1^x, \Delta_2^y, \underline{\Delta}^\bullet}\ \mathsf{Com_m}$$

(13) Suppose $\Sigma; \underline{S}$ is the conclusion of an instance of sym:

$$\dfrac{y \leq x, x \leq y, \Sigma; \underline{S}}{x \leq y, \Sigma; \underline{S}}\ \mathsf{sym}$$

Let $(y \leq x, x \leq y, \Sigma; \underline{S})^\bullet = \underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^x \Rightarrow \Delta_1^x, \Delta_2^x, \underline{\Delta}^\bullet$. Then the corresponding proof in **LG3ipm*** is derived

$$\dfrac{\dfrac{\underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^x \Rightarrow \Delta_1^x, \Delta_2^x, \underline{\Delta}^\bullet}{\underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_2^y \Rightarrow \Delta_1^x, \Delta_2^y, \underline{\Delta}^\bullet}\ \mathsf{S}}{\underline{\Gamma}^\bullet, \Gamma_1^x, \Gamma_1^y, \Gamma_2^y \Rightarrow \Delta_1^x, \Delta_2^x, \Delta_2^y, \underline{\Delta}^\bullet}\ (\mathsf{GW})$$

□

REMARK 9.49. Note that the translation *only* requires the use of the $\mathsf{Com_m}$ rule in cases where the principal label of an instance of $\mathsf{L}\vee$ is not a maximal label. In all other cases, the translation to **LG3ipm*** is within the same logic as the original **RG3ipm** proof.

EXAMPLE 9.50. *We take the following proof in* **RG3ipm***:*

$$\dfrac{\dfrac{x \le y; A^x \Rightarrow A^x}{x \le y; A^x \Rightarrow A^x, B^y}\,(\mathsf{RW}) \qquad \dfrac{x \le y; B^x, B^y \Rightarrow A^x, B^y}{x \le y; B^x \Rightarrow A^x, B^y}\,\mathsf{L}\le_0}{x \le y; (A \vee B)^x \Rightarrow A^x, B^y}\,\mathsf{L}\vee$$

The proof can be translated into **LG3ipm***:*

$$\dfrac{\dfrac{A^x \Rightarrow A^x}{A^x, A^y \Rightarrow A^x, B^x, B^y}\,(\mathsf{GW}) \qquad \dfrac{B^x, B^y, B^y \Rightarrow A^x, B^x, B^y}{B^x, B^y \Rightarrow A^x, B^x, B^y}\,(\mathsf{LC})}{(A \vee B)^x, (A \vee B)^y \Rightarrow A^x, B^x, B^y}\,(\mathsf{L}\vee_\bullet)$$

Note that in special cases, we can construct the proof using the primitive $\mathsf{L}\vee$ *rules instead of* $\mathsf{Com_m}$*:*

$$\dfrac{\dfrac{A^x \Rightarrow A^x}{A^x, A^y \Rightarrow A^x, B^x, B^y}\,(\mathsf{GW}) \qquad \dfrac{A^x \Rightarrow A^x}{A^x, B^y \Rightarrow A^x, B^x, B^y}\,(\mathsf{GW})}{A^x, (A \vee B)^y \Rightarrow A^x, B^x, B^y}\,\mathsf{L}\vee \tag{136}$$

$$\dfrac{\dfrac{B^x \Rightarrow B^x}{B^x, A^y \Rightarrow A^x, B^x, B^y}\,(\mathsf{GW}) \qquad \dfrac{B^y \Rightarrow B^y}{B^x, B^y \Rightarrow A^x, B^x, B^y}\,(\mathsf{GW})}{B^x, (A \vee B)^y \Rightarrow A^x, B^x, B^y}\,\mathsf{L}\vee \tag{137}$$

$$\dfrac{\begin{matrix} \vdots\ (136) \\ A^x, (A \vee B)^y \Rightarrow A^x, B^x, B^y \end{matrix} \qquad \begin{matrix} \vdots\ (137) \\ B^x, (A \vee B)^y \Rightarrow A^x, B^x, B^y \end{matrix}}{(A \vee B)^x, (A \vee B)^y \Rightarrow A^x, B^x, B^y}\,\mathsf{L}\vee$$

EXAMPLE 9.51. *We take the following proof in* **RG3ipm***:*

$$\dfrac{\dfrac{\dfrac{x \le y; A^x \Rightarrow B^y, A^x \qquad \dfrac{x \le y; B^x, B^y \Rightarrow B^y, A^x}{x \le y; B^x \Rightarrow B^y, A^x}\,\mathsf{L}\le_0}{x \le y; (A \vee B)^x, \Rightarrow B^y, A^x}\,\mathsf{L}\vee}{x \le y; (A \vee B)^x, C^y \Rightarrow B^y, A^x}\,(\mathsf{LW})}{(A \vee B)^x \Rightarrow (C \supset B)^x, A^x}\,\mathsf{R}\supset_\le$$

The proof can be translated into **LG3ipm***:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^x \Rightarrow A^x}{A^x, A^y \Rightarrow B^x, B^y, (C \supset B)^x, A^x}\,(\mathsf{GW}) \qquad \dfrac{B^x, B^y, B^y \Rightarrow B^x, B^y, (C \supset B)^x, A^x}{B^x, B^y \Rightarrow B^x, B^y, (C \supset B)^x, A^x}\,(\mathsf{LC})}{(A \vee B)^x, (A \vee B)^y \Rightarrow B^x, B^y, (C \supset B)^x, A^x}\,\mathsf{L}\vee_\bullet}{(A \vee B)^x, (A \vee B)^y, C^y \Rightarrow B^x, B^y, (C \supset B)^x, A^x}\,(\mathsf{LW})}{(A \vee B)^x \Rightarrow B^x, (C \supset B)^x, A^x}\,\mathsf{R}\supset_\iota}{(A \vee B)^x \Rightarrow (C \supset B)^x, A^x}\,\mathsf{RC}\supset$$

Some ideas for extending the translation of relational proofs to simply labelled proofs in logics weaker than **GD** are examined in Appendix H.

9.3.1. On Translating Pure Relational Rules to Labelled Rules. We conjecture that any relational sequent calculus for **Int***/Geo can be translated into a simply labelled calculus.

Note that the procedure for obtaining hypersequent rules from geometric rules in Section 4.4.5 on page 93 can be adapted to labelled rules.

Recall that the pure relational rules of such a calculus are geometric rules (Section 3.5.3 on page 60) of the form:

$$\frac{\mathbf{x'_{1_1}} \leq \mathbf{y'_{1_1}}, \ldots, \mathbf{x'_{m_1}} \leq \mathbf{y'_{m_1}}, \Sigma', \Sigma; \underline{S} \quad \ldots \quad \mathbf{x'_{1_k}} \leq \mathbf{y'_{1_k}}, \ldots, \mathbf{x'_{m_n}} \leq \mathbf{y'_{m_n}}, \Sigma', \Sigma; \underline{S}}{\Sigma', \Sigma; \underline{S}} \rho_{\leq}$$

where the relational formulae in bold are active relational formulae. (There may also be active relational formulae in Σ', but these in general these can be ignored.)

We present a technique for translating pure relational rules into simply labelled rules, which we conjecture are sound and preserve cut admissibility. Although this work is incomplete, it is worth noting for future work.

For each active label x_i in a rule $\rho_{\leq}$, we associate the metavariables Γ^{x_i} and Δ^{x_i} in the corresponding simply labelled rule.

We apply the transitive unfolding algorithm to each premiss, unfolding the metavariables using the active relational formulae. (In the unfolding, we also include principal relational formulae in Σ' only if they form a cycle.)

EXAMPLE 9.52. *Suppose we have the relational rule* BC_2 *for* **Sm***:*

$$\frac{y \leq x, x \leq y, x \leq z, \Sigma; \underline{S} \quad y \leq z, z \leq y, x \leq y, x \leq z, \Sigma; \underline{S}}{x \leq y, x \leq z, \Sigma; \underline{S}} \mathsf{BC}_{2\leq}$$

Applying the above unfolding procedure, we obtain the following simply labelled rule

$$\frac{\Gamma^x_1, \Gamma^x_2, \Gamma^z_3, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, \Delta^x_1, \Delta^x_2, \Delta^z_3 \quad \Gamma^x_1, \Gamma^y_2, \Gamma^y_3, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, \Delta^x_1, \Delta^y_2, \Delta^y_3}{\Gamma^x_1, \Gamma^y_2, \Gamma^z_3, \underline{\Gamma'} \Rightarrow \underline{\Delta'}, \Delta^x_1, \Delta^y_2, \Delta^z_3} \mathsf{BC}_2$$

This corresponds to a multisuccedent form of the hypersequent rule

$$\frac{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1 \quad \mathcal{H} \mid \Gamma_2, \Gamma_3 \Rightarrow \Delta_2}{\mathcal{H} \mid \Gamma_1, \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \Gamma_3 \Rightarrow \Delta_3} \ \mathsf{BC}_2'$$

from page 89. A notable difference is that the third principal component allows a non-empty succedent (Δ_3). This form of the rule is derivable with RW*, so it is not unsound.*

We conjecture that the resulting rules are sound, although we are concerned that transitive unfolding preserves linear models, and that there might be an implicit linearity property in the rules that we are not aware of.

We conjecture that the resulting simply labelled rules preserve the admissibility of cut. Our justification is that the rules have the following properties (defined in [CGT08], adapted for simply labelled sequents):

Linear Conclusion: Each metavariable occurs at most once in the conclusion. (This holds because no unfolding occurred in the conclusion.)

Separation: No metavariable in the antecedent (viz. succedent) of the conclusion occurs in the succedent (viz. antecedent) of a premiss. (This holds because transitive unfolding does not move formulae between antecedents and succedents.)

Strong Subformula: Every metavariable that occurs in the antecedent (viz. succedent) of a premiss occurs in the antecedent (viz. succedent) of the conclusion. (Again, this holds because no variables were removed.)

(Note that the separation and strong subformula properties are not necessarily the same properties of rules defined in Wansing [Wan94].)

In [CGT08], a semantic proof is given for single-succedent hypersequent calculi where these properties do not affect the validity of cut. We conjecture that this can be extended to multisuccedent calculi.

REMARK 9.53. We also note that [CGT08] gives an algorithm for transforming hypersequent rules that do not have these properties into rules which do.

It may be that the exception made for including in the unfolding principal relational formulae that form a cycle is unnecessary. We note that the sym rule

$$\frac{x \leq y, y \leq x, \Sigma; \underline{S}}{x \leq y, \Sigma; \underline{S}} \ \mathsf{sym}$$

can be transformed into the rule

$$\frac{\Gamma_1^x, \Gamma_2^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_1^y, \Delta_2^y}{\Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}\ \mathsf{S}'$$

which is not the same as the S rule

$$\frac{\Gamma_1^z, \Gamma_2^z, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^z, \Delta_2^z}{\Gamma_1^x, \Gamma_2^y, \underline{\Gamma}' \Rightarrow \underline{\Delta}', \Delta_1^x, \Delta_2^y}\ \mathsf{S}$$

However, the S' rule is adequate for proving excluded middle:

$$\cfrac{\cfrac{A^y, A^x \Rightarrow A^x, A^y}{A^y \Rightarrow A^x}\ \mathsf{S}'}{\Rightarrow A^x, \neg A^x}\ \mathsf{R}\neg_\iota$$

9.4. Conclusion

In this chapter we have followed up on the semantic relationship shown in Chapter 8 with a translation between rules and proofs in labelled calculi and rules and proofs in relational calculi. The translation is consistent with the semantics: labelled proofs are straightforwardly translatable into relational proofs, but relational proofs can only be translated into proofs in systems that require linear models.

Specifically, we have provided two methods for translating a labelled calculus into a relational calculus that is complete with respect to the extended language of relational calculi. The first method produced a calculus **RG3ipm′** simply by adding relational contexts to the rules and a basic set of relational rules. It had the noteworthy property of the refl and trans rules being admissible, but it requires a manual proof of cut admissibility using a form of multicut due to the symmetry between $\mathsf{L}\leq$ and $\mathsf{R}\leq$ rules. The calculus was shown to be equivalent to **G3I**.

The second method involves transforming non-invertible rules and their symmetric rules to produced a calculus **RG3ipm** that absorbed the some of the relational rules, and requires primitive refl and trans rules, but with the advantage that admissibility of cut, weakening and contraction is preserved. The method also allowed the structural multilabelled rules of **LG3ipm*** to be transformed into relational rules which were identical to, or could derive, the corresponding rules in **G3I***. The resulting framework **RG3ipm*** was shown equivalent to **G3I***.

We conjecture that this mechanical method is generalisable for translating any labelled calculus in **Int***/Geo into a corresponding relational calculus that preserves admissibility of cut and other structural rules. However, this is an ear for further investigation.

In the translation, we have not addressed the issue of whether the translation from labelled to relational calculi preserves invertibility, although we conjecture that this is indeed the case. We note that the translations which make use of trivially invertible rules produce invertible relational rules.

A minor result of this work is the recognition that the root rule can play an important role in the proof theory of relational calculi for intermediate logics. However, we are not aware of the rule in any other the relational calculi that we have surveyed.

In the second half of the chapter, we have shown how to translate proofs in a relational calculus of a sequent $\Sigma;\underline{S}$ into proofs in a labelled calculus of $(\Sigma;\underline{S})^\bullet$, that in specific conditions requires the use of the $\mathsf{Com_m}$ rule for **GD**, even when the original relational proof was in a weaker logic that did not use the corresponding lin rule. It is an open question as to whether a different unfolding method can be developed that preserves the original semantics.

We have not given methods for translating relational calculi into labelled calculi in general. The translation method assumes that one already has both kinds of calculi, and can translate proofs. The results of embedding a relational proof inside a labelled proof (Theorem 9.38 on page 207) suggest that this may be straightforward for translating pure relational rules into structural rules. We have outlined a method near the end of the chapter, but have not proven this method to be sound and preserve cut admissibility. We are also unclear on how to apply this method to logical rules such as $\mathsf{R}{\supset_\leq}$.

One potential method of translating relational calculi into simply labelled and hypersequent calculi is to determine a sound method for translating impure relational rules such as $\mathsf{L}{\supset_\leq}$ and $\mathsf{R}{\supset_\leq}$ into their non-relational counterparts, and obtain a calculus such as **RG3ipm′** (Figure 9.1 on page 195). From there the pure relational rules can be transformed into simply labelled or hypersequent rules, if the conjecture in Section 9.3.1 on page 219 can be proven.

Although we give this work for multisuccedent calculi, we believe that the work also holds for single succedent variants of these calculi. One reason why we have not written about single succedent relational calculi is because it is unclear how the accessibility relations which imply a formula holds in a related sequent interacts with the limitation on the number of formulae in the sequent, for example, $x \leq y; \underline{\Gamma} \Rightarrow A^x, B^y$. It would appear that the translation of that relational sequent into a labelled sequent would violate the single-succedent restriction.

We also note that a single-succedent form of transitive unfolding also seems to require a form of the Com rule. For example, the sequent $x \leq y; (A \vee B)^x, (A \supset C)^y \Rightarrow C^y, B^x$, even treated as a single-succedent sequent and transitively unfolded in the antecedent only, cannot be derived without Com. (Again, this is due to the different general semantics of the formalisms.)

As with work investigating the translation between hypersequents and labelled sequents, we have not extended the correspondence for formalisms are based on lists rather than multisets, and where the exchange rules are restricted. That too is an area for future work.

CHAPTER 10

Conclusion

10.1. Overview

Recall that there are several kinds of extensions to Gentzen sequent calculi, which we outlined in Section 1.1.2. The formal relationships between various alternative formalisms is of interest for several reasons:

(1) A relationship that allows for a mechanical transformation allows for a separation of interface from implementation. Thus, a programmer can use a formalism that is more suited for automated theorem proving, while the user can use a different formalism that is more suited for human interaction.

As noted in the introduction, relational sequents can be seen as a simplification of hypersequents that may be more suited for automated proof search. However, it is worth noting that formalisms such as hypersequents were conceived in part to model concurrency, e.g. [Avr91a, Fer03]. A translation from other formalisms into hypersequents may be useful for implementing theorem provers in multi-core systems. Having such translations helps to motivate research in this area.

(2) A formal relationship may help to translate proofs of meta-properties of a calculus or logic into alternative formalisms. For example, proofs of the interpolation property of various intermediate logics generally use single-succedent sequent calculi. Using the relationship to translate such proofs into alternative formalisms may yield novel proofs, particular if the features of particular extensions such as hypersequents or relational sequents allow for novel shortcuts in such proofs.

(3) Formal relationships between formalisms gives concrete and measurable criteria for comparing and evaluating proof systems, with reference to notions of what makes a “good” proof system, as discussed in Section 2.3.

(4) A formal relationship may allow one to create an abstraction of a notation that can express the various formalisms. Such a notation may elucidate on the meaning of various syntactic elements in logical formalisms, and may be of interest in developing notions of proof equality. Manipulation of that notation may allow one to obtain new formalisms.

We note that these various formalisms can be thought of as different kinds of data structures, and that one kind of extension, labelled calculi, can be thought of (informally) as alternative notations for other kinds of extensions, where labels denote the location of a formulae in a data structure, and relational formulae indicate the relationships between locations or formulae in those locations. Thus, labelled calculi can be useful as an intermediary formalism for showing the relationships between other formalisms. In other words, labelled calculi are a notation which can be used to express various proof theoretic formalisms.

The work in this thesis is a small contribution towards showing the various relationships between these formalisms.

10.2. Results

10.2.1. Main Results. In this thesis, we have given a formal description of the relationship between hypersequent calculi for logics in $\mathbf{Int}^*/\mathtt{Geo}$ and two types of labelled sequent calculi, (simply) labelled sequents and (simple) relational sequents.

We originally aimed to show that there is an isomorphism between hypersequents and relational sequents. However, due to differences in the expressivity of the languages of various kinds of sequent calculi, were were unable to show as strong a result. Instead we have shown a correspondence between the formalisms that preserves *linear* models.

The main results of our work are:

(1) There is a *semantic correspondence* between hypersequent calculi and simply labelled sequent calculi, such that one formalism can be translated into the other in a way that preserves models. This holds for labelled sequents with a disjunctive semantics (similar to that of hypersequents) or with a conventional sequent semantics.

(2) The semantic correspondence can be extended to a *syntactic correspondence*, so that one can translate between instantiated sequents and proofs without reference to the underlying semantics. However, the language of labelled sequents is not expressive enough to accommodate empty components. The language of labelled sequents can be extended with a symbol to denote empty lists of formulae (discussed in Appendix F), and give a syntactic *isomorphism* between hypersequents and labelled sequents.

(3) There is no one-to-one relationship between hypersequents (by way of labelled sequents) and relational sequents. The relationship between components of a hypersequent (and labels of a corresponding labelled sequent) are *implicit*, and is determined by set inclusion, whereas the relationships between labels is explicit for relational sequents. When the relationship between components is ambiguous (because two components have the same antecedent or succedent), the choice of how to transform it into a relational sequent is not deterministic.

(4) Hypersequent (by way of labelled sequent) calculi can translated into relational calculi for the same subset of logics in $\mathbf{Int}^*/\mathsf{Geo}$, such that the resulting calculi are complete with respect to the extended language of logic that the added relational formulae allow for.

(5) Relational sequents can be translated into hypersequents (by way of labelled sequents) using a method that is based on the persistence property of logics in $\mathbf{Int}^*/\mathsf{Geo}$ (and which we consider to be a natural method). The sequents in both formalisms share the same *linear* Kripke models (which correspond to the logic **GD**), although in some cases the relational sequents may be derivable in weaker logics.

(6) Proofs in relational sequent calculi can be translated into proofs in hypersequent calculi that have some form of the communication rule (Com). (This corresponds to the semantics results, as communication corresponds to linear Kripke frames.)

These last two points, (5) and (6), that relational sequents and proofs in logics weaker than **GD** can be translated into sequents and proofs in **GD**, seem strange. But that is due to the difference in expressive power of the *languages* of these formalisms. While arbitrary hypersequents for the logics we have studied have translations into the formulae of the logic, it is not clear what formulae in **Int** the relational sequent $x \leq y; (A \vee B)^x \Rightarrow A^x, B^y$ corresponds to, although the sequent is derivable in relational calculi for **Int**, (in an extended notion of a logic that we have called $\mathbf{Int}_{\leq}$). In other words, $\mathbf{Int} \subset \mathbf{Int}_{\leq}$, so formulae from $\mathbf{Int}_{\leq}$ that are not in **Int** have corresponding formulae in **GD**.

Note that we have not provided a translation from relational calculi to labelled or to hypersequent calculi, as it is unclear how to translate arbitrary relational rules, particularly rules that contain active or principal relational formulae, into hypersequent rules. We do give a method for translating geometric rules corresponding to frame axioms in $\mathbf{Int}^*/\mathsf{Geo}$ into hypersequent rules (Section 4.4.5 on page 93), and we discuss conjectures regarding possible translations in Section 9.3.1.

Similarly, we have not given a general method for translating arbitrary geometric frame axioms of Kripke frames into hypersequent rules.

In one sense, this omission is consistent with our results. Recall the equivalence (116):

$$\bigvee_{i=1}^{n} (A_i \supset B_i) \equiv (\bigwedge_{i=1}^{n} A_i) \supset (\bigvee_{i=1}^{n} B_i)$$

The left-to-right direction can be proven in **Int**. Likewise, we are able to construct relational calculi from hypersequent calculi. But the converse direction requires a stronger logic than **Int**. Although the equivalence is classical, we believe that it is worth investigating whether there is an alternative form of the equivalence that only requires linearity, e.g. $(A \supset B) \vee (B \supset A)$, and is derivable in **GD**.

We are not aware of hypersequent or labelled calculi that are cut free and have structural rules which correspond to non-geometric axioms of Kripke frames. However, we conjecture that the translations applied to such rules would yield relational rules which admit cut, weakening and contraction.

We also note that the translation from simply labelled calculi to relational calculi that preserves cut admissibility is given informally.

10.2.2. Other Results. We have identified a class of intermediate logics with corresponding geometric Kripke models ($\mathbf{Int}^*/\mathsf{Geo}$), and a subset of first-order logic called PSF that is adequate to expressing translated formula from logics in $\mathbf{Int}^*/\mathsf{Geo}$ that are valid in the corresponding Kripke models.

We have also introduced a framework of cut-free sequent calculi for PSF, called **G3c/PSF***, that allows for proof-theoretic treatment of the translated formulae. We have used the calculus **G3c/PSF*** to give a correctness proof of the translations of hypersequents (and implicitly labelled sequents) into PSF. We believe this is a novel technique, as the correctness of such translations is generally given model-theoretically.

The naming of components in labelled sequent calculi allows the distinction between standard external and internal structural rules in hypersequents to be reduced to whether the names of labels fresh for the premiss or conclusion.

We have introduced new hypersequent calculi, by hyperextending a well-known sequent calculus **m-G3i** into a base calculus for the framework of hypersequent calculi **HG3ipm*** for some logics **Int***/Geo, and by translating an unusual labelled calculus **O** into a hypersequent calculus **HO**.

We have also introduced novel frameworks of labelled calculi for **Int***/Geo, **LG3ipm***, **RG3ipm′** and **RG3ipm***, by translating the rules of the hypersequent calculi in **HG3ipm*** using methods given here. The translations of hypersequent between labelled calculi, and one of the translations of labelled calculi to relational calculi preserve cut admissibility of the calculi. The calculus **RG3ipm′** has the interesting property that the reflexivity and transitivity rules are admissible, rather than primitive.

We have identified the root rule as a sound rule for relational sequents for **Int***/Geo, which we have not seen used in other calculi. The rule allows one to show connections between hypersequent and relational rules such as LQ and dir, where the root rule (which roughly corresponds to the elimination of empty components in hypersequent calculi) is implicit.

We developed a notion of normal relational sequent, called a grounded sequent, and shown that one can derive a grounded sequent from a labelled sequent, and that, more importantly, that it is a unique grounded sequent. (However, multiple simply labelled sequents may have the same grounded sequent, so it is not a bijection.)

We have also shown that from relational sequents one can derive simply labelled sequents by a (trivial) process called flattening. This process essentially merges labels, and so removes much of the structure in the sequent. It does not preserve models, and is not useful for translating proofs or calculi.

10.3. Open Problems

We have not investigated the hierarchy of logics that are in $\mathbf{Int}^*/\mathsf{Geo}$, or what subset of them have the finite model property (FMP). Nor have we provided complexity or decidability results about formulae in PSF, or closely examined the relationship between PSF and GF (Guarded Formula) [AvN96] or related subsets of first-order formulae, e.g. LGF (Loosely Guarded Formulae) [Hod02]. An understanding of the hierarchy of logics in $\mathbf{Int}^*/\mathsf{Geo}$ between **GD** and **Cl** (includes the logics in the class $\mathbf{G_k}$) is of interest with respect to the translation from relational proofs to labelled and hypersequent proofs.

In the relational calculus **RG3ipm′**, we have identified a connection between contraction and reflexivity, and between weakening and transitivity. This suggests that substructural logics may be used to provide hypersequent or simply labelled calculi for logics which are not in $\mathbf{Int}^*/\mathsf{Geo}$, e.g. subintuitionistic logics and their non-intuitionistic extensions.

The translation has assumed the presence of permutation (by using multisets), as well as contraction and weakening. We have not investigated whether the translation can be extended to substructural logics. However, we note that the relational semantics of substructural logics is much more complex, e.g. the “residuated Kripke frames” in [MS03]. This may require an extension of the language of relational calculi.

The translation has only been given for multisuccedent calculi, whereas hypersequent calculi for intermediate logics in the literature are generally single-succedent calculi. We believe that this work can be extended to such calculi without difficulty, although we note that the relational formulae may impose additional restrictions on the succedent, e.g. $x \le y; \underline{\Gamma} \Rightarrow A^x, B^y$ may not be a well-formed sequent because B^x implicitly occurs in the succedent, which is restricted to at most one formula per label. This has yet to be investigated.

An open area of investigation is to followup on the conjecture in Section 9.3.1 on page 219 about the relationship between the rules generated by the “completion” procedure in [CGT08] and geometric rules discussed in [Neg03]. If the conjecture is true, then these works may be adapted to obtain an automated procedure for generating hypersequent and relational calculi for a variety of logics.

We have not given a general hypersequent framework for logics in **Int***/Geo. We note that work in [CGT08] discusses the properties of single-succedent hypersequent structural rules which admit cut (noted in Section 9.3.1). We believe that if a connection between these properties and geometric rules can be shown, then we can describe such a framework in general terms, and use this to obtain novel hypersequent calculi for logics in **Int***/Geo.

We have not analysed beyond a superficial manner how the translation between formalisms affects the complexity of proofs. Nor have we fully analysed how it affects properties of rules, such as invertibility, or how it affects properties related to criteria for analysing proofs, such as explicitness or separation. (The conjecture about the preservation of invertibility of rules when translated labelled rules to relational rules, noted in Remark 9.21 on page 202, is of interest.)

We have not actually used the translation from hypersequents to relational calculi to translate proofs of properties about the logics. For example, a translation of a proof of the Craig interpolation theorem in the seven interpolable intermediate logics (c.f. [Mak79]) into relational calculi would be an interesting result.

10.3.1. Future Work. The work in this thesis suggests several directions for future work.

Clearly the conjectures noted in this thesis need to be explored. In particular, the conjectures about translating pure relational rules into labelled rules requires further investigation.

One idea is to investigate the hierarchy of labelled calculi, and what minimum complexity of labelling expressions or relational formulae is needed for corresponding substructural logics, or for a correspondence with other proof formalisms, such as higher order sequents or display calculi. In part this requires an analysis of the various types of notation.

Of particular interest is an examination of relational calculi that allow relational formulae in the succedent, such as the systems in [Vig00], or connection systems which not only allow relational formulae in the succedent, but allow variables in place of labels, e.g. [GM07]. This may also allow us to develop cut-free relational calculi for logics outside of **Int***/Geo, such as **KP**.

Another idea is to combine the work in [Gab96] and [CGT08], and to investigate the relationship between Hilbert-style axiomatisations, structural rules and frame conditions of Kripke models.

We would like to extend this work to other superintuitionistic systems with additional connectives, such as logics of strong negation, e.g. [Nel49] or bi-intuitionistic logic (**BiInt**) [Rau74], using the translation to PSF and the calculus **G3c/PSF***. **BiInt** in particular has historically been difficult to provide cut-free calculi for (c.f. [PU09]), and an analysis using **G3c/PSF*** may prove useful.

APPENDIX A

Calculi from the Referenced Literature

For reference, we include the rules of various calculi from elsewhere in the literature. Cut admissibility has been shown for these calculi in the relevant sources, unless otherwise noted.

The calculus **G1c** [TS00, Nv01] is given in Figure A.1.

$$\frac{}{\Gamma, P \Rightarrow P, \Delta}\ \mathsf{Ax} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \mathsf{L}\bot$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \mathsf{L}\wedge_1 \qquad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \mathsf{L}\wedge_2 \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ \mathsf{R}\wedge$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ \mathsf{R}\vee_1 \qquad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ \mathsf{R}\vee_2$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}\ \mathsf{L}\supset \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}\ \mathsf{R}\supset$$

$$\frac{\Gamma, [t/x]A \Rightarrow \Delta}{\Gamma, \forall x.A \Rightarrow \Delta}\ \mathsf{L}\forall \qquad \frac{\Gamma \Rightarrow [y/x]A, \Delta}{\Gamma \Rightarrow \forall x.A, \Delta}\ \mathsf{R}\forall$$

$$\frac{\Gamma, [y/x]A \Rightarrow \Delta}{\Gamma, \exists x.A \Rightarrow \Delta}\ \mathsf{L}\exists \qquad \frac{\Gamma \Rightarrow [t/x]A, \Delta}{\Gamma \Rightarrow \exists x.A, \Delta}\ \mathsf{R}\exists$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathsf{LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\ \mathsf{RW} \qquad \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathsf{LC} \qquad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}\ \mathsf{RC}$$

where P is atomic, and y is fresh in the conclusions of the $\mathsf{R}\forall$ and $\mathsf{L}\exists$ rules.

FIGURE A.1. The calculus **G1c** for **Cl**.

For **G1i**, the succedent is limited to at most one formulae, so the side formula in the succedent of the succedent rules is omitted. **G1m** is **G1i** without the $\mathsf{L}\bot$ rule.

The calculus **G3c** [TS00, Nv01] is given in Figure A.2 on the following page.

$$\frac{}{\Gamma, P \Rightarrow P, \Delta}\ \mathsf{Ax} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \mathsf{L}\bot$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ \mathsf{R}\wedge$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ \mathsf{R}\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}\ \mathsf{L}\supset \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}\ \mathsf{R}\supset$$

$$\frac{\Gamma, \forall x.A, [t/x]A \Rightarrow \Delta}{\Gamma, \forall x.A \Rightarrow \Delta}\ \mathsf{L}\forall \qquad \frac{\Gamma \Rightarrow [y/x]A, \Delta}{\Gamma \Rightarrow \forall x.A, \Delta}\ \mathsf{R}\forall$$

$$\frac{\Gamma, [y/x]A \Rightarrow \Delta}{\Gamma, \exists x.A \Rightarrow \Delta}\ \mathsf{L}\exists \qquad \frac{\Gamma \Rightarrow [t/x]A, \exists x.A, \Delta}{\Gamma \Rightarrow \exists x.A, \Delta}\ \mathsf{R}\exists$$

where P is atomic, and y is fresh in the conclusions of the $\mathsf{R}\forall$ and $\mathsf{L}\exists$ rules.

FIGURE A.2. The calculus **G3c** for **Cl**.

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}\ \mathsf{L}\supset \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B, \Delta}\ \mathsf{R}\supset \qquad \frac{\Gamma \Rightarrow [y/x]A}{\Gamma \Rightarrow \forall x.A, \Delta}\ \mathsf{R}\forall$$

where in $\mathsf{R}\forall$ $x \notin FV(\Gamma)$, $y = x$ or $y \notin FV(\Gamma, A)$.

FIGURE A.3. Rules from the calculus **m-G3i** for **Int**.

The calculus **m-G3i** [TS00] is a multisuccedent variant of the calculus **G3i** for **Int**. The rules are identical to **G3c** except for the rules in Figure A.3, where the $\mathsf{R}\supset$ and $\mathsf{R}\forall$ rules are restricted to single-succedent premisses and the principal formula in the $\mathsf{L}\supset$ rule is also active in the left premiss. Cut admissibility for **m-G3i** is actually shown in [Dyc97].

PROPOSITION A.1 (Weakening and Contraction). *The weakening and contraction rules*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (\mathsf{LW}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\ (\mathsf{RW}) \qquad \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (\mathsf{LC}) \qquad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}\ (\mathsf{RC})$$

are depth-preserving admissible in **G3c** *and* **m-G3i**.

Proof. See [TS00]. □

PROPOSITION A.2. *The* $\mathsf{L}\forall^*$ *rule*

$$\frac{\Gamma, \forall \bar{x}.A, [\bar{y}/\bar{x}]A \Rightarrow \Delta}{\Gamma, \forall \bar{x}.A \Rightarrow \Delta}\ (\mathsf{L}\forall^*)$$

is derivable in **G3c** *and* **m-G3i**.

Proof. By induction on the number of variables in $\bar{x}$. The base case (a single variables) is equivalent to $\mathsf{L}\forall$. For $n \geq 2$ variables, we use alternating applications of $\mathsf{L}\forall$ and LW:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[t_n/x_n]\ldots[t_1/x_1]A, \forall x_1,\ldots,x_n.A, \Gamma \Rightarrow \Delta \\ \vdots \\ \forall x_3,\ldots,x_n.[t_2/x_2][t_1/x_1]A, \forall x_1,\ldots,x_n.A, \Gamma \Rightarrow \Delta\end{array}}{\forall x_3,\ldots,x_n.[t_2/x_2][t_1/x_1]A, \forall x_2,\ldots,x_n.[t_1/x_1]A, \forall x_1,\ldots,x_n.A, \Gamma \Rightarrow \Delta}\ (\mathsf{LW})}{\forall x_2,\ldots,x_n.[t_1/x_1]A, \forall x_1,\ldots,x_n.A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\forall}{\forall x_1,\ldots,x_n.A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\forall$$

□

PROPOSITION A.3. *The* $\mathsf{L}\exists^*$ *rule*

$$\frac{\Gamma, \exists \bar{x}.A, [\bar{z}/\bar{x}]A \Rightarrow \Delta}{\Gamma, \exists \bar{x}.A \Rightarrow \Delta}\ (\mathsf{L}\exists^*)$$

is derivable in **G3c** *and* **m-G3i**.

Proof. Similar to Proposition A.2. □

PROPOSITION A.4 (Substitution). *Variable substitution*

$$\frac{\Gamma \Rightarrow \Delta}{[t/x]\Gamma \Rightarrow [t/x]\Delta}\ [\mathsf{t/x}]$$

where t is free for x in Γ, Δ (that is, t does not contain a free variable y that would become bound by replacing x with t in $\Gamma \Rightarrow \Delta$) is depth-preserving admissible in **G3c** *and* **m-G3i**.

Proof. See [TS00]. □

NOTATION A.5. For readability, we omit parentheses from instances of the substitution rule.

The calculus **LJpm** [Min00] (Figure A.4 on the following page) is similar to **m-G3i**, except that the structural rules are primitive and the axioms have no side formulae.

In [Min00], cut admissibility for **LJpm** is shown semantically.

$$\frac{}{P \Rightarrow P}\ \mathsf{Ax} \qquad \frac{}{\bot \Rightarrow P}\ \mathsf{L}\bot$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathsf{LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\ \mathsf{RW} \qquad \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathsf{LC} \qquad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}\ \mathsf{RC}$$

FIGURE A.4. Rules from the calculus **LJpm** for **Int**.

PROPOSITION A.6 (Interderivability). **m-G3i** $\vdash \Gamma \Rightarrow \Delta$ *iff* **LJpm** $\vdash \Gamma \Rightarrow \Delta$.

Proof. By induction on the derivation depths. We note that the structural rules of **LJpm** are admissible in **m-G3i**. ◻

APPENDIX B

Partially Shielded Formulae

Below are proofs regarding PSF and the framework **G3c/PSF*** from Section 3.5 on page 52.

PROPOSITION B.1. *The* L∀⊃′ *rule*

$$\frac{\Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z}, \forall\bar{y}\ldots, B\bar{x}\bar{z} \Rightarrow \Delta}{\Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z}, \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge \bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{y}) \supset B\bar{x}\bar{y} \Rightarrow \Delta} \text{(L}\forall\supset'\text{)}$$

is derivable in **G3c/PSF**.

Proof. Context formulae omitted for brevity:

$$\frac{\langle\, A_i\bar{x}\bar{z} \Rightarrow A_i\bar{x}\bar{z}\,\rangle_{i=1}^{n} \quad \Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z}, \forall\bar{y}\ldots, B\bar{x}\bar{z} \Rightarrow \Delta}{\Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z}, \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge \bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{y}) \supset B\bar{x}\bar{y} \Rightarrow \Delta} \text{(L}\forall\supset\text{)}$$

□

LEMMA B.2 (Embedding). *If* **G3c/PSF** $\vdash \Gamma \Rightarrow \Delta$, *then* **G3c** $\vdash \Gamma \Rightarrow \Delta$.

Proof. The R⊤, L∀, R∀, L∀⊃ and R∀⊃ rules of **G3c/PSF** are derivable in **G3c**. The other rules are identical.

(1) The R⊤ axiom is derivable in **G3c**.

$$\cfrac{\cfrac{\Gamma, \bot \Rightarrow \bot, \Delta}{\Gamma \Rightarrow \bot \supset \bot, \Delta}\,\text{R}\supset}{\Gamma \Rightarrow \top, \Delta}\,\top_{\text{def}}$$

See Remark B.3 below.

(2) The L∀ rule is equivalent to the L∀* rule in **G3c** (Proposition A.2 on page 235).

(3) The R∀ rule is equivalent to multiple applications of the R∀ rule in **G3c**.

(4) The L∃ rule is equivalent to the L∃* rule in **G3c** (Proposition A.3 on page 235).

(5) The R∃ rule is equivalent to multiple applications of the R∃ rule in **G3c**.

(6) The L∀⊃ can be derived, (omitting some context formulae below for brevity).

If $n = 0$, then (138) is the axiom $\Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.\ldots \Rightarrow \mathcal{R}\bar{x}\bar{y}, \Delta$. Otherwise, (138) is derived from

$$\dfrac{\mathcal{R}\bar{x}\bar{z} \Rightarrow \mathcal{R}\bar{x}\bar{z} \qquad \dfrac{\Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.\ldots \Rightarrow A_1\bar{x}\bar{y}, \Delta \quad \ldots \quad \Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.\ldots \Rightarrow A_n\bar{x}\bar{y}, \Delta}{\Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.\ldots \Rightarrow A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}, \Delta}\,\mathsf{R}\wedge^*}{\Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.\ldots \Rightarrow \mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}, \Delta}\,\mathsf{R}\wedge \tag{138}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\ (138)\\ \Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.\ldots \Rightarrow \mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}, \Delta\end{matrix} \quad \Gamma, \mathcal{R}\bar{x}\bar{z}, B\bar{x}\bar{z}, \forall\bar{y}.\ldots \Rightarrow \Delta}{\Gamma, \mathcal{R}\bar{x}\bar{z}, (\mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}) \supset B\bar{x}\bar{z}, \forall\bar{y}.\ldots \Rightarrow \Delta}\,\mathsf{L}\supset}{\Gamma, \mathcal{R}\bar{x}\bar{z}, \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}) \supset B\bar{x}\bar{y} \Rightarrow \Delta}\,(\mathsf{L}\forall^*)$$

(7) The $\mathsf{R}\forall\supset$ rule can be derived.

$$\dfrac{\dfrac{\dfrac{\Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z} \Rightarrow B\bar{x}\bar{z}, \Delta}{\Gamma, \mathcal{R}\bar{x}\bar{z} \wedge A_1\bar{x}\bar{z} \wedge \ldots \wedge A_n\bar{x}\bar{y} \Rightarrow B\bar{x}\bar{z}, \Delta}\,\mathsf{L}\wedge^*}{\Gamma \Rightarrow (\mathcal{R}\bar{x}\bar{z} \wedge A_1\bar{x}\bar{z} \wedge \ldots \wedge A_n\bar{x}\bar{y}) \supset B\bar{x}\bar{z}, \Delta}\,\mathsf{R}\supset}{\Gamma \Rightarrow \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}) \supset B\bar{x}\bar{y}, \Delta}\,\mathsf{R}\forall^+$$

□

REMARK B.3. Because $\top$ is primitive in **G3c/PSF** but defined in **G3c**, we should use a translation of formulae in the former to formulae in the latter, such as

$$\mathtt{f}\,A =_{def} \begin{cases} \bot \supset \bot & \text{if } A = \top \\ A & A \text{ is atomic or } \bot \\ (\mathtt{f}\,B) \odot (\mathtt{f}\,C) & A = B \odot C \end{cases}$$

for Lemma B.2. We omit it for brevity.

LEMMA B.4 (Nullary Connective Deletion). *The following constant deletion rules*

$$\dfrac{\top, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\mathsf{L}\top) \qquad \dfrac{\Gamma \Rightarrow \Delta, \bot}{\Gamma \Rightarrow \Delta}\,(\mathsf{R}\bot)$$

are depth-preserving admissible in **G3c/PSF**.

Proof. By induction on the derivation depth, with the following cases:

(1) Suppose the premiss of $\mathsf{L}\top$ (viz. $\mathsf{R}\bot$) is an axiom. Then so is the conclusion.

(2) For all other rules, $\top$ (viz. $\bot$) is not the principal formulae (being a nullary connective). So by the induction hypothesis, we can permute $\mathsf{L}\top$ (viz. $\mathsf{R}\bot$) to lower derivation depth.

□

LEMMA B.5 (Contraction). *The contraction rules*

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (\mathsf{LC}) \qquad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}\ (\mathsf{RC})$$

are depth-preserving admissible in **G3c/PSF**.

Proof. Straightforward simultaneous induction on the derivation depth. Recall that the rules are invertible (Lemma 3.55 on page 59). We show the interesting cases below:

(1) Suppose one of the active formulae is the principal formula of an instance of the $\mathsf{L}\forall\supset$ rule—either the shield $\mathcal{R}\bar{x}\bar{z}$ or $\forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}) \supset B\bar{x}\bar{y}$.

Because the principal formula are also active formulae, by the induction hypothesis we permute the instance(s) of LC to a lower depth.

(2) Suppose one of the active formula is the principal formula of an instance of the $\mathsf{R}\forall$ rule:

$$\cfrac{\cfrac{\Gamma \Rightarrow \Delta, \forall\bar{x}.A\bar{x}, A\bar{z}}{\Gamma \Rightarrow \Delta, \forall\bar{x}.A\bar{x}, \forall\bar{x}.A\bar{x}}\ \mathsf{R}\forall}{\Gamma \Rightarrow \Delta, \forall\bar{x}.A\bar{x}}\ (\mathsf{RC})$$

From the depth-preserving invertibility of $\mathsf{R}\forall$ (Lemma 3.55), we can derive the following, and apply an instance of RC at lower derivation depth:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \Rightarrow \Delta, \forall\bar{x}.A\bar{x}, A\bar{z}}{\Gamma \Rightarrow \Delta, A\bar{z}', A\bar{z}}\ (\mathsf{R}\forall^{-1})}{\Gamma \Rightarrow \Delta, A\bar{z}, A\bar{z}}\ [\mathsf{z/z'}]}{\Gamma \Rightarrow \Delta, A\bar{z}}\ (\mathsf{RC})}{\Gamma \Rightarrow \Delta, \forall\bar{x}.A\bar{x}}\ \mathsf{R}\forall$$

(3) Suppose one of the active formulae is the principal formula of an instance of the $\mathsf{R}\forall\supset$ rule:

$$\cfrac{\cfrac{\Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z} \Rightarrow \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge \bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{y}) \supset B\bar{x}\bar{y}, B\bar{x}\bar{z}, \Delta}{\Gamma \Rightarrow \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge \bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{y}) \supset B\bar{x}\bar{y}, \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge \bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{y}) \supset B\bar{x}\bar{y}, \Delta}\ \mathsf{R}\forall\supset}{\Gamma \Rightarrow \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge \bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{y}) \supset B\bar{x}\bar{y}, \Delta}\ (\mathsf{RC})$$

where $\bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{y} = A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}$. From the invertibility of $\mathsf{R}\forall\supset$ (Lemma 3.55), we can derive the following, and apply the LC and RC rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z} \Rightarrow \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge \bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{y}) \supset B\bar{x}\bar{y}, B\bar{x}\bar{z}, \Delta}{\Gamma, \mathcal{R}\bar{x}\bar{z}, \mathcal{R}\bar{x}\bar{z}', A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z}, A_1\bar{x}\bar{z}', \ldots, A_n\bar{x}\bar{z}' \Rightarrow B\bar{x}\bar{z}', B\bar{x}\bar{z}, \Delta}\ (\mathsf{R}\forall\supset^{-1})}{\Gamma, \mathcal{R}\bar{x}\bar{z}, \mathcal{R}\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z}, A_1\bar{x}\bar{z}, \ldots, A_n\bar{x}\bar{z} \Rightarrow B\bar{x}\bar{z}, B\bar{x}\bar{z}, \Delta}\ [\bar{z}/\bar{z}']}{\Gamma, \mathcal{R}\bar{x}\bar{z}, \bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{z} \Rightarrow B\bar{x}\bar{z}, B\bar{x}\bar{z}, \Delta}\ (\mathsf{LC})^{+}}{\Gamma, \mathcal{R}\bar{x}\bar{z}, \bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{z} \Rightarrow B\bar{x}\bar{z}, \Delta}\ (\mathsf{RC})}{\Gamma \Rightarrow \forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge \bigwedge\!\!\!\bigwedge A_i\bar{x}\bar{y}) \supset B\bar{x}\bar{y}, \Delta}\ \mathsf{R}\forall\supset$$

□

THEOREM B.6 (Cut). *The context-splitting cut rule*

$$\dfrac{\Gamma \Rightarrow A, \Delta \quad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathsf{cut})$$

is admissible in **G3c/PSF**.

Proof. By induction on the cut rank (a lexically-ordered pair consisting of the size of the cut formula and sum of the depths of the premisses). We give the interesting cases below:

(1) Suppose the cut formula is the principal formula from an instance of the $\mathsf{L}\forall\supset$ rule, $\forall\bar{y}.(\mathcal{R}\bar{x}\bar{y} \wedge A_1\bar{x}\bar{y} \wedge \ldots \wedge A_n\bar{x}\bar{y}) \supset B\bar{x}\bar{y}$ (which we abbreviate as $\forall\bar{y}\ldots.$ in the derivation fragments below):

$$\dfrac{\overset{\vdots\ \delta_0}{\Gamma \Rightarrow \Delta, \forall\bar{y}\ldots.} \qquad \dfrac{\overset{\vdots\ \delta_i}{\mathcal{R}\bar{x}\bar{z}, \forall\bar{y}\ldots., \Gamma' \Rightarrow A_i\bar{x}\bar{z}, \Delta'} \quad \ldots \quad \overset{\vdots\ \delta_{n+1}}{\mathcal{R}\bar{x}\bar{z}, \forall\bar{y}\ldots., B\bar{x}\bar{z}, \Gamma' \Rightarrow \Delta'}}{\mathcal{R}\bar{x}\bar{z}, \forall\bar{y}\ldots., \Gamma' \Rightarrow \Delta'}\ \mathsf{L}\forall\supset}{\mathcal{R}\bar{x}\bar{z}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathsf{cut})$$

The cut rank is $\langle\ |\forall\bar{y}\ldots|,\ h(\delta_0) + 1 + \max\{h(\delta_1), \ldots, h(\delta_{n+1})\}\ \rangle$. We can replace this cut with $2n+2$ cuts of smaller rank. We start (if $n > 0$) with n cuts of the first premiss of the original cut with the nth premiss of the $\mathsf{L}\forall\supset$ instance:

$$\dfrac{\overset{\vdots\ \delta_0}{\Gamma \Rightarrow \Delta, \forall\bar{y}\ldots.} \quad \overset{\vdots\ \delta_i}{\forall\bar{y}\ldots., \mathcal{R}\bar{x}\bar{z}, \Gamma' \Rightarrow A_i\bar{x}\bar{z}, \Delta'}}{\mathcal{R}\bar{x}\bar{z}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', A_i\bar{x}\bar{z}}\ (\mathsf{cut}) \tag{139}$$

each of which has rank $\langle\ |\forall\bar{y}\ldots|,\ h(\delta_0) + h(\delta_i)\ \rangle$, for $i = 1, \ldots, n$. We also add a cut of the first premiss of the original cut with the last premiss:

$$\dfrac{\begin{array}{c}\vdots\;\delta_0\\ \Gamma\Rightarrow\Delta,\forall\bar{y}.\ldots\end{array}\qquad \begin{array}{c}\vdots\;\delta_{n+1}\\ \forall\bar{y}.\ldots,B\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma'\Rightarrow\Delta'\end{array}}{B\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\;(\mathsf{cut}) \tag{140}$$

which has rank $\langle\ |\forall\bar{y}\ldots|,\ h(\delta_0)+h(\delta_{n+1})\ \rangle$. We then add a cut of the first result of (139) with the result of applying $\mathsf{R}\forall\supset^{-1}$ to the first premiss of the original cut:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\;(139)\\ \mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta',A_1\bar{x}\bar{z}\end{array}\qquad \dfrac{\dfrac{\begin{array}{c}\vdots\;\delta_0\\ \Gamma\Rightarrow\Delta,\forall\bar{y}.\ldots\end{array}}{A_1\bar{x}\bar{z}',\ldots,A_n\bar{x}\bar{z}',\mathcal{R}\bar{x}\bar{z}',\Gamma\Rightarrow\Delta,B\bar{x}\bar{z}'}\;(\mathsf{R}\forall\supset^{-1})}{A_1\bar{x}\bar{z},\ldots,A_n\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma\Rightarrow\Delta,B\bar{x}\bar{z}}\;[\mathsf{z/z'}]}{A_2\bar{x}\bar{z},\ldots,A_n\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma,\Gamma'\Rightarrow\Delta,\Delta,\Delta',B\bar{x}\bar{z}}\;(\mathsf{cut})}{A_2\bar{x}\bar{z},\ldots,A_n\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta,\Delta',B\bar{x}\bar{z}}\;(\mathsf{LC})^+}{A_2\bar{x}\bar{z},\ldots,A_n\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta',B\bar{x}\bar{z}}\;(\mathsf{LR})^* \tag{141}$$

each of which has cut rank $\langle\ |A_1\bar{x}\bar{z}|,\ \ldots\ \rangle$ (the depths of premisses are irrelevant). For each $1 < i \leq n$, we repeatedly cut the result with

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\;(139)\\ \mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta',A_i\bar{x}\bar{z}\end{array}\qquad \begin{array}{c}\vdots\;(142)\\ A_i\bar{x}\bar{z},\ldots,A_n\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta',B\bar{x}\bar{z}\end{array}}{A_{i+1}\bar{x}\bar{z},\ldots,A_n\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma,\Gamma'\Rightarrow\Delta,\Delta,\Delta',B\bar{x}\bar{z}}\;(\mathsf{cut})}{A_{i+1}\bar{x}\bar{z},\ldots,A_n\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta,\Delta',B\bar{x}\bar{z}}\;(\mathsf{LC})^+}{A_{i+1}\bar{x}\bar{z},\ldots,A_n\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta',B\bar{x}\bar{z}}\;(\mathsf{LR})^* \tag{142}$$

which will have a cut rank of $\langle\ |A_i\bar{x}\bar{z}|,\ \ldots\ \rangle$, and apply LC and RC as appropriate. The result after n cuts will be $\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta',A_i\bar{x}\bar{z}$. We apply a final cut,

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\;(142)\\ \mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta',B\bar{x}\bar{z}\end{array}\qquad \begin{array}{c}\vdots\;(140)\\ B\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta'\end{array}}{\mathcal{R}\bar{x}\bar{z},\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma,\Gamma',\Gamma'\Rightarrow\Delta,\Delta,\Delta',\Delta'}\;(\mathsf{cut})}{\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta,\Delta',\Delta'}\;(\mathsf{LC})^+}{\mathcal{R}\bar{x}\bar{z},\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\;(\mathsf{RC})^* \tag{143}$$

which has rank $\langle\ |B\bar{x}\bar{z}|,\ \ldots\ \rangle$, and apply LC and RC as appropriate.

(2) Suppose the cut formula is the other principal formula from an instance of the $\mathsf{L}\forall\supset$ rule, $\mathcal{R}\bar{x}\bar{z}$ (the shield):

$$\dfrac{\begin{array}{c}\vdots\;\delta_0\\ \Gamma\Rightarrow\Delta,\mathcal{R}\bar{x}\bar{z}\end{array}\qquad \dfrac{\begin{array}{c}\vdots\;\delta_i\\ \mathcal{R}\bar{x}\bar{z},\forall\bar{y}.\ldots,\Gamma'\Rightarrow\Delta',A_i\bar{x}\bar{z}\end{array}\qquad \begin{array}{c}\vdots\;\delta_{n+1}\\ \mathcal{R}\bar{x}\bar{z},B\bar{x}\bar{z},\forall\bar{y}.\ldots,\Gamma'\Rightarrow\Delta'\end{array}}{\mathcal{R}\bar{x}\bar{z},\forall\bar{y}.\ldots,\Gamma'\Rightarrow\Delta'}\;\mathsf{L}\forall\supset}{\forall\bar{y}.\ldots,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\;(\mathsf{cut})$$

The cut rank is $\langle\ |\mathcal{R}\bar{x}\bar{z}|, h(\delta_0)+1+\max\{h(\delta_1),\ldots,h(\delta_{n+1}\}\ \rangle$. We have the following subcases:

(a) Suppose the left premiss is an axiom with $\mathcal{R}\bar{x}\bar{z}$ principal. Then we can obtain $\forall\bar{y}.\ldots,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'$ from the second premiss by weakening.

(b) Suppose the left premiss is an axiom where $\mathcal{R}\bar{x}\bar{z}$ is not principal (either as an instance of Ax or $\mathsf{L}\bot$). Then the conclusion of the cut is also an axiom.

(c) Otherwise the second premiss is the result of a rule where $\mathcal{R}\bar{x}\bar{z}$ is not principal (since it is atomic). We replace the cut with cuts on the $n+1$ premiss(es) of that inference, applying weakening, contractions or substitution of free variables as needed. Those cuts will have a cut rank of $\langle\, |\mathcal{R}\bar{x}\bar{z}|, h(\delta_0)+h(\delta_i)\,\rangle$, for $i = 1,\ldots n+1$, each of which is lower.

We note there is a similar case in [DS06].

□

LEMMA B.7. **G3c/PSF*** $\vdash \mathcal{R}xy, A, \Gamma \Rightarrow \Delta, [y/x]A$, *for all* $A \in \mathsf{SPSF}$, *where* $y\#A$.

Proof. By induction on the structure of the formulae A. We assume, as the induction hypothesis (IH), that for all subformulae $A' \in \mathsf{SPSF}$ of A, **G3c/PSF*** $\vdash \mathcal{R}xy, A', \Gamma \Rightarrow \Delta, [y/x]A'$. We have the following cases:

(1) Suppose $A = Px$ (an atomic formula). Then we have the following derivation:

$$\dfrac{\mathcal{R}xy, Px, Py, \Gamma \Rightarrow \Delta, Py}{\mathcal{R}xy, Px, \Gamma \Rightarrow \Delta, Py}\ \mathsf{LF}_0$$

(2) Suppose $A = \bot$. Trivial.

(3) Suppose $A = Bx \wedge Cx$. Then we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdots\ (\mathsf{IH})}{\mathcal{R}xy, Bx, \Gamma \Rightarrow \Delta, By} \quad \dfrac{\vdots\ (\mathsf{IH})}{\mathcal{R}xy, Cx\Gamma \Rightarrow \Delta, Cy}}{\mathcal{R}xy, Bx, Cx, \Gamma \Rightarrow By \wedge \Delta, Cy}\ \mathsf{R}\wedge}{\mathcal{R}xy, Bx \wedge Cx, \Gamma \Rightarrow By \wedge \Delta, Cy}\ \mathsf{L}\wedge$$

(4) Suppose $A = Bx \vee Cx$. Then we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ (\mathsf{IH})}{\mathcal{R}xy, Bx, \Gamma \Rightarrow \Delta, By}}{\mathcal{R}xy, Bx, \Gamma \Rightarrow \Delta, By, Cy}\ (\mathsf{RW})}{\mathcal{R}xy, Bx, \Gamma \Rightarrow \Delta, By \vee Cy}\ \mathsf{R}\vee \quad \dfrac{\dfrac{\dfrac{\vdots\ (\mathsf{IH})}{\mathcal{R}xy, Cx, \Gamma \Rightarrow \Delta, Cy}}{\mathcal{R}xy, Cx, \Gamma \Rightarrow \Delta, By, Cy}\ (\mathsf{RW})}{\mathcal{R}xy, Cx, \Gamma \Rightarrow \Delta, By \vee Cy}\ \mathsf{R}\vee}{\mathcal{R}xy, Bx \vee Cx, \Gamma \Rightarrow \Delta, By \vee Cy}\ \mathsf{L}\vee$$

(5) Suppose $A = \forall w.(\mathcal{R}xw \wedge Bw) \supset Cw$. Then we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\mathcal{R}xz, \mathcal{R}xy, \mathcal{R}yz, \forall w.(\mathcal{R}xw \wedge Bw) \supset Cw, Bz, Cz, \Gamma \Rightarrow \Delta, Cz}{\mathcal{R}xz, \mathcal{R}xy, \mathcal{R}yz, \forall w.(\mathcal{R}xw \wedge Bw) \supset Cw, Bz, \Gamma \Rightarrow \Delta, Cz}\,(\mathsf{L}\forall\supset')}{\mathcal{R}xy, \mathcal{R}yz, \forall w.(\mathcal{R}xw \wedge Bw) \supset Cw, Bz, \Gamma \Rightarrow \Delta, Cz}\,\mathsf{trans}}{\mathcal{R}xy, \forall w.(\mathcal{R}xw \wedge Bw) \supset Cw, \Gamma \Rightarrow \Delta, \forall w.(\mathcal{R}yw \wedge Bw) \supset Cw}\,\mathsf{R}\forall\supset$$

(6) Suppose $A = \forall \bar{w}.B\bar{w}x$. Then we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ (\mathsf{IH})\\ \mathcal{R}xy, B\bar{z}x, \Gamma \Rightarrow \Delta, B\bar{z}y\end{array}}{\mathcal{R}xy, \forall \bar{w}.B\bar{w}x, B\bar{z}x\Gamma, \Rightarrow \Delta, B\bar{z}y}\,(\mathsf{LW})}{\mathcal{R}xy, \forall \bar{w}.B\bar{w}x, \Gamma \Rightarrow \Delta, B\bar{z}y}\,\mathsf{L}\forall}{\mathcal{R}xy, \forall \bar{w}.B\bar{w}x, \Gamma \Rightarrow \Delta, \forall \bar{w}.B\bar{w}y}\,\mathsf{R}\forall$$

(7) Suppose $A = \exists \bar{w}.B\bar{w}x$. Then the derivation is similar to case 6:

$$\dfrac{\dfrac{\dfrac{\mathcal{R}xy, B\bar{z}x\Gamma \Rightarrow \Delta, B\bar{z}y}{\mathcal{R}xy, B\bar{z}x\Gamma \Rightarrow \Delta, \exists \bar{w}.B\bar{w}y, B\bar{z}y}\,(\mathsf{RW})}{\mathcal{R}xy, B\bar{z}x\Gamma \Rightarrow \Delta, \exists \bar{w}.B\bar{w}y}\,\mathsf{R}\exists}{\mathcal{R}xy, \exists \bar{w}.B\bar{w}x, \Gamma \Rightarrow \Delta, \exists \bar{w}.B\bar{w}y}\,\mathsf{L}\exists$$

□

PROPOSITION B.8. *The* ***atomic right-folding*** *rule*

$$\dfrac{\mathcal{R}xy, \Gamma \Rightarrow \Delta, Px, Py}{\mathcal{R}xy, \Gamma \Rightarrow \Delta, Py}\,\mathsf{RF}_0$$

where Px and Py are atomic, is admissible in **G3c/PSF***.

Proof. By induction on the derivation depth, with the following cases:

(1) Suppose the premiss is an axiom. If Px is principal, then the conclusion of the rule is derivable using the LF_0 rule,

$$\dfrac{\mathcal{R}xy, Px, Py, \Gamma' \Rightarrow \Delta, Py}{\mathcal{R}xy, Px, \Gamma' \Rightarrow \Delta, Py}\,\mathsf{LF}_0$$

which is a base case of Lemma B.7 on the preceding page. Otherwise the conclusion is also an axiom.

(2) Otherwise since Px is not the principal formula of any other rule. So by the induction hypothesis, RF_0 can be permuted towards the axiom(s) and applied to the premiss(es) of that rule.

□

LEMMA B.9 (Folding). *The **general folding rules***

$$\frac{\mathcal{R}xy, A, [y/x]A, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, A, \Gamma \Rightarrow \Delta}\ (\mathsf{LF}) \qquad \frac{\mathcal{R}xy, \Gamma \Rightarrow A, [y/x]A, \Delta}{\mathcal{R}xy, \Gamma \Rightarrow [y/x]A, \Delta}\ (\mathsf{RF})$$

where $A \in \mathsf{SPSF}$*, are admissible in* **G3c/PSF***.

Proof. Proof by simultaneous induction on the rank determined by the formula size and derivation depth, with the induction hypothesis being that the rules are admissible for instances of lower rank.

We note the cases below:

(1) For atomic formulae, we have the following subcases:
   (a) For LF, we use the LF_0 rule.
   (b) For RF, we use the RF_0 rule (Proposition B.8 on the preceding page).
   (c) The other subcases are trivial.
(2) For derivations by axioms, we have the following subcases:
   (a) For the LF rule, if the premiss is derived by an axiom with $[y/x]A$ as a principal formula, then the conclusion of the rule is derivable by Lemma B.7 on page 242.
   (b) For the RF rule, if the premiss is derived by an axiom with A as a principal formula, then the conclusion of the rule is derivable by Lemma B.7.
   (c) Otherwise, the conclusion is also an axiom.
(3) $A = B \wedge C$ and either A or $[y/x]A$ is the principal formula in an instance of $\mathsf{L}\wedge$: Then the following is derivable by $\mathsf{L}\wedge^{-1}$:

$$\dfrac{\dfrac{\mathcal{R}xy, B, C, [y/x]B, [y/x]C, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, B, C, \Gamma \Rightarrow \Delta}\ (\mathsf{LF})^+}{\mathcal{R}xy, B \wedge C, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge$$

A similar proof for the case of derivations ending in an instance of $\mathsf{R}\vee$ uses the RF rule.

(4) $A = B \vee C$ and either A or $[y/x]A$ is the principal formula in an instance of $\mathsf{L}\vee$. Then the following is derivable by $\mathsf{L}\vee^{-1}$:

$$\dfrac{\dfrac{\mathcal{R}xy, B, [y/x]B, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, B, \Gamma \Rightarrow \Delta}\ (\mathsf{LF}) \quad \dfrac{\mathcal{R}xy, C, [y/x]C, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, C, \Gamma \Rightarrow \Delta}\ (\mathsf{LF})}{\mathcal{R}xy, B \vee C, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge$$

A similar proof for the case of derivations ending in an instance of $\mathsf{R}\wedge$ uses the RF rule.

(5) $A = \forall\bar{w}.Bx\bar{w}$ (w.l.o.g., we ignore other free variables in A) and either A or $[y/x]A$ is the principal formula by an instance of $\mathsf{L}\forall$.

Because the principal formula is also an active formula, we refer to the induction hypothesis and permute instances of LF towards the axiom(s) at lower derivation depths.

(6) $A = \forall\bar{w}.Bx\bar{w}$ (w.l.o.g., we ignore other free variables in A) and either A or $[y/x]A$ is the principal formula by an instance of $\mathsf{R}\forall$. Then the following is derivable by $\mathsf{R}\forall^{-1}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{R}xy, \Gamma \Rightarrow \Delta', \forall\bar{w}.Bx\bar{w}, \forall\bar{w}.By\bar{w}}{\mathcal{R}xy, \Gamma \Rightarrow \Delta', Bx\bar{z}, By\bar{z}'}\ (\mathsf{R}\forall^{-1})^{+}}{\mathcal{R}xy, \Gamma \Rightarrow \Delta', Bx\bar{z}, By\bar{z}}\ [z/z']}{\mathcal{R}xy, \Gamma \Rightarrow \Delta', By\bar{z}}\ (\mathsf{RF})}{\mathcal{R}xy, \Gamma \Rightarrow \Delta', \forall\bar{w}.By\bar{w}}\ \mathsf{R}\forall$$

(7) $A = \forall w.(\mathcal{R}xw \wedge Cw) \supset Bw$ (recall there are no other free variables in A) and either A or $[y/x]A$ is the principal formula by an instance of $\mathsf{L}\forall\supset$.

Because the principal formula is also an active formula, we refer to the induction hypothesis and permute instances of LF towards the axiom(s) at lower derivation depths.

(8) $A = \forall w.(\mathcal{R}xw \wedge Cw) \supset Bw$ (recall there are no other free variables in A) and either A or $[y/x]A$ is the principal formula by an instance of $\mathsf{R}\forall\supset$. Then the following

is derivable by $\mathsf{R}\forall\supset^{-1}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{R}xy,\Gamma\Rightarrow\Delta',\forall w.(\mathcal{R}xw\wedge Cw)\supset Bw,\forall w.(\mathcal{R}yw\wedge Cw)\supset Bw}{\mathcal{R}xy,\mathcal{R}yz',\mathcal{R}xz,Cz,Cz',\Gamma\Rightarrow\Delta',Bz,Bz'}(\mathsf{R}\forall\supset^{-1})^{+}}{\mathcal{R}xy,\mathcal{R}yz,\mathcal{R}xz,Cz,Cz,\Gamma\Rightarrow\Delta',Bz,Bz}[z/z']}{\mathcal{R}xy,\mathcal{R}yz,\mathcal{R}xz,Cz,Cz,\Gamma\Rightarrow\Delta',Bz}(\mathsf{RC})}{\mathcal{R}xy,\mathcal{R}yz,\mathcal{R}xz,Cz,\Gamma\Rightarrow\Delta',Bz}(\mathsf{LC})}{\mathcal{R}xy,\mathcal{R}yz,Cz,\Gamma\Rightarrow\Delta',Bz}(\text{trans.})}{\mathcal{R}xy,\Gamma\Rightarrow\Delta',\forall w.(\mathcal{R}yw\wedge Cw)\supset Bw}\mathsf{R}\forall\supset$$

(9) For the extended rules from Figures 3.3 and 3.5 on page 63 (other than the LF_0 rule, which is already covered by the atomic case), we refer to the induction hypothesis and permute instances of LF towards the axiom(s) at lower derivation depths.

For the inverted forms of these rules, we use LW and RW, respectively. □

PROPOSITION B.10 (Merge). *The* cyc_2 *(simple cycle merging) rule*

$$\dfrac{\mathcal{R}xy,\mathcal{R}yx,\Gamma\Rightarrow\Delta}{[y/x]\Gamma\Rightarrow[y/x]\Delta}(\mathsf{cyc}_2)$$

where y is free for x in Γ,Δ, *is derivable in* **G3c/PSF***.

Proof.

$$\dfrac{\dfrac{\mathcal{R}xy,\mathcal{R}yx,\Gamma\Rightarrow\Delta}{\mathcal{R}yy,\mathcal{R}yy,[y/x]\Gamma\Rightarrow[y/x]\Delta}[y/x]}{[y/x]\Gamma\Rightarrow[y/x]\Delta}\mathsf{refl}^{+}$$

□

REMARK B.11. The cyc_2 rule is akin to an antisymmetry rule: because two parameters are equal (by antisymmetry), they are interchangeable. This rule is also useful because $\mathcal{R}$-formulae are removed from the conclusion, which allows for deriving labelled sequents from relational sequents, as will be shown in the next section.

COROLLARY B.12 (Cycle merging). *The* ***cycle merging*** *rule*

$$\dfrac{\mathcal{R}x_1x_2,\ldots,\mathcal{R}x_{n-1}x_n,\mathcal{R}x_nx_1,\Gamma\Rightarrow\Delta}{[x_1/x_2]\ldots[x_1/x_n]\Gamma\Rightarrow[x_1/x_2]\ldots[x_1/x_n]\Delta}(\mathsf{cyc}_n)$$

where x_1 *is free for all* x_i *(for* $1\le i\le n$*), is admissible in* **G3c/PSF***.

Proof. By induction on n. For the base case, we use refl. For the induction step, we assume as the induction hypothesis that the rule holds for cycles of n $\mathcal{R}$-formulae. Then for $n+1$ $\mathcal{R}$-formulae, we can derive:

$$\dfrac{\dfrac{\dfrac{\mathcal{R}x_1x_2,\dots,\mathcal{R}x_nx_{n+1},\mathcal{R}x_{n+1}x_1,\Gamma\Rightarrow\Delta}{\mathcal{R}x_1x_2,\dots,\mathcal{R}x_nx_1,\mathcal{R}x_nx_{n+1},\mathcal{R}x_{n+1}x_1,\mathcal{R}x_1x_{n+1},\Gamma\Rightarrow\Delta}\,(\mathsf{LW})^+}{\mathcal{R}x_{n+1}x_1,\mathcal{R}x_1x_{n+1},[x_1/x_2]\dots[x_1/x_n]\Gamma\Rightarrow[x_1/x_2]\dots[x_1/x_n]\Delta}\,(\mathsf{cyc_n})}{[x_1/x_2]\dots[x_1/x_{n+1}]\Gamma\Rightarrow[x_1/x_2]\dots[x_1/x_{n+1}]\Delta}\,(\mathsf{cyc_2})$$

□

Proposition B.13. *The extended transitivity rule*

$$\dfrac{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,\mathcal{R}xz,\Gamma\Rightarrow\Delta}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,\Gamma\Rightarrow\Delta}\,(\mathsf{trans_n})$$

where $n\geq 0$, is admissible in **G3c/PSF***.

Proof. For $n=0$, this is an instance of LC. For $n\geq 1$, by derivation:

$$\dfrac{\dfrac{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,\mathcal{R}xz,\Gamma\Rightarrow\Delta}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,\mathcal{R}xy_2,\dots,\mathcal{R}xy_n,\mathcal{R}xz,\Gamma\Rightarrow\Delta}\,(\mathsf{LW})^{n-1}}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,\Gamma\Rightarrow\Delta}\,\mathsf{trans}^n$$

□

Corollary B.14 (Transitive Folding). *The **transitive folding** rules*

$$\dfrac{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,A,[z/x]A,\Gamma\Rightarrow\Delta}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,A,\Gamma\Rightarrow\Delta}\,(\mathsf{LF}_\tau)$$

$$\dfrac{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,\Gamma\Rightarrow A,[z/x]A,\Delta}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,\Gamma\Rightarrow[z/x]A,\Delta}\,(\mathsf{RF}_\tau)$$

where $n\geq 0$ and $A\in$ SPSF, are admissible in **G3c/PSF***.

Proof. The case for $n=0$ is shown using the LF and RF rules. For $n\geq 1$, by derivation:

$$\dfrac{\dfrac{\dfrac{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,A,[z/x]A,\Gamma\Rightarrow\Delta}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,A,[z/x]A,\Gamma\Rightarrow\Delta}\,(\mathsf{LW})}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,\mathcal{R}xz,A,\Gamma\Rightarrow\Delta}\,(\mathsf{LF})}{\mathcal{R}xy_1,\mathcal{R}y_1y_2,\dots,\mathcal{R}y_nz,A,\Gamma\Rightarrow\Delta}\,(\mathsf{trans_n})$$

The proof for RF_τ is similar. For the inverted forms of these rules, we use LW and RW, respectively. $\square$

LEMMA B.15. *Cycle merging is invertible:*

$$\frac{[x_1/x_2]\dots[x_1/x_n]\Sigma;[x_1/x_2]\dots[x_1/x_n]\underline{S}}{x_1 \preceq x_2,\dots,x_{n-1} \preceq x_n, x_n \preceq x_1, \Sigma; \underline{S}}\ (\mathsf{cyc}_\mathsf{n}^{-1})$$

Proof. By using $\mathsf{LW} \preceq$ to add the relational formulae, then LW with $\mathsf{L} \preceq_\tau$ or RW with $\mathsf{R} \preceq_\tau$ to add replace A^{x_1} with A^{x_i} where $2 \le i \le n$. For example,

$$\dfrac{\dfrac{\dfrac{\dfrac{\Sigma;\Gamma_1^x,\Gamma_2^x,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta_1^x,\Delta_2^x}{x \preceq y, y \preceq x, \Sigma;\Gamma_1^x,\Gamma_2^x,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta_1^x,\Delta_2^x}\ (\mathsf{LW} \preceq)^+}{x \preceq y, y \preceq x, \Sigma;\Gamma_1^x,\Gamma_2^x,\Gamma_2^y,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta_1^x,\Delta_2^x,\Delta_2^y}\ (\mathsf{GW})^*}{x \preceq y, y \preceq x, \Sigma;\Gamma_1^x,\Gamma_2^y,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta_1^x,\Delta_2^x,\Delta_2^y}\ (\mathsf{L} \preceq_\tau)^*}{x \preceq y, y \preceq x, \Sigma;\Gamma_1^x,\Gamma_2^y,\underline{\Gamma'} \Rightarrow \underline{\Delta'},\Delta_1^x,\Delta_2^y}\ (\mathsf{R} \preceq_\tau)^*$$

To reverse the substitutions in Σ, use $\mathsf{LW} \preceq$ and $\mathsf{trans_n}$ similarly. $\square$

APPENDIX C

Hypersequent Calculi

Below are proofs from Chapter 4.

LEMMA C.1 (Hypercontext-Sharing and Splitting). *For calculi with the standard external rules, the hypercontext-splitting and hypercontext-sharing rules are interderivable.*

Proof. Hypercontext-splitting rules can be derived from hypercontext sharing rules using EW:

$$\dfrac{\dfrac{\mathcal{H}_1 \mid \Gamma_1' \Rightarrow \Delta_1'}{\mathcal{H}_1 \mid \ldots \mid \mathcal{H}_n \mid \Gamma_1' \Rightarrow \Delta_1'}\,\mathsf{EW}^* \quad \ldots \quad \dfrac{\mathcal{H}_n \mid \Gamma_n' \Rightarrow \Delta_n'}{\mathcal{H}_1 \mid \ldots \mid \mathcal{H}_n \mid \Gamma_n' \Rightarrow \Delta_n'}\,\mathsf{EW}^*}{\mathcal{H}_1 \mid \ldots \mid \mathcal{H}_n \mid \Gamma \Rightarrow \Delta}\,\rho_2$$

Hypercontext-sharing rules can be derived from hypercontext-splitting rules using EC:

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1' \Rightarrow \Delta_1' \quad \ldots \quad \mathcal{H} \mid \Gamma_n' \Rightarrow \Delta_n'}{\mathcal{H}^n \mid \Gamma \Rightarrow \Delta}\,\rho_1}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\,\mathsf{EC}^*$$

□

LEMMA C.2 (Rule composition). *Let* **HGS** *be a hypersequent calculus with the* EC *rule. Let $\rho_1, \ldots, \rho_n$, for $n \geq 2$, be hypercontext-sharing rules of* **HGS**, *and let $\sigma_1, \ldots, \sigma_n$ be partial variable substitutions such that the rules $\sigma_1\rho_1, \ldots, \sigma_n\rho_n$ have the same conclusions. (Note that each σ_i is a replacement of variables by other variables, such that $\sigma_i\rho_i$ is a schematic rule, and not a rule instance.) Then the rules may be composed into a single rule, $\sigma_2\rho_1 \circ \ldots \circ \sigma_n\rho_n$.*

Proof. By induction on n, we show that the rule is derivable. For the base case ($n = 2$), we derive the rule $\rho_1\sigma_2 \circ \rho_2\sigma_2$. Suppose the two rules are

$$\dfrac{\mathcal{H} \mid \mathcal{X}_1 \quad \ldots \quad \mathcal{H} \mid \mathcal{X}_m}{\mathcal{H} \mid \mathcal{G}}\,\rho_1\sigma_1 \qquad \dfrac{\mathcal{H} \mid \mathcal{Y}_1 \quad \ldots \quad \mathcal{H} \mid \mathcal{Y}_n}{\mathcal{H} \mid \mathcal{G}}\,\rho_2\sigma_2$$

where each $\mathcal{X}_i$ and $\mathcal{Y}_j$ are active components in ρ_1 and ρ_2, respectively. Then the following can be derived:

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \mathcal{Y}_1 \mid \mathcal{X}_1 \quad \ldots \quad \mathcal{H} \mid \mathcal{Y}_1 \mid \mathcal{X}_m}{\mathcal{H} \mid \mathcal{G} \mid \mathcal{Y}_1}\rho_1\sigma_1 \qquad \ldots \qquad \dfrac{\mathcal{H} \mid \mathcal{Y}_n \mid \mathcal{X}_1 \quad \ldots \quad \mathcal{H} \mid \mathcal{Y}_n \mid \mathcal{X}_m}{\mathcal{H} \mid \mathcal{G} \mid \mathcal{Y}_n}\rho_1\sigma_1}{\mathcal{H} \mid \mathcal{G} \mid \mathcal{G}}\rho_2\sigma_2}{\mathcal{H} \mid \mathcal{G}}\mathsf{EC}^*$$

Note that the composed rule will have $m \cdot n$ premisses. In practise this may be reduced where some premisses can be shown to be derivable from other premisses.

For the induction step, apply the same procedure as the base case on the composed rule $\sigma_2\rho_1 \circ \ldots \circ \sigma_{n-1}\rho_{n-1}$ and $\sigma_n\rho_n$.

□

LEMMA C.3 (Merge). *Let* **HGS** *be a hypersequent calculus with the* LW*,* RW *(internal weakening) and* EC *rules. Then the merge rule is admissible in* **HGS***.*

Proof. By derivation.

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta \mid \Gamma, \Gamma' \Rightarrow \Delta'}\mathsf{LW}^*}{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\mathsf{RW}^*}{\mathcal{H} \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\mathsf{EC}$$

□

LEMMA C.4 (External Contraction). *Let* **HGS** *be a hypersequent calculus with the* LC*,* RC *(internal contraction) and* M *rules. Then the* EC *rule is admissible in* **HGS***.*

Proof. By derivation.

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, \Gamma \Rightarrow \Delta, \Delta}\mathsf{M}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Delta}\mathsf{LC}^*}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\mathsf{RC}^*$$

□

REMARK C.5. The single-succedent variant of EC is similarly derivable.

LEMMA C.6 (External Weakening). *Let* **HGS** *be a hypersequent calculus with the* S *(split),* LW *and* RW *(internal weakening) rules. Then the* EW *(external weakening) rule is admissible in* **HGS***.*

Proof. By derivation.

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow}\,\mathsf{S}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \Delta'}\,\mathsf{RW}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}\,\mathsf{LW}$$

□

LEMMA C.7. *The* Com *and* Com′ *rules are interderivable.*

Proof.

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1, \Gamma_1' \Rightarrow A_1}{\mathcal{H} \mid \Gamma_1, \Gamma_1', \Gamma_2, \Gamma_2' \Rightarrow A_1}\,\mathsf{LW}^* \qquad \dfrac{\mathcal{H} \mid \Gamma_2, \Gamma_2' \Rightarrow A_2}{\mathcal{H} \mid \Gamma_1, \Gamma_1', \Gamma_2, \Gamma_2' \Rightarrow A_2}\,\mathsf{LW}^*}{\mathcal{H} \mid \Gamma_1, \Gamma_2' \Rightarrow A_1 \mid \Gamma_2, \Gamma_1' \Rightarrow A_2}\,\mathsf{Com}'$$

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A_1 \qquad \mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A_2}{\mathcal{H} \mid \Gamma_1, \Gamma_1 \Rightarrow A_1 \mid \Gamma_2, \Gamma_2 \Rightarrow A_2}\,\mathsf{Com}}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2}\,\mathsf{LC}^*$$

□

LEMMA C.8. *The* $\mathsf{S_I}$ *and* Com″ *rules are derivable from* Com′ *(or* Com*) and vice versa.*

Proof.

$$\dfrac{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A \qquad \mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A}{\mathcal{H} \mid \Gamma_1 \Rightarrow A \mid \Gamma_2 \Rightarrow A}\,\mathsf{Com}' \tag{144}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{H}_1 \mid \Gamma_1 \Rightarrow A_1}{\mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow A_1}\,\mathsf{LW}^*}{\mathcal{H}_1 \mid \Gamma_1, \Gamma_2 \Rightarrow A_1}\,\mathsf{EW}^* \qquad \dfrac{\dfrac{\mathcal{H}_2 \mid \Gamma_2 \Rightarrow A_2}{\mathcal{H}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow A_2}\,\mathsf{LW}^*}{\mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow A_2}\,\mathsf{EW}^*}{\mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1 \Rightarrow A_2 \mid \Gamma_2 \Rightarrow A_1}\,\mathsf{Com}' \tag{145}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A_1}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_1}\,\mathsf{S_I} \qquad \dfrac{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A_2}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_2 \mid \Gamma_2 \Rightarrow A_2}\,\mathsf{S_I}}{\mathcal{H} \mid \mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2 \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2}\,\mathsf{Com}''}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2}\,\mathsf{EC}^+ \tag{146}$$

□

LEMMA C.9. Com′ *can be derived using* $\mathsf{S_C}$.

Proof.

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A_1}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow}\,\mathsf{S_C}}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2}\,\mathsf{RW} \qquad \dfrac{\dfrac{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A_2}{\mathcal{H} \mid \Gamma_1 \Rightarrow\ \mid \Gamma_2 \Rightarrow A_2}\,\mathsf{S_C}}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2}\,\mathsf{RW}}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2}$$

□

Lemma C.10. *Variants of* Com'' *with more than two premisses are admissible systems with* Com'' *and* EW.

$$\dfrac{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \quad \ldots \quad \mathcal{H} \mid \Gamma_n \Rightarrow A_n}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n \mid \Gamma_{n+1} \Rightarrow A_{n+1}}\ (\mathsf{Com}''_{n\geq 2})$$

Proof. By induction on n. The base case ($n = 2$) is the Com'' rule. For the induction step, we have have

$$\dfrac{\dfrac{\vdots\ (\mathrm{IH})}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n}\ (\mathsf{Com}''_n) \qquad \dfrac{\mathcal{H} \mid \Gamma_{n+1} \Rightarrow A_{n+1}}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_{n-1} \Rightarrow A_{n-1} \mid \Gamma_{n+1} \Rightarrow A_{n+1}}\ \mathsf{EW}^{+}}{\mathcal{H} \mid \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n \mid \Gamma_{n+1} \Rightarrow A_{n+1}}\ \mathsf{Com}''$$

□

Lemma C.11. $\mathbf{HGipm}_{\sqsupset} \vdash \mathcal{H}$ *iff* $\mathbf{HG3ipm} \vdash \mathcal{H}$.

Proof. By induction on the derivation depth. From left-to-right, we show that the rules of $\mathbf{HGipm}_{\sqsupset}$ are admissible in **HG3ipm**:

(1) Suppose a proof in $\mathbf{HGipm}_{\sqsupset}$ ends in an instance of $\mathsf{L}{\supset}_{\iota}$. Then the following can be derived in **HG3ipm**:

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid A \supset B, \Gamma \Rightarrow \Delta, A}{\mathcal{H} \mid A \supset B, A \supset B, \Gamma \Rightarrow \Delta, A}\ (\mathsf{LW}) \qquad \mathcal{H} \mid B, A \supset B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \supset B, A \supset B, \Gamma \Rightarrow \Delta}\ \mathsf{L}{\supset}}{\mathcal{H} \mid A \supset B, \Gamma \Rightarrow \Delta}\ (\mathsf{LC})$$

(2) Suppose a proof in $\mathbf{HGipm}_{\sqsupset}$ ends in an instance of $\mathsf{R}{\supset}_{\mathsf{I}}$. Then the rule is a special case of $\mathsf{R}{\supset}$ in **HG3ipm**.

(3) Suppose a proof in $\mathbf{HGipm}_{\sqsupset}$ ends in an instance of $\sqsupset_1$. The $\sqsupset$ rule is admissible in **HG3ipm** by Corollary 4.59 on page 80, and $\sqsupset_1$ is a special case of $\sqsupset$.

From right-to-left, we show that the rules of **HG3ipm** are admissible in $\mathbf{HGipm}_{\sqsupset}$:

(1) Suppose a proof in **HG3ipm** ends in an instance of $\mathsf{L}{\supset}$. Then the following can be derived in $\mathbf{HGipm}_{\sqsupset}$:

$$\dfrac{\mathcal{H} \mid A \supset B, \Gamma \Rightarrow A, \Delta \qquad \dfrac{\mathcal{H} \mid B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \supset B, B, \Gamma \Rightarrow \Delta}\ (\mathsf{LW})}{\mathcal{H} \mid A \supset B, \Gamma \Rightarrow \Delta}\ \mathsf{L}{\supset}_{\iota}$$

(2) Suppose a proof in **HG3ipm** ends in an instance of $\mathsf{R}\supset$. Then the following can be derived in $\mathbf{HGipm}_{\supset}$:

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid A, \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \supset B}\ \mathsf{R}\supset_{\mathsf{I}}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \supset B}\ (\mathsf{RW})^*$$

□

LEMMA C.12. *The cut rule is admissible in* **HG3ipm**+LQ.

Proof. By induction on the rank defined by the size of the cut formula and the sum of derivation depths of the premisses. We note the relevant case below:

(1) Suppose a derivation ends with

$$\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \qquad \dfrac{\mathcal{H}' \mid A, \Gamma', \Gamma'' \Rightarrow}{\mathcal{H}' \mid A, \Gamma' \Rightarrow \ \mid \Gamma'' \Rightarrow}\ \mathsf{LQ}}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta \mid \Gamma'' \Rightarrow}\ (\mathsf{cut})$$

Then the cut rule can be permuted to a lower depth:

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \qquad \mathcal{H}' \mid A, \Gamma', \Gamma'' \Rightarrow}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma', \Gamma'' \Rightarrow \Delta}\ (\mathsf{cut})}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta \mid \Gamma'' \Rightarrow}\ \mathsf{LQ}$$

Note that there is no case where the left premiss of cut is the conclusion of an instance of LQ, because LQ requires an empty succedent. □

LEMMA C.13. *The cut rule is admissible in* **HG3ipm**+$\mathsf{Com_m}$.

Proof. By induction on the rank defined by the size of the cut formula, the number of instances of $\mathsf{Com_m}$ in the premisses, and the sum of the derivation depths of the premisses.

We note the relevant case below:

(1) Suppose a derivation ends with an instance of $\mathsf{Com_m}$ deriving the left premiss of an instance of cut:

$$\dfrac{\dfrac{\overset{\vdots\ \delta_1}{\mathcal{H}|\Gamma_1 \Rightarrow \Delta_1, \Delta_2, A|\Gamma_1, \Gamma_2 \Rightarrow \Delta_2, A} \qquad \overset{\vdots\ \delta_2}{\mathcal{H}|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1|\Gamma_2 \Rightarrow \Delta_1, \Delta_2, A}}{\mathcal{H}|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2 \Rightarrow \Delta_2, A} \qquad \overset{\vdots\ \delta_3}{\mathcal{H}'|A, \Gamma_3 \Rightarrow \Delta_3}}{\mathcal{H}|\mathcal{H}'|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3}\ (\mathsf{cut})$$

The cut rank is $\langle |A|, n+1, \max\{h(\delta_1), h(\delta_2)\} + h(\delta_3)\rangle$. We can replace the derivation with multiple cuts at lower derivation depths (which are of lower rank):

$$\dfrac{\dfrac{\overset{\vdots\ \delta_2}{\mathcal{H}|\Gamma_1,\Gamma_2 \Rightarrow \Delta_1|\Gamma_2 \Rightarrow \Delta_1,\Delta_2,A} \quad \overset{\vdots\ \delta_3}{\mathcal{H}'|A,\Gamma_3 \Rightarrow \Delta_3}}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1,\Gamma_2 \Rightarrow \Delta_1 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,\textsf{(cut)}}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1,\Gamma_2,\Gamma_3 \Rightarrow \Delta_1 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,\textsf{(LW)}^* \tag{147}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\ \delta_1}{\mathcal{H} \mid \Gamma_1 \Rightarrow \Delta_1,\Delta_2,A \mid \Gamma_1,\Gamma_2 \Rightarrow \Delta_2,A} \quad \overset{\vdots\ \delta_3}{\mathcal{H}'|A,\Gamma_3 \Rightarrow \Delta_3}}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1 \Rightarrow \Delta_1,\Delta_2,A \mid \Gamma_1,\Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,\textsf{(cut)} \quad \overset{\vdots\ \delta_3}{\mathcal{H}'|A,\Gamma_3 \Rightarrow \Delta_3}}{\mathcal{H} \mid \mathcal{H}'^2 \mid \Gamma_1,\Gamma_3 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_1,\Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,\textsf{(cut)}}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1,\Gamma_3 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_1,\Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,\textsf{EC}^*}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1,\Gamma_3 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_1 \Rightarrow \Delta_2,\Delta_3 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,(\textsf{S}_\textsf{l})}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1,\Gamma_3 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_1 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,\textsf{(RW)}^*}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_3 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_1 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_1 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,(\textsf{S}_\textsf{l})}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_3 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_1 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,\textsf{EC} \tag{148}$$

$$\dfrac{\dfrac{\dfrac{\vdots\ (148) \qquad \vdots\ (147)}{\mathcal{H}^2 \mid \mathcal{H}'^2 \mid \Gamma_3 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,(\textsf{Com}'_\textsf{m})}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_3 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,\textsf{EC}^+}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_1,\Delta_2,\Delta_3 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,\textsf{(LW)}^* \tag{149}$$

$$\dfrac{\dfrac{\vdots\ (149) \qquad \vdots\ (147)}{\mathcal{H}^2 \mid \mathcal{H}'^2 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,(\textsf{Com}'_\textsf{m})}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2,\Gamma_3 \Rightarrow \Delta_2,\Delta_3}\,\textsf{EC}^+$$

where $\textsf{Com}'_\textsf{m}$ is the hypercontext-splitting form of $\textsf{Com}_\textsf{m}$. (Recall by Lemma 4.47 on page 76.)

The rank of the cut in (147) is $\langle |A|, n, h(\delta_2) + h(\delta_3)\rangle$, which is clearly smaller. The rank of the upper cut in (148) is $\langle |A|, n, h(\delta_1) + h(\delta_3)\rangle$, and the rank of the lower cut in (148) is $\langle |A|, n, \max\{h(\delta_1), h(\delta_3)\} + 1 + h(\delta_3)\rangle$, both of which are also clearly smaller. Note that including the number of occurrences of $\textsf{Com}_\textsf{m}$ in the premisses of the cut as part of the cut rank is crucial for this proof.

Since there is symmetry between the antecedents and succedents of active components of the premisses of $\textsf{Com}_\textsf{m}$, the case where the conclusion of an instance of the $\mathsf{Com_m}$ rule is the right premiss of a cut is the symmetric version of this case.

$\square$

APPENDIX D

The Relationship between Hypersequents and Labelled Sequents

Here we give proofs from Chapter 6.

LEMMA D.1. *Let* $x \in \mathtt{Term}_0$ *and* $A \in \mathtt{Prop}$. *Then* $\mathtt{psf}^{-1}\,(\mathtt{psf}\ x\ A) = \langle x,\ A\rangle$.

Proof. By induction on the structure of A. Our induction hypothesis is that for all formula A' where $|A'| < |A|$, $\mathtt{psf}^{-1}\,(\mathtt{psf}\ x\ A') = \langle x,\ A'\rangle$. We show the cases below:

(1) Suppose $A = P$ (a propositional variable, the base case). Then

$$\mathtt{psf}^{-1}\,(\mathtt{psf}\ x\ P) = \mathtt{psf}^{-1}\ \hat{P}x = \langle x,\ P\rangle$$

(2) Suppose $A = \bot$. Then the case is similar to case 1.

(3) Suppose $A = B \wedge C$. Then

$$\begin{aligned}
\mathtt{psf}^{-1}\,(\mathtt{psf}\ x\ B \wedge C) &= \mathtt{psf}^{-1}\,((\mathtt{psf}\ x\ B) \wedge_0 (\mathtt{psf}\ x\ C)) \\
&= \langle x,\ (\pi_2\ \mathtt{psf}^{-1}\,(\mathtt{psf}\ x\ B)) \wedge (\pi_2\ \mathtt{psf}^{-1}\,(\mathtt{psf}\ x\ C))\rangle \\
&= \langle x,\ (\pi_2\ \langle x, B\rangle) \wedge (\pi_2\ \langle x, C\rangle)\rangle \\
&= \langle x,\ B \wedge C\rangle
\end{aligned}$$

(4) Suppose $A = B \vee C$. Then the case is similar to case 3.

(5) Suppose $A = B \supset C$. Then

$$\begin{aligned}
\mathtt{psf}^{-1}\,(\mathtt{psf}\ x\ B \supset C) &= \mathtt{psf}^{-1}\ \forall y.(\mathcal{R}xy \wedge (\mathtt{psf}\ y\ B)) \supset (\mathtt{psf}\ y\ C)) \\
&= \langle x,\ (\pi_2\ \mathtt{psf}^{-1}\,(\mathtt{psf}\ y\ B)) \supset (\pi_2\ \mathtt{psf}^{-1}\,(\mathtt{psf}\ y\ C))\rangle \\
&= \langle x,\ (\pi_2\ \langle y, B\rangle) \supset (\pi_2\ \langle y, C\rangle)\rangle \\
&= \langle x,\ B \supset C\rangle
\end{aligned}$$

$\square$

LEMMA D.2. *Let* $x \in \mathtt{Term}_0$ *and* $\Gamma \in \mathtt{Prop}^*$. *Then* $\mathtt{psf}_\wedge^{-1}\ \mathtt{psf}_\wedge\ x\,\Gamma = \langle x,\ \Gamma\rangle$.

Proof. We have two cases.

(1) Suppose $\Gamma = \emptyset$. Then $\mathtt{psf}_{\wedge}^{-1}\ \mathtt{psf}_{\wedge}\ x\ \emptyset = \mathtt{psf}_{\wedge}^{-1}\ \hat{\top} x = \langle x,\ \emptyset\rangle$.

(2) Otherwise,

$$\begin{aligned}
\mathtt{psf}_{\wedge}^{-1}\ \mathtt{psf}_{\wedge}\ x\,\Gamma &= \mathtt{psf}_{\wedge}^{-1}\ \wedge\!\!\wedge_1(\mathtt{psf}\ x) \circledast \Gamma \\
&= \langle x,\ \pi_2 \circledast \mathtt{psf}^{-1} \circledast \wedge\!\!\wedge_1^{-1} \wedge\!\!\wedge_1(\mathtt{psf}\ x) \circledast \Gamma\rangle \\
&= \langle x,\ \pi_2 \circledast \mathtt{psf}^{-1} \circledast (\mathtt{psf}\ x) \circledast \Gamma\rangle \\
&= \langle x,\ (\pi_2 \circ \mathtt{psf}^{-1}) \circledast (\mathtt{psf}\ x) \circledast \Gamma\rangle \\
&= \langle x,\ ((\pi_2 \circ \mathtt{psf}^{-1}) \circ (\mathtt{psf}\ x)) \circledast \Gamma\rangle \\
&= \langle x,\ id \circledast \Gamma\rangle \\
&= \langle x,\ \Gamma\rangle
\end{aligned}$$

□

LEMMA D.3. $\mathtt{psf}_{\dagger}'^{-1}\mathtt{psf}_{\dagger}' A^x = A^x$.

Proof.

$$\begin{aligned}
\mathtt{psf}_{\dagger}'^{-1}\mathtt{psf}_{\dagger}' A^x &= \mathtt{psf}_{\dagger}'^{-1}\ \mathtt{psf}\ \hat{x}\ A \\
&= \mathtt{aplab}\ (\mathtt{psf}^{-1}\ \mathtt{psf}\ \hat{x}\ A) \\
&= \mathtt{aplab}\ \langle \hat{x}.\ A\rangle \\
&= A^x
\end{aligned}$$

□

LEMMA D.4. $\mathtt{psf}_{\dagger\Rightarrow}^{-1}\ \mathtt{psf}_{\dagger\Rightarrow} S^x = S^x$.

Proof. Let $S^x = (\Gamma^x \Rightarrow \Delta^x)$.

$$\begin{aligned}
\mathtt{psf}^{-1}_{\dagger\Rightarrow}\ \mathtt{psf}_{\dagger\Rightarrow}(\Gamma^x \Rightarrow \Delta^x) &= \mathtt{psf}^{-1}_{\dagger\Rightarrow}\ \mathtt{psf}_{\Rightarrow}\ \hat{x}\ ((\pi_{\mathtt{form}} \circledast \Gamma^x) \Rightarrow (\pi_{\mathtt{form}} \circledast \Delta^x)) \\
&= \mathtt{psf}^{-1}_{\dagger\Rightarrow}\ \mathtt{psf}_{\Rightarrow}\ \hat{x}\ (\Gamma \Rightarrow \Delta) \\
&= \mathtt{aplab}\ (\mathtt{psf}^{-1}_{\Rightarrow}\ \mathtt{psf}_{\Rightarrow}\ \hat{x}\ (\Gamma \Rightarrow \Delta)) \\
&= \mathtt{aplab}\ \langle \hat{x},\ (\Gamma \Rightarrow \Delta) \rangle \\
&= \Gamma^x \Rightarrow \Delta^x
\end{aligned}$$

□

LEMMA D.5. $\mathtt{psf}^{-1}_{\dagger}\ \mathtt{psf}_{\dagger}\ \underline{S} = \underline{S}$.

Proof.

$$\begin{aligned}
\mathtt{psf}^{-1}_{\dagger}\ \mathtt{psf}_{\dagger}\ \underline{S} &= \mathtt{psf}^{-1}_{\dagger}\ \mathbb{W}_2\ (\mathtt{psf}_{\dagger\Rightarrow} \circledast (\lambda x.\underline{S} \mathbin{/\!/} x) \circledast \mathtt{lab}(\underline{S})) \\
&= \bigsqcup (\mathtt{psf}^{-1}_{\dagger\Rightarrow} \circledast \mathbb{W}_2^{-1}\mathbb{W}_2\ (\mathtt{psf}_{\dagger\Rightarrow} \circledast (\lambda x.\underline{S} \mathbin{/\!/} x) \circledast \mathtt{lab}(\underline{S}))) \\
&= \bigsqcup (\mathtt{psf}^{-1}_{\dagger\Rightarrow}\ (\mathtt{psf}_{\dagger\Rightarrow} \circledast (\lambda x.\underline{S} \mathbin{/\!/} x) \circledast \mathtt{lab}(\underline{S}))) \\
&= \bigsqcup (\lambda x.\underline{S} \mathbin{/\!/} x) \circledast \mathtt{lab}(\underline{S})) \\
&= \underline{S}
\end{aligned}$$

□

THEOREM D.6 (Model Theoretic Correctness). *Let* $\mathfrak{M} = \langle W, R, v \rangle \in \mathcal{K}_{\mathbf{Int}}$. *Then* $\mathfrak{M} \models A$ *iff* $\mathfrak{M} \models \mathtt{psf}\ A$*, for all* $\hat{x} \in \mathtt{Term}_0$. $\mathfrak{M} \models \mathtt{psf}\ \hat{x}\ A'$.

Proof. By induction on the structure of A, where the induction hypothesis is that for all A' such that $|A'| < |A|$, $\mathfrak{M} \models A'$.

(1) Suppose $A = P$ (a proposition variable). From left-to-right, suppose $\mathfrak{M} \models P$. Then for all $w \in W$, $w \Vdash P$. $\mathtt{psf}\ \hat{x}\ P = \hat{P}\hat{x}$, which by Definition 3.41 on page 55, $\mathfrak{M} \models \hat{P}\hat{x}$ iff $x \Vdash P$. From right-to-left, Let $\mathfrak{M} \models \hat{P}\hat{x}$, for all $\hat{x}$. So for all $w \in W$, $w \Vdash P$. That is, $\mathfrak{M} \models P$.

(2) Suppose $A = B \wedge C$. From left-to-right, suppose $\mathfrak{M} \models B \wedge C$. From Definition 3.10 on page 45, both $\mathfrak{M} \models B$ add $\mathfrak{M} \models C$. By the induction hypothesis, both $\mathfrak{M} \models \mathtt{psf}\ \hat{x}\ B$ and $\mathfrak{M} \models \mathtt{psf}\ \hat{x}\ C$. So (from Definition 6.6 on page 139) $\mathfrak{M} \models (\mathtt{psf}\ \hat{x}\ B) \wedge$

($\mathtt{psf}\ \hat{x}\ C$). From right-to-left, suppose $\mathfrak{M} \vDash (\mathtt{psf}\ \hat{x}\ B) \wedge (\mathtt{psf}\ \hat{x}\ C)$. Then $\mathfrak{M} \vDash \mathtt{psf}\ \hat{x}\ B$ and $\mathfrak{M} \vDash \mathtt{psf}\ \hat{x}\ C$. So by the induction hypothesis, both $\mathfrak{M} \vDash B$ add $\mathfrak{M} \vDash C$, and thus $\mathfrak{M} \vDash B \wedge C$.

(3) Suppose $A = B \vee C$. From left-to-right, suppose $\mathfrak{M} \vDash B \vee C$. From Definition 3.10 on page 45, either $\mathfrak{M} \vDash B$ or $\mathfrak{M} \vDash C$. By the induction hypothesis, either $\mathfrak{M} \vDash \mathtt{psf}\ \hat{x}\ B$ or $\mathfrak{M} \vDash \mathtt{psf}\ \hat{x}\ C$. So $\mathfrak{M} \vDash (\mathtt{psf}\ \hat{x}\ B) \vee (\mathtt{psf}\ \hat{x}\ C)$. From right-to-left, suppose (from Definition 6.6) that $\mathfrak{M} \vDash (\mathtt{psf}\ \hat{x}\ B) \vee (\mathtt{psf}\ \hat{x}\ C)$. Then either $\mathfrak{M} \vDash \mathtt{psf}\ \hat{x}\ B$ or $\mathfrak{M} \vDash \mathtt{psf}\ \hat{x}\ C$. So by the induction hypothesis, either $\mathfrak{M} \vDash B$ or $\mathfrak{M} \vDash C$, and thus $\mathfrak{M} \vDash B \vee C$.

(4) Suppose $A = B \supset C$. From left-to-right, suppose $\mathfrak{M} \vDash B \supset C$. Let $\mathfrak{M}$ be a rooted model , we can assume without loss of generality that $\mathfrak{M}$ is a rooted model (from Theorem 3.15 on page 46), with $x \in W$ as the root. $x \Vdash B \supset C$ iff for all $w \in W$, $(x, w) \in R$ (true because x is the root and the frames are transitive) and $w \Vdash B$ implies $w \Vdash C$ (from Definition 3.10 on page 45).

$\mathtt{psf}\ \hat{x}\ A = \forall \hat{y}.(\mathcal{R}\hat{x}\hat{y} \wedge \hat{B}\hat{y}) \supset \hat{C}\hat{y})$, where $\hat{B}\hat{y} = \mathtt{psf}\ \hat{y}\ B$ and $\hat{C}\hat{y} = \mathtt{psf}\ \hat{y}\ C$, by definition. For all $w \in W$, $\mathfrak{M} \vDash \mathcal{R}\hat{x}\hat{w}$ and $\mathfrak{M} \vDash \hat{B}\hat{w}$ imply $\mathfrak{M} \vDash \hat{C}\hat{w}$, which is equivalent to $(x, w) \in R$ and $w \Vdash B$ implies $w \Vdash C$.

From right-to-left: $\mathfrak{M} \vDash \mathtt{psf}\ \hat{x}\ A$ iff for all $\hat{w}$, $\mathfrak{M} \vDash \mathcal{R}\hat{x}\hat{w}$ and $\mathfrak{M} \vDash \hat{B}\hat{w}$ imply $\mathfrak{M} \vDash \hat{C}\hat{w}$, i.e. $\mathfrak{M} \vDash B \supset C$.

□

THEOREM D.7 (Proof Theoretic Correctness). *If* **HG3ipm**$^* \vdash \mathcal{H}$, **G3c/PSF**$^* \vdash \mathtt{psf}\ \mathcal{H}$.

Proof. By induction on the derivation depth.

We use the $\mathtt{psf}'$ function, noting Lemma 6.50 on page 149.

NOTATION D.8. We will represent $\mathtt{psf}'\ \mathcal{H} \mid \Gamma \Rightarrow \Delta$ as $\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x\} \Rightarrow \hat{\Delta}_{\mathcal{H}}, \hat{\Delta}\{x\}$, where $\hat{\Gamma}_{\mathcal{H}}, \hat{\Delta}_{\mathcal{H}}$ denote the antecedent and succedent formulae from the translated hypercontext $\mathcal{H}$ and $\hat{\Gamma}\{x\}, \hat{\Delta}\{x\}$ denote the antecedent and succedent formulae from the translated component $\Gamma \Rightarrow \Delta$.

We show the interesting cases below:

(1) Suppose we have a derivation which ends in an instance of R$\supset$:

$$\frac{\mathcal{H} \mid A, \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \supset B} \ \mathsf{R}\supset$$

We can derive the following in **G3c/PSF***:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x\}, \hat{A}x \Rightarrow \hat{\Delta}_{\mathcal{H}}, \hat{B}x}{\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x\}, \hat{\Gamma}\{x'\}, \mathcal{R}x'x, \hat{A}x \Rightarrow \hat{\Delta}_{\mathcal{H}}, \hat{B}x} \ (\mathsf{LW})^{+}}{\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x'\}, \mathcal{R}x'x, \hat{A}x \Rightarrow \hat{\Delta}_{\mathcal{H}}, \hat{B}x} \ (\mathsf{LF})^{*}}{\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x'\} \Rightarrow \hat{\Delta}_{\mathcal{H}}, \forall y.(\mathcal{R}x'y \wedge \hat{A}y) \supset \hat{B}y} \ \mathsf{R}\forall\supset}{\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x\} \Rightarrow \hat{\Delta}_{\mathcal{H}}, \forall y.(\mathcal{R}xy \wedge \hat{A}y) \supset \hat{B}y} \ [\mathsf{x}/\mathsf{x}']}{\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x\} \Rightarrow \hat{\Delta}_{\mathcal{H}}, \hat{\Delta}\{x\}, \forall y.(\mathcal{R}xy \wedge \hat{A}y) \supset \hat{B}y} \ (\mathsf{RW})^{*}$$

where $x, x' \# \hat{\Gamma}_{\mathcal{H}}, \hat{\Delta}_{\mathcal{H}}$.

(2) Suppose we have a derivation that ends in an instance of EC:

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta} \ \mathsf{EC}$$

We can derive the following in **G3c/PSF***:

$$\dfrac{\dfrac{\dfrac{\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x\}, \hat{\Gamma}\{y\} \Rightarrow \hat{\Delta}_{\mathcal{H}}, \hat{\Delta}\{x\}, \hat{\Delta}\{y\}}{\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x\}, \hat{\Gamma}\{x\} \Rightarrow \hat{\Delta}_{\mathcal{H}}, \hat{\Delta}\{x\}, \hat{\Delta}\{x\}} \ [\mathsf{x}/\mathsf{y}]}{\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x\} \Rightarrow \hat{\Delta}_{\mathcal{H}}, \hat{\Delta}\{x\}, \hat{\Delta}\{x\}} \ (\mathsf{LC})^{*}}{\hat{\Gamma}_{\mathcal{H}}, \hat{\Gamma}\{x\} \Rightarrow \hat{\Delta}_{\mathcal{H}}, \hat{\Delta}\{x\}} \ (\mathsf{RC})^{*}$$

where $x, y \# \hat{\Gamma}_{\mathcal{H}}, \hat{\Delta}_{\mathcal{H}}$.

$\square$

APPENDIX E

Translation between Hypersequents and Simple Labelled Sequents

Here we give proofs from Chapter 7 on page 151

We show the translation is syntactically correct by showing that the two translation functions are the inverse of one another (modulo equivalence of labels).

PROPOSITION E.1. *The function* $\mathtt{seq}_3^+$ *distributes over sequent union, that is,*

$$\mathtt{seq}_3^+\ (\underline{S}_1 \sqcup \underline{S}_2) = (\mathtt{seq}_3^+\ \underline{S}_1) \mid (\mathtt{seq}_3^+\ \underline{S}_2)$$

where $\mathtt{lab}(\underline{S}_1) \cap \mathtt{lab}(\underline{S}_2) = \emptyset$.

Proof. From the Definition 7.23 on page 159 and Definition 5.10 on page 106. □

LEMMA E.2. *For all* $M \in \mathtt{Lab}^+$ *and solid* $\mathcal{H} \in \mathtt{Seq}^+$, $\mathtt{seq}_3^+\ (\mathtt{sls}_3\ M\ \mathcal{H}) = \mathcal{H}$.

Proof. By induction on the number of components. Without loss of generality, fix $M = x_1, \ldots, x_n$.

(1) The base case. Suppose $\mathcal{H}$ is $\Gamma \Rightarrow \Delta$ (one component). Then

$$\begin{aligned}
\mathtt{seq}_3^+\ \mathtt{sls}_3\ M\ \mathcal{H} &= \mathtt{seq}_3^+\ \mathtt{sls}_3\ M\ \Gamma \Rightarrow \Delta \\
&= \mathtt{seq}_3^+\ \mathtt{sls}_2\ x_1\ \Gamma \Rightarrow \Delta \\
&= \mathtt{seq}_3^+\ (\mathtt{sls}_1\ x_1\ \Gamma) \Rightarrow (\mathtt{sls}_1\ x_1\ \Delta) \\
&= \mathtt{seq}_3^+\ \Gamma^{x_1} \Rightarrow \Delta^{x_1} \\
&= \mathtt{seq}_2^+\ \Gamma^{x_1} \Rightarrow \Delta^{x_1} \\
&= (\mathtt{seq}_1^+\ \Gamma^{x_1}) \Rightarrow (\mathtt{seq}_1^+\ \Delta^{x_1}) \\
&= \Gamma \Rightarrow \Delta
\end{aligned}$$

(2) The induction step. Suppose $\mathcal{H}$ is $\mathcal{H}' \mid \Gamma_n \Rightarrow \Delta_n$ and, as induction hypothesis $\mathtt{seq}_3^+\ \mathtt{sls}_3\ M\ \mathcal{H}' = \mathcal{H}'$. Then

$$\begin{aligned}
\mathtt{seq}_3^+\ \mathtt{sls}_3\ \mathcal{H} &= \mathtt{seq}_3^+\ \mathtt{sls}_3\ M\ (\mathcal{H}' \mid \Gamma_n \Rightarrow \Delta_n) \\
&= \mathtt{seq}_3^+\ (\mathtt{sls}_3\ M\ \mathcal{H}') \sqcup (\mathtt{sls}_2\ x_n\ \Gamma_n \Rightarrow \Delta_n) \\
&= (\mathtt{seq}_3^+\ \mathtt{sls}_3\ M\ \mathcal{H}') \mid (\mathtt{seq}_3^+\ \mathtt{sls}_2\ x_n\ \Gamma_n \Rightarrow \Delta_n) \\
&= \mathcal{H}' \mid (\mathtt{seq}_3^+\ \mathtt{sls}_2\ x_n\ \Gamma_n \Rightarrow \Delta_n) \\
&= \mathcal{H}' \mid \mathtt{seq}_3^+\ (\mathtt{sls}_1\ x_n\ \Gamma_n) \Rightarrow (\mathtt{sls}_1\ x_n\ \Delta_n) \\
&= \mathcal{H}' \mid \mathtt{seq}_3^+\ \Gamma_n^{x_n} \Rightarrow \Delta_n^{x_n} \\
&= \mathcal{H}' \mid \mathtt{seq}_2^+\ \Gamma_n^{x_n} \Rightarrow \Delta_n^{x_n} \\
&= \mathcal{H}' \mid (\mathtt{seq}_1^+\ \Gamma_n^{x_n}) \Rightarrow (\mathtt{seq}_1^+\ \Delta_n^{x_n}) \\
&= \mathcal{H}' \mid \Gamma_n \Rightarrow \Delta_n
\end{aligned}$$

Therefore $\mathtt{seq}_3^+\ \mathtt{sls}_3\ M\ \mathcal{H} = \mathcal{H}$. □

LEMMA E.3. *For all $M \in \mathsf{Lab}^+$ and non-empty $\underline{\Gamma} \Rightarrow \underline{\Delta} \in \mathsf{SLS}$,*
$(\mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ \underline{\Gamma} \Rightarrow \underline{\Delta}) \approx (\underline{\Gamma} \Rightarrow \underline{\Delta})$.

Proof. By induction on the number of labels. Without loss of generality, fix $M = x_1, \ldots, x_n$.

(1) The base case. Suppose $\underline{\Gamma} \Rightarrow \underline{\Delta}$ is $\Gamma_1^{y_1} \Rightarrow \Delta_1^{y_1}$ (one label). Then

$$\begin{aligned}
\mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ \underline{\Gamma} \Rightarrow \underline{\Delta} &= \mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ \Gamma_1^{y_1} \Rightarrow \Delta_1^{y_1} \\
&= \mathtt{sls}_3\ M\ \mathtt{seq}_2^+\ \Gamma_1^{y_1} \Rightarrow \Delta_1^{y_1} \\
&= \mathtt{sls}_3\ M\ (\mathtt{seq}_1^+\ \Gamma_1^{y_1}) \Rightarrow (\mathtt{seq}_1^+\ \Delta_1^{y_1}) \\
&= \mathtt{sls}_3\ M\ \Gamma_1 \Rightarrow \Delta_1 \\
&= \mathtt{sls}_2\ x_1\ \Gamma_1 \Rightarrow \Delta_1 \\
&= (\mathtt{sls}_1\ x_1\ \Gamma_1) \Rightarrow (\mathtt{sls}_1\ x_1\ \Delta_1) \\
&= \Gamma_1^{x_1} \Rightarrow \Delta_1^{x_1} \\
&\approx \Gamma_1^{y_1} \Rightarrow \Delta_1^{y_1}
\end{aligned}$$

(2) The induction step. Suppose $\underline{\Gamma} \Rightarrow \underline{\Delta}$ is $\underline{\Gamma'}, \Gamma_n^{y_n} \Rightarrow \underline{\Delta'}, \Delta_n^{y_n}$ where $y_n \# \underline{\Gamma'}, \underline{\Delta'}$ and, as induction hypothesis, $(\mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ \underline{\Gamma'} \Rightarrow \underline{\Delta'}) \approx (\underline{\Gamma'} \Rightarrow \underline{\Delta'})$. Then

$$
\begin{aligned}
\mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ \underline{\Gamma} \Rightarrow \underline{\Delta} &= \mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ \underline{\Gamma'}, \Gamma_n^{y_n} \Rightarrow \underline{\Delta'}, \Delta_n^{y_n} \\
&= \mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ (\underline{\Gamma'} \Rightarrow \underline{\Delta'}) \sqcup (\Gamma_n^{y_n} \Rightarrow \Delta_n^{y_n}) \\
&= \mathtt{sls}_3\ M\ (\mathtt{seq}_3^+\ \underline{\Gamma'} \Rightarrow \underline{\Delta'}) \mid (\mathtt{seq}_3^+\ \Gamma_n^{y_n} \Rightarrow \Delta_n^{y_n}) \\
&= \mathtt{sls}_3\ M\ (\mathtt{seq}_3^+\ \underline{\Gamma'} \Rightarrow \underline{\Delta'}) \mid (\mathtt{seq}_2^+\ \Gamma_n^{y_n} \Rightarrow \Delta_n^{y_n}) \\
&= \mathtt{sls}_3\ M\ (\mathtt{seq}_3^+\ \underline{\Gamma'} \Rightarrow \underline{\Delta'}) \mid (\mathtt{seq}_1^+\ \Gamma_n^{y_n}) \Rightarrow (\mathtt{seq}_1^+\ \Delta_n^{y_n}) \\
&= \mathtt{sls}_3\ M\ (\mathtt{seq}_3^+\ \underline{\Gamma'} \Rightarrow \underline{\Delta'}) \mid \Gamma_n \Rightarrow \Delta_n \\
&= (\mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ \underline{\Gamma'} \Rightarrow \underline{\Delta'}) \sqcup (\mathtt{sls}_2\ x_n\ \Gamma_n \Rightarrow \Delta_n) \\
&= (\mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ \underline{\Gamma'} \Rightarrow \underline{\Delta'}) \sqcup (\mathtt{sls}_1\ x_n\ \Gamma_n) \Rightarrow (\mathtt{sls}_1\ x_n\ \Delta_n) \\
&= (\mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ \underline{\Gamma'} \Rightarrow \underline{\Delta'}) \sqcup (\Gamma_n^{x_n} \Rightarrow \Delta_n^{x_n}) \\
&\approx (\underline{\Gamma'} \Rightarrow \underline{\Delta'}) \sqcup (\Gamma_n^{x_n} \Rightarrow \Delta_n^{x_n}) \\
&\approx (\underline{\Gamma'} \Rightarrow \underline{\Delta'}) \sqcup (\Gamma_n^{y_n} \Rightarrow \Delta_n^{y_n}) \\
&= \underline{\Gamma'}, \Gamma_n^{y_n} \Rightarrow \underline{\Delta'}, \Delta_n^{y_n}
\end{aligned}
$$

Therefore $(\mathtt{sls}_3\ M\ \mathtt{seq}_3^+\ \underline{\Gamma} \Rightarrow \underline{\Delta}) \approx (\underline{\Gamma} \Rightarrow \underline{\Delta})$. □

THEOREM E.4. **HG1ip** $\vdash \mathcal{H}$ *iff* **HO** $\vdash \mathcal{H}$.

Proof. By induction on the derivation depths. We show the non-trivial cases only. From left-to-right:

(1) Suppose a derivation in **HG1ip** ends with an instance of $\mathsf{L}\wedge_1$. Then the corresponding rule can be applied in **HO** using LW and $\mathsf{L}\wedge_\mathsf{O}$, with $n = 1$.

The case for $\mathsf{L}\wedge_2$ is similar.

(2) Suppose a derivation in **HG1ip** ends with an instance of $\mathsf{R}\vee_1$. This can be derived using EW and $\mathsf{R}\vee_{\mathsf{O}_1}$ in **HO**.

The cases for $\mathsf{R}\vee_2$ and $\mathsf{L}\supset$ are similar .

(3) Suppose a derivation in **HG1ip** ends with an instance of $\mathsf{R}\supset$. This is a special case of $\mathsf{R}\supset_\mathsf{O}$ in **HO**, with $n = 1$.

From right-to-left:

(1) Suppose a derivation in **HO** ends with an instance of $\mathsf{L}\wedge_{\mathsf{O}}$. We note the $\mathsf{L}\wedge$ rule can be derived in **HG1ip**:

$$\dfrac{\dfrac{\dfrac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge_1}{A \wedge B, A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge_2}{A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{LC}$$

Likewise, the corresponding rule can be derived in **HG1ip** using multiple instances of $\mathsf{L}\wedge$.

(2) Suppose a derivation in **HO** ends with an instance of $\mathsf{R}\vee_{\mathsf{O}_1}$. Then the corresponding rule can be derived in **HG1ip** using $\mathsf{R}\vee_1$ and EC.

The case for $\mathsf{R}\vee_{\mathsf{O}_2}$ is similar.

(3) Suppose a derivation in **HO** ends with an instance of $\mathsf{R}\supset_{\mathsf{O}}$. Then the corresponding rule can be derived in **HG1ip** using multiple instances of $\mathsf{R}\supset$.

□

APPENDIX F

Extending the Translation to Cover Empty Components

As noted in Chapter 7, there are hypersequent calculi where the empty component is meaningful, for example, [MOG05]. There is now way to specify empty slices using the notation for simple labelled sequents. The simplest way of addressing this is to alter the notation to use the hypersequents with named components, akin to the "indexed sequents" from [Min97], so that instead of

$$\Gamma_1^{x_1}, \ldots, \Gamma_1^{x_n} \Rightarrow \Delta_1^{x_1}, \ldots, \Delta_n^{x_1}$$

we use

$$\Gamma_1 \Rightarrow^{x_1} \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow^{x_n} \Delta_n$$

With labels attached to sequent arrows rather than formulae, it is possible to represent empty slices. Unless the labels contain additional information beyond names for components (such as relationships between labels), it is of no advantage over hypersequents.

An alternative is to extend the language of formulae to by adding a placeholder symbol such as ε. (The new type will be denoted by $\mathtt{Prop} + \varepsilon$.) Semantically, ε has the same meaning as an empty slice, so that the following invertible rules can be added to a simple labelled calculus which requires empty slices:

$$\frac{\underline{\Gamma} \Rightarrow \underline{\Delta}}{\varepsilon^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\varepsilon \qquad \frac{\underline{\Gamma} \Rightarrow \underline{\Delta}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, \varepsilon^x}\ \mathsf{R}\varepsilon$$

where the restrictions that $A^x \notin \underline{\Gamma}$ for $\mathsf{L}\varepsilon$ and $A^x \notin \underline{\Delta}$ for $\mathsf{R}\varepsilon$. Note that context-splitting rules may also need the restriction regarding the occurrence of ε^x in any of the contexts.

The extension of the language of formulae will require extending the definitions of labelled multisets, the notion of equivalence modulo permutation accordingly before extending the translations.

Definition F.1 (ε-Labelled Multisets and Sequents). **ε-labelled multisets** are non-empty labelled multisets of formulae or a labelled ε symbol, written as ε^x. The set of

ε-labelled multisets is denoted by $((\mathtt{Form}_1+\varepsilon)\times\mathtt{Lab})^+$. **$\varepsilon$-labelled sequents** are pairs of ε-labelled multisets. The set of ε-labelled sequents is represented by $\mathtt{SLS}_\varepsilon$.

DEFINITION F.2 (Well-Formed ε-Labelled Multisets and Sequents). A multiset of ε-labelled formulae $\underline{\Gamma}$ is **well-formed** iff $\underline{\Gamma}=\varepsilon^x,\underline{\Gamma}'$ implies $A^x\notin\underline{\Gamma}'$.

A ε-labelled sequent $\underline{\Gamma}\Rightarrow\underline{\Delta}$ is **well-formed** iff (1) both $\underline{\Gamma}$ and $\underline{\Delta}$ are well-formed; and (2) for all $x\in\mathtt{lab}(\underline{\Gamma},\underline{\Delta})$, $\underline{\Gamma}\,/\!\!/\,x\neq\emptyset$ and $\underline{\Delta}\,/\!\!/\,x\neq\emptyset$.

The definitions of slice, multiset and sequent union for ε-labelled multisets or sequents must be updated accordingly:

DEFINITION F.3 (Slice). Let $\underline{\Gamma}$ be a well-formed ε-labelled multiset:

$$\underline{\Gamma}\,/\!\!/_{\varepsilon}\,x =_{def} \begin{cases} \underline{\Gamma}\,/\!\!/\,x & \text{if } \varepsilon^x\notin\underline{\Gamma} \\ \varepsilon^x & \text{otherwise} \end{cases}$$

The slice of a well-formed ε-labelled sequent is defined naturally from the slice of a well-formed ε-labelled multiset.

DEFINITION F.4 (Multiset Union). Let $\Gamma^x,\Delta^x,\underline{\Gamma},\underline{\Delta}$ be well-formed ε-labelled multisets.

$$\varepsilon^x\cup_\varepsilon\Gamma^x =_{def} \Gamma^x$$

$$\Gamma^x\cup_\varepsilon\varepsilon^x =_{def} \Gamma^x$$

$$\Gamma^x\cup_\varepsilon\Delta^x =_{def} \Gamma^x\cup\Delta^x$$

$$\underline{\Gamma}\cup_\varepsilon\underline{\Delta} =_{def} \bigcup_{i=1}^{n}(\underline{\Gamma}\,/\!\!/_{\varepsilon}\,x_i)\cup_\varepsilon(\underline{\Delta}\,/\!\!/_{\varepsilon}\,x_i) \qquad \text{where } \mathtt{lab}(\underline{\Gamma},\underline{\Delta})=\{x_1,\ldots,x_n\}$$

PROPOSITION F.5. $\cup_\varepsilon$ *is commutative and associative.*

Proof. From the definition. □

DEFINITION F.6 (Sequent Union). Let $\underline{\Gamma}_1\Rightarrow\underline{\Delta}_1$ and $\underline{\Gamma}_2\Rightarrow\underline{\Delta}_2$ be well-formed ε-labelled sequents.

$$(\underline{\Gamma}_1\Rightarrow\underline{\Delta}_1)\sqcup_\varepsilon\underline{\Gamma}_2\Rightarrow\underline{\Delta}_2 =_{def} (\underline{\Gamma}_1\cup_\varepsilon\underline{\Gamma}_2)\Rightarrow(\underline{\Delta}_1\cup_\varepsilon\underline{\Delta}_2)$$

PROPOSITION F.7. $\sqcup_\varepsilon$ *is commutative and associative.*

Proof. From the definition. □

Definition F.8 (Equivalence Modulo Permutation for ε-Labelled Multisets and Sequents). We define the following extension of $\approx$ for well-formed ε-labelled multisets:

$$\begin{aligned} \varepsilon^x, \underline{\Gamma} &\approx_\varepsilon \underline{\Delta} && \text{if } \underline{\Gamma} \approx_\varepsilon \underline{\Delta} \\ \underline{\Gamma} &\approx_\varepsilon \underline{\Delta}, \varepsilon^y && \text{if } \underline{\Gamma} \approx_\varepsilon \underline{\Delta} \\ \underline{\Gamma} &\approx_\varepsilon \underline{\Delta} && \text{if } \underline{\Gamma} \approx \underline{\Delta} \end{aligned}$$

and for well-formed ε-labelled sequents:

$$\begin{aligned} (\Gamma_1^x \Rightarrow \Delta_1^x) &\approx_\varepsilon (\Gamma_2^y \Rightarrow \Delta_2^y) && \text{iff } \Gamma_1^x \approx_\varepsilon \Gamma_2^y \text{ and } \Delta_1^x \approx_\varepsilon \Delta_2^y \\ \underline{S}'_1, S_1^x &\approx_\varepsilon \underline{S}'_2, S_2^y && \text{iff } x \# \underline{S}'_1, y \# \underline{S}'_2, \underline{S}'_1 \approx_\varepsilon \underline{S}'_2 \text{ and } S_1^x \approx_\varepsilon S_2^y \end{aligned}$$

Proposition F.9. *The relation $\approx_\varepsilon$ is an equivalence relation—that is, it is reflexive, transitive and symmetric.*

Proof. From Definition F.8. □

Proposition F.10. *If $\underline{S}_1 \approx \underline{S}_2$ then $\underline{S}_1 \approx_\varepsilon \underline{S}_2$.*

Proof. From Definition F.8. □

Proposition F.11. *Let $\underline{S}_1 \approx_\varepsilon \underline{S}_2$, where $\underline{S}_1 \# \underline{S}_3$ and $\underline{S}_2 \# \underline{S}_3$.*
Then $\underline{S}_1 \sqcup \underline{S}_3 \approx_\varepsilon \underline{S}_2 \sqcup \underline{S}_3$.

Proof. From Definition F.6 on the preceding page. □

Definition F.12 (Translation of Hypersequents to ε-Labelled Sequents). We fix $M = x_1, \ldots, x_n$ as a non-empty list of simple labels in $\texttt{Lab}^+$.

$$\texttt{sls}_1^\varepsilon\ x\ \Gamma = \begin{cases} \varepsilon^x & \text{if } x \in \texttt{Lab} \text{ and } \Gamma = \emptyset \\ \texttt{sls}_1\ x\ \Gamma & \text{otherwise} \end{cases}$$

$$\texttt{sls}_2^\varepsilon\ x\ S = (\lambda y.\texttt{sls}_1^\varepsilon\ x\ y) \circledast S \text{ where } S \in \mathsf{SLS} \mathbin{/\!\!/} x$$

$$\texttt{sls}_3^\varepsilon\ M\ (S_1 \mid \ldots \mid S_n) = \begin{cases} \texttt{sls}_2^\varepsilon\ x_1\ S_1 & \text{if } n = 1 \\ (\texttt{sls}_3^\varepsilon\ M\ S_1 \mid \ldots \mid S_{n-1}) \sqcup (\texttt{sls}_2^\varepsilon\ x_n\ S_n) & \text{otherwise} \end{cases}$$

PROPOSITION F.13. *The function* $\mathtt{sls}_3^{\varepsilon}$ *distributes over hypersequent pipes, i.e.*

$$\mathtt{sls}_3^{\varepsilon}\ (\mathcal{H}_1 \mid \mathcal{H}_2) \approx_{\varepsilon} (\mathtt{sls}_3^{\varepsilon}\ \mathcal{H}_1) \sqcup \varepsilon (\mathtt{sls}_3^{\varepsilon}\ \mathcal{H}_2)$$

Proof. From Definitions F.6 and F.12. □

DEFINITION F.14 (Translation of ε-Labelled Sequents to Hypersequents). We define a translation $\mathtt{seq}_3^{\varepsilon+}$ of well-formed labelled sequents to hypersequents:

$$\mathtt{seq}_1^{\varepsilon+}\ \Gamma^x = \begin{cases} \emptyset & \text{if } \Gamma^x = \varepsilon^x \\ \mathtt{seq}_1^{+}\ \Gamma^x & \text{otherwise} \end{cases}$$

$$\mathtt{seq}_2^{\varepsilon+}\ S^x = \mathtt{seq}_1^{\varepsilon+} \circledast S^x \text{ where } S^x = \Gamma^x \Rightarrow \Delta^x$$

$$\mathtt{seq}_3^{\varepsilon+}\ S_1^{x_1}, \ldots, S_n^{x_n} = \begin{cases} \mathtt{seq}_2^{\varepsilon+}\ S_1^{x_1} & \text{if } n = 1 \\ (\mathtt{seq}_3^{\varepsilon+}\ S_1^{x_1}, \ldots, S_{n-1}^{x_{n-1}}) \mid (\mathtt{seq}_2^{\varepsilon+}\ S_n^{x_n}) & \text{otherwise} \end{cases}$$

PROPOSITION F.15. *The function* $\mathtt{seq}_3^{\varepsilon+}$ *distributes over sequent union, i.e.*

$$\mathtt{seq}_3^{\varepsilon+}\ (\underline{S}_1 \sqcup_{\varepsilon} \underline{S}_2) = (\mathtt{seq}_3^{\varepsilon+}\ \underline{S}_1) \mid (\mathtt{seq}_3^{\varepsilon+}\ \underline{S}_2) \textit{ if } \underline{S}_1 \# \underline{S}_2$$

Proof. From Definitions F.6 and F.14. □

THEOREM F.16. *For all* $M \in \mathtt{Lab}^+$ *and* $\mathcal{H} \in \mathtt{Seq}^+$, $\mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}) = \mathcal{H}$.

Proof. By induction on the number of components where either the antecedent or succedent is empty. Without loss of generality, fix $M = x_1, \ldots, x_n$.

(1) The base case. Since there are no components where either the antecedent or succedent is empty, then

$$\mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}) = \mathtt{seq}_3^{+}\ (\mathtt{sls}_3\ M\ \mathcal{H}) = \mathcal{H}$$

by Lemma E.2 on page 264.

(2) The induction step. Suppose $\mathcal{H}$ is $\mathcal{H}' \mid S_n$ and, as the induction hypothesis, $\mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}') = \mathcal{H}'$. There are three sub-cases for S_n:

(a) S_n is $\emptyset \Rightarrow \emptyset$. Then

$$\begin{aligned}
\mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}) &= \mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ (\mathcal{H}' \mid \emptyset \Rightarrow \emptyset)) \\
&= \mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}') \sqcup (\mathtt{sls}_2^{\varepsilon}\ \emptyset \Rightarrow \emptyset) \\
&= \mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}') \sqcup (\mathtt{sls}_1^{\varepsilon} \circledast \emptyset \Rightarrow \emptyset) \\
&= \mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}') \sqcup (\varepsilon^{x_n} \Rightarrow \varepsilon^{x_n}) \\
&= \mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}') \mid \mathtt{seq}_3^{\varepsilon+}\ \varepsilon^{x_n} \Rightarrow \varepsilon^{x_n} \\
&= \mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}') \mid \mathtt{seq}_2^{\varepsilon+}\ \varepsilon^{x_n} \Rightarrow \varepsilon^{x_n} \\
&= \mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}') \mid \mathtt{seq}_1^{\varepsilon+} \circledast \varepsilon^{x_n} \Rightarrow \varepsilon^{x_n} \\
&= \mathtt{seq}_3^{\varepsilon+}\ (\mathtt{sls}_3^{\varepsilon}\ M\ \mathcal{H}') \mid \emptyset \Rightarrow \emptyset \\
&= \mathcal{H}' \mid \emptyset \Rightarrow \emptyset
\end{aligned}$$

(b) S_n is $\emptyset \Rightarrow \Delta$. Similar.

(c) S_n is $\Gamma \Rightarrow \emptyset$. Similar.

□

THEOREM F.17. *For all* $M \in \mathtt{Lab}^+$ *and* $\underline{S} \in \mathrm{SLS}_{\varepsilon}$, $\mathtt{sls}_3^{\varepsilon}\ M\ (\mathtt{seq}_3^{\varepsilon+}\ \underline{S}) \approx_{\varepsilon} \underline{S}$.

Proof. By induction on the number of labels where either the antecedent or succedent contains ε^x. Without loss of generality, fix $M = x_1, \ldots, x_n$.

(1) The base case. Since there are no labels x such that the antecedent or succedent contains ε^x, then $\mathtt{sls}_3^{\varepsilon}\ M\ (\mathtt{seq}_3^{\varepsilon+}\ \underline{S}) = \mathtt{sls}_3\ M\ (\mathtt{seq}_3^{+}\ \underline{S}) \approx \underline{S} \approx_{\varepsilon} \underline{S}$.

(2) The induction step. Suppose $\underline{S} = \underline{S'}, S_n^y$, where $y \# \underline{S'}$ and, as the induction hypothesis, $\mathtt{sls}_3^{\varepsilon}\ M\ (\mathtt{seq}_3^{\varepsilon+}\ \underline{S'}) \approx_{\varepsilon} \underline{S'}$. There are three sub-cases for S_n^y:

(a) S^y_n is $\varepsilon^y \Rightarrow \varepsilon^y$. Then

$$\begin{aligned}
\mathtt{sls}^{\varepsilon}_{3}\ M\ (\mathtt{seq}^{\varepsilon+}_{3}\ \underline{S}) &= \mathtt{sls}^{\varepsilon}_{3}\ M\ (\mathtt{seq}^{\varepsilon+}_{3}\ \underline{S}' \sqcup \varepsilon^y \Rightarrow \varepsilon^y) \\
&= \mathtt{sls}^{\varepsilon}_{3}\ M\ (\mathtt{seq}^{\varepsilon+}_{3}\ \underline{S}' \mid \mathtt{seq}^{\varepsilon+}_{3}\ \varepsilon^y \Rightarrow \varepsilon^y) \\
&= \mathtt{sls}^{\varepsilon}_{3}\ M\ (\mathtt{seq}^{\varepsilon+}_{3}\ \underline{S}' \mid \mathtt{seq}^{\varepsilon+}_{2}\ \varepsilon^y \Rightarrow \varepsilon^y) \\
&= \mathtt{sls}^{\varepsilon}_{3}\ M\ (\mathtt{seq}^{\varepsilon+}_{3}\ \underline{S}' \mid \mathtt{seq}^{\varepsilon+}_{1}\ \circledast\ \varepsilon^y \Rightarrow \varepsilon^y) \\
&= \mathtt{sls}^{\varepsilon}_{3}\ M\ (\mathtt{seq}^{\varepsilon+}_{3}\ \underline{S}' \mid \Rightarrow) \\
&= (\mathtt{sls}^{\varepsilon}_{3}\ M\ \mathtt{seq}^{\varepsilon+}_{3}\ \underline{S}') \sqcup (\mathtt{sls}^{\varepsilon}_{2}\ \Rightarrow) \\
&= (\mathtt{sls}^{\varepsilon}_{3}\ M\ \mathtt{seq}^{\varepsilon+}_{3}\ \underline{S}') \sqcup (\mathtt{sls}^{\varepsilon}_{1} \circledast \Rightarrow) \\
&= (\mathtt{sls}^{\varepsilon}_{3}\ M\ \mathtt{seq}^{\varepsilon+}_{3}\ \underline{S}') \sqcup (\varepsilon^{x_n} \Rightarrow \varepsilon^{x_n}) \\
&\qquad \text{where } x_n \# (\mathtt{sls}^{\varepsilon}_{3}\ M\ (\mathtt{seq}^{\varepsilon+}_{3}\ \underline{S}')) \\
&\approx_{\varepsilon} \underline{S}' \sqcup (\varepsilon^{x_n} \Rightarrow \varepsilon^{x_n}) \\
&\approx_{\varepsilon} \underline{S}' \sqcup (\varepsilon^{y} \Rightarrow \varepsilon^{y}) = \underline{S}
\end{aligned}$$

(b) S^y_n is $\varepsilon^y \Rightarrow \Delta^y$. Similar.

(c) S^y_n is $\Gamma^y \Rightarrow \varepsilon^y$. Similar.

□

APPENDIX G

Translating between Labelled Sequents and Relational Sequents

Here we give proofs from Chapter 9.

LEMMA G.1. *Let* **RGS** *be a relational extension of a calculus* **LGS** *with contraction and weakening. Then the reflexivity and transitivity rules*

$$\frac{x \le x, \Sigma; \underline{S}}{\Sigma; \underline{S}}\ (\mathsf{refl}) \qquad \frac{x \le y \le z, x \le z, \Sigma; \underline{S}}{x \le y \le z, \Sigma; \underline{S}}\ (\mathsf{trans})$$

can be shown admissible in **RGS**.

Proof. By induction on the derivation depth. We suppose that if an instance of a rule refl or trans occurs at a particular depth, then it can be permuted to a lower depth or eliminated.

For refl, suppose the following cases, supposing there is a derivation of $x \le x, \Sigma; \underline{S}$:

(1) If $x \le x, \Sigma; \underline{S}$ is an axiom, then so is $\Sigma; \underline{S}$.
(2) The active formula $x \le x$ is principal in an instance of $\mathsf{LW} \le$. Then by applying the induction hypothesis, $\Sigma; \underline{S}$ is derivable.
(3) The active formula $x \le x$ is principal in an instance of $\mathsf{LC} \le$. Then by applying the induction hypothesis twice, $\Sigma; \underline{S}$ is derivable.
(4) The active formula $x \le x$ is principal in an instance of $\mathsf{L} \le'$. Then the instance $\mathsf{L} \le'$ can be replaced by LC.
(5) The active formula $x \le x$ is principal in an instance of $\mathsf{R} \le'$. Then the instance $\mathsf{R} \le'$ can be replaced by RC.

For trans, suppose the following cases, supposing there is a derivation of $x \le y \le z, x \le z, \Sigma; \underline{S}$:

(1) If $x \le y \le z, x \le z, \Sigma; \underline{S}$ is an axiom, then so is $x \le y \le z, \Sigma; \underline{S}$.
(2) The active formula $x \le z$ is principal in an instance of $\mathsf{LW} \le$. Then by applying the induction hypothesis, $x \le y \le z, \Sigma; \underline{S}$ is derivable.

(3) The active formula $x \leq z$ is principal in an instance of $\mathsf{LC} \leq$. Then by applying the induction hypothesis twice, $x \leq y \leq z, \Sigma; \underline{S}$ is derivable.

(4) The active formula $x \leq z$ is principal in an instance of $\mathsf{L} \leq'$:

$$\dfrac{\dfrac{x \leq y \leq z, \Sigma; A^x, A^z, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y \leq z, x \leq z, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L} \leq'}{x \leq y \leq z, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ (\mathsf{trans})$$

Then the proof can be rewritten as

$$\dfrac{\dfrac{\dfrac{\dfrac{x \leq y \leq z, \Sigma; A^x, A^z, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y \leq z, \Sigma; A^x, A^y, A^z, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{LW}}{x \leq y \leq z, y \leq z, \Sigma; A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L} \leq'}{x \leq y \leq z, x \leq y \leq z, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L} \leq'}{x \leq y \leq z, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{LC} \leq^2$$

eliminating the instance of trans.

(5) The active formula $x \leq z$ is principal in an instance of $\mathsf{R} \leq'$. Then the case is similar to the case for $\mathsf{L} \leq'$, using RW instead.

□

THEOREM G.2 (Interderivability). **RG3ipm′** $\vdash \Sigma; \underline{S}$ *iff* **G3I** $\vdash \Sigma; \underline{S}$.

Proof. By showing that the rules of **RG3ipm′** are admissible in **G3I** and viz. that the rules of **G3I** are admissible in **RG3ipm′**.

(1) The axiom of **RG3ipm′** is derivable in **G3I**, by refl:

$$\dfrac{x \leq x, \Sigma; \underline{\Gamma}, A^x \Rightarrow A^x, \underline{\Delta}}{\Sigma; \underline{\Gamma}, A^x \Rightarrow A^x, \underline{\Delta}}\ \mathsf{refl}$$

(2) The $\mathsf{L} \leq'$ and $\mathsf{R} \leq'$ rules of **RG3ipm′** is admissible in **G3I**. By Lemma 5.100 (page 124) and $\mathsf{LW} \leq$.

(3) The axiom of **G3I** is derivable in **RG3ipm′**, by $\mathsf{L} \leq$:

$$\dfrac{x \leq y, \Sigma; \underline{\Gamma}, A^x, A^y \Rightarrow A^y, \underline{\Delta}}{x \leq y, \Sigma; \underline{\Gamma}, A^x \Rightarrow A^y, \underline{\Delta}}\ \mathsf{L} \leq$$

□

Theorem G.3 (Polycut Admissibility).

$$\frac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},n_1\cdot A^{x_1},\ldots,n_k\cdot A^{x_k} \quad \Sigma';m_1\cdot A^{y_1},\ldots,m_{k'}\cdot A^{y_{k'}},\underline{\Gamma}'\Rightarrow\underline{\Delta}'}{\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma}'\Rightarrow\underline{\Delta},\underline{\Delta}'}\ \text{(pcut)}$$

where for all $1\leq i\leq k$ *and* $1\leq j\leq k'$, $x_i\leq y_j\in\Sigma,\Sigma'$, *is admissible in* **RG3ipm′**. *Recall the notation on page 21, that* $n\cdot A^x$ *denotes* n *copies of* A^x.

Terminology G.4. Each x_i is a **left cut label** and each y_i is a **right cut label**. A is called the **cut formula**, and each A^{x_i} and A^{y_i} (for $1\leq i\leq k$) are called the **left cut formulae** and **right cut formula**, respectively.

Remark G.5. Without the restriction on the relational context, polycut would be unsound, e.g.

$$\frac{A^y\Rightarrow A^y,A^x \quad A^y,A^x\Rightarrow A^x}{A^y\Rightarrow A^x}\ \text{(pcut)}$$

Proof. By induction on the rank defined by (a) the size of the cut formula; and (b) the sum of the derivation depths of the premisses. We suppose the derivation has been rewritten according to Proposition 9.14 on page 197. The cases are listed below:

(1) For the base case, where both premisses are axioms, we have the following subcases:
   (a) If the cut formula is not principal in either of the axioms, then the conclusion of the cut $\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma}'\Rightarrow\underline{\Delta},\underline{\Delta}'$ is an axiom.
   (b) Otherwise, the cut formula is principal in both axioms. Without loss of generality, we assume P^{x_i} is principal in the left premiss, and P^{y_j} is principal in the right premiss. So $P^{x_i}\in\underline{\Gamma}$ and $P^{y_j}\in\underline{\Gamma}'$. Recall that $x_i\leq y_j\in\Sigma,\Sigma'$. The following can be derived without an instance of polycut:

$$\dfrac{\dfrac{\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma}',P^{y_j}\Rightarrow\underline{\Delta},\underline{\Delta}'}{x_j\leq y_j,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma}'\Rightarrow\underline{\Delta},\underline{\Delta}'}\ \mathsf{L}\leq'}{\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma}'\Rightarrow\underline{\Delta},\underline{\Delta}'}\ \mathsf{LC}\leq$$

(2) Suppose the right premiss is the conclusion of an instance of the $\mathsf{L}\leq$ rule. There are the following subcases:

(a) if the left cut formula is principal in $\mathsf{L}\leq$,

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ \Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^x\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\delta_2\\ x\leq y,x\leq z,\Sigma';A^x,A^z,A^y,\underline{\Gamma'}\Rightarrow\underline{\Delta'}\end{array}}{x\leq y,x\leq z,\Sigma';A^x,A^y,\underline{\Gamma'}\Rightarrow\underline{\Delta'}}\,\mathsf{L}\leq}{x\leq y,x\leq z,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma'}\Rightarrow\underline{\Delta},\underline{\Delta'}}\,(\mathsf{pcut})$$

(where w.l.o.g. we omit the additional right cut formulae for brevity), then the cut of rank $\langle|A|,\ h(\delta_1)+1+h(\delta_2)\rangle$ can be permuted to the premiss of the $\mathsf{L}\leq$ rule, giving it a rank of $\langle|A|,\ h(\delta_1)+h(\delta_2)\rangle$. (Note that this eliminates that instance of the $\mathsf{L}\leq$ rule.)

(b) if a right cut formula is principal in $\mathsf{L}\leq$,

$$\dfrac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^x \qquad \dfrac{x\leq y\leq z,\Sigma';A^x,A^z,A^y,\underline{\Gamma'}\Rightarrow\underline{\Delta'}}{x\leq y,x\leq z,\Sigma';A^x,A^y,\underline{\Gamma'}\Rightarrow\underline{\Delta'}}\,\mathsf{L}\leq}{x\leq y\leq z,,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma'}\Rightarrow\underline{\Delta},\underline{\Delta'}}\,(\mathsf{pcut})$$

then the cut with a rank of $\langle|A|,\ h(\delta_1)+1+h(\delta_2)\rangle$ can be permuted upwards and the derivation fragment be rewritten as

$$\dfrac{\dfrac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},A^x \qquad \dfrac{x\leq y\leq z,\Sigma';A^x,A^z,A^y,\underline{\Gamma'}\Rightarrow\underline{\Delta'}}{x\leq z,x\leq y\leq z,\Sigma';A^x,A^z,A^y,\underline{\Gamma'}\Rightarrow\underline{\Delta'}}\,\mathsf{LW}\leq}{x\leq z,x\leq y\leq z,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma'}\Rightarrow\underline{\Delta},\underline{\Delta'}}\,(\mathsf{pcut})}{x\leq y\leq z,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma'}\Rightarrow\underline{\Delta},\underline{\Delta'}}\,\mathsf{trans}$$

with a rank of $\langle|A|,\ h(\delta_1)+h(\delta_2)\rangle$. Again, note that this eliminates that instance of the $\mathsf{L}\leq$ rule. Also recall that instances of $\mathsf{LW}\leq$ can be eliminated, so the rule is considered to have no effect on derivation depth.

(3) Suppose the left premiss is the conclusion of an instance of the $\mathsf{R}\leq$ rule. Then the case is similar to case 2.

(4) Suppose the cut formulae are of the form $B\wedge C$:

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ \Sigma,\underline{\Gamma}\Rightarrow\underline{\Delta},(B\wedge C)^x\end{array} \qquad \begin{array}{c}\vdots\,\delta_2\\ x\leq y,\Sigma';(B\wedge C)^x,(B\wedge C)^y,\underline{\Gamma'}\Rightarrow\underline{\Delta'}\end{array}}{x\leq y,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma'}\Rightarrow\underline{\Delta},\underline{\Delta'}}\,(\mathsf{pcut})$$

with a rank of $\langle\ |B\wedge C|,\ h(\delta_1)+h(\delta_1)\ \rangle$, (where the additional cut formulae are omitted for brevity). The cut can be replaced with two cuts on smaller formulae, by using the invertibility of the L$\wedge$ and R$\wedge$ rules:

$$\dfrac{\dfrac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(B\wedge C)^x}{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},B^x}\,(\mathsf{R}\wedge_1^{-1}) \qquad \dfrac{x\le y,\Sigma',(B\wedge C)^x,(B\wedge C)^y,\underline{\Gamma}'\Rightarrow\underline{\Delta}'}{x\le y,\Sigma',B^x,B^y,C^x,C^y,\underline{\Gamma}'\Rightarrow\underline{\Delta}'}\,(\mathsf{L}\wedge^{-1})}{x\le y,\Sigma,\Sigma';C^x,C^y,\underline{\Gamma},\underline{\Gamma}'\Rightarrow\underline{\Delta},\underline{\Delta}'}\,(\mathsf{pcut}) \tag{150}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(B\wedge C)^x}{\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},C^x}\,(\mathsf{R}\wedge_2^{-1}) \qquad \begin{array}{c}\vdots\;(150)\\ x\le y,\Sigma,\Sigma';C^x,C^y,\underline{\Gamma},\underline{\Gamma}'\Rightarrow\underline{\Delta},\underline{\Delta}'\end{array}}{x\le y,\Sigma,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma},\underline{\Gamma}'\Rightarrow\underline{\Delta},\underline{\Delta},\underline{\Delta}'}\,(\mathsf{pcut})}{x\le y,\Sigma,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma}'\Rightarrow\underline{\Delta},\underline{\Delta},\underline{\Delta}'}\,(\mathsf{LC})^*}{x\le y,\Sigma,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma}'\Rightarrow\underline{\Delta},\underline{\Delta}'}\,(\mathsf{RC})^*}{x\le y,\Sigma,\Sigma';\underline{\Gamma},\underline{\Gamma}'\Rightarrow\underline{\Delta},\underline{\Delta}'}\,(\mathsf{LC}\le)^*$$

The cuts have ranks $\langle|B|,\ldots\rangle$ and $\langle|C|,\ldots\rangle$, respectively.

(5) Suppose the cut formulae are of the form $B\vee C$. The cuts can be rewritten similar to case 4 above.

(6) Suppose the cut formula is of the form $B\supset C$. Because $B\supset C$ occurs in the left premiss of the L$\supset$ rule, we can permute the cut up the right premiss until we reach an axiom and eliminate the cut.

(7) For all other cases where the the cut formulae in a premiss are side formulae of a rule, the cut can be permuted upwards to the premiss(es) of that rule, thus reducing the cut rank.

□

COROLLARY G.6 ($\sqsupset'_{\le}$). *The rule*

$$\dfrac{x\le y,\Sigma;\Gamma^x,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}{\Sigma;\Gamma^x,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x}\,(\sqsupset'_{\le})$$

where $y\#\Sigma,\underline{\Gamma}',\underline{\Delta}$*, is admissible in* **RG3ipm**.

Proof. By derivation.

$$\dfrac{\dfrac{\dfrac{x\le y,\Sigma;\Gamma^x,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}{x\le y,\Sigma;\Gamma^x,\Gamma^y,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x,A^y}\,\mathsf{LW}^*}{x\le x,\Sigma;\Gamma^x,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x}\,(\sqsupset_{\le})}{\Sigma;\Gamma^x,\underline{\Gamma}'\Rightarrow\underline{\Delta},A^x}\,\mathsf{refl}$$

□

LEMMA G.7 (Folding Rules). *The **folding rules** are admissible in* **G3I***:*

$$\frac{x \le y, \Sigma; A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \le y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ (\mathsf{L} \le) \qquad \frac{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ (\mathsf{R} \le)$$

Proof. By induction on the derivation depth. We show the interesting cases below:

(1) For $\mathsf{L} \le$, suppose that A^y is principal in the axiom

$$x \le y, y \le z, \Sigma'; A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}', A^z$$

then we can derive the conclusion of the $\mathsf{L} \le$ rule:

$$\frac{x \le z, x \le y, y \le z, \Sigma'; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}', A^z}{x \le y, y \le z, \Sigma'; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}', A^z}\ \mathsf{trans}$$

(2) For $\mathsf{R} \le$, the case is similar. Suppose that A^x is principal in the axiom

$$w \le x, x \le y, \Sigma'; A^w, \underline{\Gamma'} \Rightarrow \underline{\Delta}, A^x, A^y$$

then we can derive the conclusion of the $\mathsf{R} \le$ rule:

$$\frac{w \le y, w \le x, x \le y, \Sigma'; A^w, \underline{\Gamma'} \Rightarrow \underline{\Delta}, A^y}{w \le x, x \le y, \Sigma'; A^w, \underline{\Gamma'} \Rightarrow \underline{\Delta}, A^y}\ \mathsf{trans}$$

(3) Suppose the premiss of an instance of the $\mathsf{R} \le$ rule is the conclusion of an instance of the $\mathsf{R}\wedge$ rule, such that the principal formula of the former is an active formula of the latter:

$$\cfrac{\cfrac{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^x, A^y \qquad x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^x, B^y}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^x, (A \wedge B)^y}\ \mathsf{R}\wedge}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^y}\ (\mathsf{R} \le)$$

(w.l.o.g. we assume that $(A \wedge B)^y$ is the principal formula.) Using the invertibility of $\mathsf{R}\wedge$, we can permute $\mathsf{R} \le$ above $\mathsf{R}\wedge$:

$$\cfrac{\cfrac{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ (\mathsf{R} \le) \qquad \cfrac{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, B^x, B^y}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, B^y}\ (\mathsf{R} \le)}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^y}\ \mathsf{R}\wedge$$

The case for when the $\mathsf{L} \le$ rule is below an instance of the $\mathsf{L}\vee$ rule is similar.

(4) Suppose the premiss of an instance of the $\mathsf{R} \le$ rule is the conclusion of an instance of the $\mathsf{R}\vee$ rule, such that the principal formula of the former is an active

formula of the latter:

$$\dfrac{\dfrac{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \vee B)^x, A^y, B^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \vee B)^x, (A \vee B)^y} \text{R}\vee}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \vee B)^y} (\text{R} \leq)$$

(w.l.o.g. we assume that $(A \vee B)^y$ is the principal formula.) Using the invertibility of R$\vee$, we can permute R $\leq$ above R$\vee$:

$$\dfrac{\dfrac{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, B^x, A^y, B^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y, B^y} (\text{R} \leq)^+}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \vee B)^y} \text{R}\vee$$

The case for when the L $\leq$ rule is below an instance of the L$\wedge$ rule is similar.

(5) Suppose the premiss of an instance of the L $\leq$ rule is the conclusion of an instance of the L$\supset_\leq$ rule, where $(A \supset B)^x$ is principal (using AB as an abbreviation for $A \supset B$):

$$\dfrac{\dfrac{x \leq y, \Sigma; (AB)^x, (AB)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y \quad x \leq y, \Sigma; (AB)^x, (AB)^y, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; (AB)^x, (AB)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}} \text{L}\supset_\leq}{x \leq y, \Sigma; (AB)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}} (\text{L} \leq)$$

The L $\leq$ rule can be permuted above the L$\supset_\leq$ rule:

$$\dfrac{\dfrac{x \leq y, \Sigma; (AB)^x, (AB)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}{x \leq y, \Sigma; (AB)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y} (\text{L} \leq) \quad \dfrac{x \leq y, \Sigma; (AB)^x, (AB)^y, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y, \Sigma; (AB)^x, (AB)^y, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}} (\text{L} \leq)}{x \leq y, \Sigma; (AB)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}} \text{L}\supset_\leq$$

(6) Suppose the premiss of an instance of the L $\leq$ rule is the conclusion of an instance of the L$\supset_\leq$ rule, where $(A \supset B)^y$ is principal (using AB as an abbreviation for $A \supset B$):

$$\dfrac{\dfrac{x \leq y \leq z, \Sigma; (AB)^x, (AB)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^z \quad x \leq y \leq z, \Sigma; (AB)^x, (AB)^y, B^z, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \leq y \leq z, \Sigma; (AB)^x, (AB)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}} \text{L}\supset_\leq}{x \leq y \leq z, \Sigma; (AB)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}} (\text{L} \leq)$$

The $\mathsf{L}\leq$ rule can be permuted above the $\mathsf{L}\supset_{\leq}$ rule:

$$\dfrac{\dfrac{\dfrac{x\leq y\leq z,\Sigma;(AB)^x,(AB)^y,\underline{\Gamma}\Rightarrow\underline{\Delta},A^z}{x\leq y\leq z,\Sigma;(AB)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^z}\,(\mathsf{L}\leq)}{x\leq z,x\leq y\leq z,\Sigma;(AB)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^z}\,(\mathsf{LW}\leq)\quad \dfrac{\dfrac{x\leq y\leq z,\Sigma;(AB)^x,(AB)^y,B^z,\underline{\Gamma}\Rightarrow\underline{\Delta}}{x\leq y\leq z,\Sigma;(AB)^x,B^z,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,(\mathsf{L}\leq)}{x\leq z,x\leq y\leq z,\Sigma;(AB)^x,B^z,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,(\mathsf{LW}\leq)}{\dfrac{x\leq z,x\leq y\leq z,\Sigma;(AB)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}{x\leq y\leq z,\Sigma;(AB)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,\mathsf{trans}}\,\mathsf{L}\supset_{\leq}$$

(7) Suppose the premiss of an instance of the $\mathsf{R}\leq$ rule is the conclusion of an instance of the $\mathsf{R}\supset_{\leq}$ rule, such that the principal formula of the former is an active formula of the latter:

$$\dfrac{\dfrac{x\leq y,y\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},(A\supset B)^x,B^w}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(A\supset B)^x,(A\supset B)^y}\,\mathsf{R}\supset_{\leq}}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(A\supset B)^y}\,(\mathsf{R}\leq)$$

(w.l.o.g. we assume that $(A\supset B)^y$ is the principal formula). From the invertibility of $\mathsf{R}\supset_{\leq}$, the following can be derived

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x\leq y,y\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},(A\supset B)^x,B^w}{x\leq y,y\leq w,x\leq w',\Sigma;\underline{\Gamma},A^w,A^{w'}\Rightarrow\underline{\Delta},B^w,B^{w'}}\,(\mathsf{R}\supset_{\leq}^{-1})}{x\leq y,y\leq w,x\leq w,\Sigma;\underline{\Gamma},A^w,A^w\Rightarrow\underline{\Delta},B^w,B^w}\,[w/w']}{x\leq y,y\leq w,x\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},B^w,B^w}\,(\mathsf{LC})}{x\leq y,y\leq w,x\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},B^w}\,(\mathsf{RC})}{x\leq y,y\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},B^w}\,\mathsf{trans}}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(A\supset B)^y}\,\mathsf{R}\supset_{\leq}$$

which *eliminates* that instance of $\mathsf{R}\leq$.

□

REMARK G.8. Lemma 5.100 is shown semantically using Lemma 3.71 on page 64.

PROPOSITION G.9. *The rule*

$$\dfrac{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},P^x,P^y}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},P^y}\,\mathsf{R}\leq_0$$

is admissible in **RG3ipm***.*

Proof. By induction on the derivation depth. For the base case, suppose P^x is principal in an axiom. Then the following is derivable:

$$\frac{x \leq y, \Sigma; \underline{\Gamma}', P^x, P^y \Rightarrow \underline{\Delta}, P^y}{x \leq y, \Sigma; \underline{\Gamma}', P^x \Rightarrow \underline{\Delta}, P^y}\ \mathsf{L} \leq_0$$

Otherwise, the conclusion $x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, P^y$ is also an axiom.

For all other cases, because the active formulae are atomic and therefore not principal for any other rules, we permute instances of the $\mathsf{R} \leq_0$ rule upwards. □

LEMMA G.10 (Folding Rules). *The general* ***folding rules*** *are depth-preserving admissible in* **RG3ipm***:*

$$\frac{x \leq y, \Sigma; A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{\overline{x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}}\ (\mathsf{L} \leq) \qquad \frac{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{\overline{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}}\ (\mathsf{R} \leq)$$

Proof. Semantically, using Lemma 3.71 (without the depth-preserving clause). Or by simultaneous induction on the rank determined by the size of the principal formula and the derivation depth. We show the interesting cases below:

(1) Suppose that the the active formulae in an instance of $\mathsf{R} \leq$ are atomic. Then we use the $\mathsf{R} \leq_0$ rule from Proposition G.9.

(2) Suppose the premiss of an instance of the $\mathsf{R} \leq$ rule is the conclusion of an instance of the $\mathsf{R}\wedge$ rule, such that the principal formula of the former is an active formula of the latter:

$$\cfrac{\cfrac{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^x, A^y \quad x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^x, B^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^x, (A \wedge B)^y}\ \mathsf{R}\wedge}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^y}\ (\mathsf{R} \leq)$$

(w.l.o.g. we assume that $(A \wedge B)^y$ is the principal formula.) Using the invertibility of $\mathsf{R}\wedge$, we can permute $\mathsf{R} \leq$ above $\mathsf{R}\wedge$:

$$\cfrac{\cfrac{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ (\mathsf{R} \leq) \quad \cfrac{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, B^x, B^y}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, B^y}\ (\mathsf{R} \leq)}{x \leq y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^y}\ \mathsf{R}\wedge$$

The case for when the $\mathsf{L} \leq$ rule is below an instance of the $\mathsf{L}\vee$ rule is similar.

(3) Suppose the premiss of an instance of the $\mathsf{R} \leq$ rule is the conclusion of an instance of the $\mathsf{R}\vee$ rule, such that the principal formula of the former is an active

formula of the latter:

$$\dfrac{\dfrac{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \vee B)^x, A^y, B^y}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \vee B)^x, (A \vee B)^y}\ \mathsf{R}\vee}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \vee B)^y}\ (\mathsf{R} \le)$$

(w.l.o.g. we assume that $(A \vee B)^y$ is the principal formula.) Using the invertibility of $\mathsf{R}\vee$, we can permute $\mathsf{R} \le$ above $\mathsf{R}\vee$:

$$\dfrac{\dfrac{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, B^x, A^y, B^y}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y, B^y}\ (\mathsf{R} \le)^+}{x \le y, \Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \vee B)^y}\ \mathsf{R}\vee$$

The case for when the $\mathsf{L} \le$ rule is below an instance of the $\mathsf{L}\wedge$ rule is similar.

(4) Suppose the premiss of an instance of the $\mathsf{L} \le$ rule is the conclusion of an instance of the $\mathsf{L}{\supset_\le}$ rule, where $(A \supset B)^x$ is principal (using AB as an abbreviation for $A \supset B$):

$$\dfrac{\dfrac{x \le y, \Sigma; (AB)^x, (AB)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y \quad x \le y, \Sigma; (AB)^x, (AB)^y, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \le y, \Sigma; (AB)^x, (AB)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}{\supset_\le}}{x \le y, \Sigma; (AB)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ (\mathsf{L} \le)$$

The $\mathsf{L} \le$ rule can be permuted above the $\mathsf{L}{\supset_\le}$ rule:

$$\dfrac{\dfrac{x \le y, \Sigma; (AB)^x, (AB)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}{x \le y, \Sigma; (AB)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ (\mathsf{L} \le) \quad \dfrac{x \le y, \Sigma; (AB)^x, (AB)^y, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \le y, \Sigma; (AB)^x, (AB)^y, B^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ (\mathsf{L} \le)}{x \le y, \Sigma; (AB)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}{\supset_\le}$$

(5) Suppose the premiss of an instance of the $\mathsf{L} \le$ rule is the conclusion of an instance of the $\mathsf{L}{\supset_\le}$ rule, where $(A \supset B)^y$ is principal (using AB as an abbreviation for $A \supset B$):

$$\dfrac{\dfrac{x \le y \le z, \Sigma; (AB)^x, (AB)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^z \quad x \le y \le z, \Sigma; (AB)^x, (AB)^y, B^z, \underline{\Gamma} \Rightarrow \underline{\Delta}}{x \le y \le z, \Sigma; (AB)^x, (AB)^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}{\supset_\le}}{x \le y \le z, \Sigma; (AB)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ (\mathsf{L} \le)$$

The $\mathsf{L}\leq$ rule can be permuted above the $\mathsf{L}\supset_{\leq}$ rule:

$$\dfrac{\dfrac{\dfrac{x\leq y\leq z,\Sigma;(AB)^x,(AB)^y,\underline{\Gamma}\Rightarrow\underline{\Delta},A^z}{x\leq y\leq z,\Sigma;(AB)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^z}\,(\mathsf{L}\leq)}{x\leq z,x\leq y\leq z,\Sigma;(AB)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^z}\,(\mathsf{LW}\leq)\quad\dfrac{\dfrac{x\leq y\leq z,\Sigma;(AB)^x,(AB)^y,B^z,\underline{\Gamma}\Rightarrow\underline{\Delta}}{x\leq y\leq z,\Sigma;(AB)^x,B^z,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,(\mathsf{L}\leq)}{x\leq z,x\leq y\leq z,\Sigma;(AB)^x,B^z,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,(\mathsf{LW}\leq)}{\dfrac{x\leq z,x\leq y\leq z,\Sigma;(AB)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}{x\leq y\leq z,\Sigma;(AB)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,\mathsf{trans}}\,\mathsf{L}\supset_{\leq}$$

(6) Suppose the premiss of an instance of the $\mathsf{R}\leq$ rule is the conclusion of an instance of the $\mathsf{R}\supset_{\iota\leq}$ rule, such that the principal formula of the former is an active formula of the latter. Then there are two subcases:

(a) When the principal formulae of $\mathsf{R}\leq$ is also the principal formula of $\mathsf{R}\supset_{\iota\leq}$,

$$\dfrac{\dfrac{x\leq y\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},(A\supset B)^x,(A\supset B)^y,B^w}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(A\supset B)^x,(A\supset B)^y}\,\mathsf{R}\supset_{\iota\leq}}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(A\supset B)^y}\,(\mathsf{R}\leq)$$

then the instance of $\mathsf{R}\leq$ can be permuted upwards:

$$\dfrac{\dfrac{x\leq y\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},(A\supset B)^x,(A\supset B)^y,B^w}{x\leq y\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},(A\supset B)^y,B^w}\,(\mathsf{R}\leq)}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(A\supset B)^y}\,\mathsf{R}\supset_{\iota\leq}$$

(b) Otherwise, when $(A\supset B)^x$ is the principal formula of $\mathsf{R}\supset_{\iota\leq}$,

$$\dfrac{\dfrac{x\leq y,x\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},(A\supset B)^x,(A\supset B)^y,B^w}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(A\supset B)^x,(A\supset B)^y}\,\mathsf{R}\supset_{\iota\leq}}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(A\supset B)^y}\,(\mathsf{R}\leq)$$

then the derivation fragment can be rewritten as

$$\dfrac{\dfrac{\dfrac{\dfrac{x\leq y,x\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},(A\supset B)^x,(A\supset B)^y,B^w}{x\leq y\leq w,x\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},(A\supset B)^x,(A\supset B)^y,B^w}\,(\mathsf{LW}\leq)}{x\leq y\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},(A\supset B)^x,(A\supset B)^y,B^w}\,\mathsf{trans}}{x\leq y\leq w,\Sigma;\underline{\Gamma},A^w\Rightarrow\underline{\Delta},(A\supset B)^y,B^w}\,(\mathsf{R}\leq)}{x\leq y,\Sigma;\underline{\Gamma}\Rightarrow\underline{\Delta},(A\supset B)^y}\,\mathsf{R}\supset_{\iota\leq}$$

□

LEMMA G.11 (Root). *The **root** rule*

$$\dfrac{x\leq y_1,\ldots,x\leq y_n,\Sigma;\underline{S}}{\Sigma;\underline{S}}\,(\mathsf{root})$$

where $x\#\Sigma,\underline{S}$, *is admissible in* **RG3ipm**.

Proof. Semantically from root rule in **G3c/PSF***. The rule is shown also depth-preserving admissible by induction on the derivation depth. We note the interesting cases:

(1) Suppose the premiss of the root rule is an axiom. Then the conclusion is also an axiom.
(2) Suppose the premiss of the root rule is an instance of trans, where one of the active formula of root is principal for trans:

$$\dfrac{\dfrac{x \le y \le z, x \le z, \Sigma; \underline{S}}{x \le y \le z, \Sigma; \underline{S}}\ \text{trans}}{y \le z, \Sigma; \underline{S}}\ (\text{root})$$

Then the derivation fragment can be replaced with an instance of root which is at lower depth.
(3) For all other cases, the root rule can be permuted towards the axioms.

□

The root *rule cannot be applied to the principal formulae of the* $\mathsf{L}\le_0$ *or* $\mathsf{L}\supset_\le$ *rules, because it requires that the "root" label be fresh for the logical context.*

LEMMA G.12 (Root). *The* ***root*** *rule*

$$\dfrac{x \le y_1, \dots, x \le y_n, \Sigma; \underline{S}}{\Sigma; \underline{S}}\ (\text{root})$$

where $x\#\Sigma,\underline{S}$, *is admissible in* **G3I***.

Proof. Semantically from root rule in **G3c/PSF***. □

REMARK G.13. A proof of Lemma G.12 by induction on derivation depth, similar to proof for **RG3ipm** (Lemma 9.30 on page 205), fails for the case of the dir rule, because there is no primitive wk dir rule. However, wk dir is sound for **Jan**, and therefore **RG3ipm**+dir.

THEOREM G.14 (Interderivability). **RG3ipm*** $\vdash \Sigma;\underline{\Gamma} \Rightarrow \underline{\Delta}$ *iff* **G3I*** $\vdash \Sigma;\underline{\Gamma} \Rightarrow \underline{\Delta}$.

Proof. By induction on the derivation depth. We show the interesting cases below. From left-to-right:

(1) Suppose we have an axiom $\Sigma; P^x \Rightarrow P^x$. Then we can derive it by reflexivity:

$$\frac{x \leq x, \Sigma; P^x \Rightarrow P^x}{\Sigma; P^x \Rightarrow P^x} \text{ refl}$$

(2) Suppose we have a proof that ends in an instance of $\mathsf{R}{\supset_{\iota\leq}}$:

$$\frac{x \leq y, \Sigma; \underline{\Gamma}, A^y \Rightarrow \underline{\Delta}, B^y, (A \supset B)^x}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^x} \; \mathsf{R}{\supset_{\iota\leq}}$$

Then the same can be derived in **G3I*** using RC:

$$\dfrac{\dfrac{x \leq y, \Sigma; \underline{\Gamma}, A^y \Rightarrow \underline{\Delta}, B^y, (A \supset B)^x}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^x, (A \supset B)^x} \; \mathsf{R}{\supset_{\leq}}}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^x} \; (\mathsf{RC})$$

From right-to-left:

(1) Suppose we have an axiom $x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow A^y, \underline{\Delta}$. Then the following is derivable

$$\dfrac{\vdots}{\dfrac{x \leq y, \Sigma; A^x, A^y, \underline{\Gamma} \Rightarrow A^y, \underline{\Delta}}{x \leq y, \Sigma; A^x, \underline{\Gamma} \Rightarrow A^y, \underline{\Delta}} \; (\mathsf{L} \leq)}$$

using Lemma 9.34.

(2) Suppose we have a proof that ends in an instance of $\mathsf{R}{\supset_{\leq}}$:

$$\frac{x \leq y, \Sigma; \underline{\Gamma}, A^y \Rightarrow \underline{\Delta}, B^y}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^x} \; \mathsf{R}{\supset_{\leq}}$$

Then the following is derivable in **RG3ipm** using RW:

$$\dfrac{\dfrac{x \leq y, \Sigma; \underline{\Gamma}, A^y \Rightarrow \underline{\Delta}, B^y}{x \leq y, \Sigma; \underline{\Gamma}, A^y \Rightarrow \underline{\Delta}, B^y, (A \supset B)^x} \; \mathsf{RW}}{\Sigma; \underline{\Gamma} \Rightarrow \underline{\Delta}, (A \supset B)^x} \; \mathsf{R}{\supset_{\iota\leq}}$$

□

THEOREM G.15. *If* **LG3ipm**$^* \vdash \underline{\Gamma} \Rightarrow \underline{\Delta}$, *then* **RG3ipm**$^* \vdash \underline{\Gamma} \Rightarrow \underline{\Delta}$.

Proof. By induction on the derivation depth. The interesting cases are shown below.

(1) Suppose the proof ends in an instance of $\mathsf{L}{\supset}$:

$$\frac{(A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}, A^x \quad B^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}{(A \supset B)^x, \underline{\Gamma} \Rightarrow \underline{\Delta}} \; \mathsf{L}{\supset}$$

Then the following can be derived in **RG3ipm**:

$$\dfrac{\dfrac{\dfrac{(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^x}{x\le x;(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^x}\,(\mathsf{LW}\le)\qquad \dfrac{\dfrac{B^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}{(A\supset B)^x,B^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,\mathsf{LW}}{x\le x;(A\supset B)^x,B^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,(\mathsf{LW}\le)}{x\le x;(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,\mathsf{L}{\supset_{\le}}}{(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,\mathsf{refl}$$

(2) Suppose the proof ends in an instance of $\mathsf{R}{\supset}$:

$$\dfrac{\Gamma^x,A^x,\underline{\Gamma}'\Rightarrow\underline{\Delta}',B^x}{\Gamma^x,\underline{\Gamma}'\Rightarrow\underline{\Delta}',\Delta^x,(A\supset B)^x}\,\mathsf{R}{\supset}$$

where $x\#\underline{\Gamma}',\underline{\Delta}'$. Then the following can be derived in **RG3ipm**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma^x,A^x,\underline{\Gamma}'\Rightarrow\underline{\Delta}',B^x}{\Gamma^y,A^y,\underline{\Gamma}'\Rightarrow\underline{\Delta}',B^y}\,[\mathsf{y/x}]}{\Gamma^x,\Gamma^y,A^y,\underline{\Gamma}'\Rightarrow\underline{\Delta}',B^y,(A\supset B)^x}\,(\mathsf{GW})}{x\le y;\Gamma^x,\Gamma^y,A^y,\underline{\Gamma}'\Rightarrow\underline{\Delta}',B^y,(A\supset B)^x}\,(\mathsf{LW}\le)}{x\le y;\Gamma^x,A^y,\underline{\Gamma}'\Rightarrow\underline{\Delta}',B^y,(A\supset B)^x}\,(\mathsf{L}\le)^*}{\Gamma^x,\underline{\Gamma}'\Rightarrow\underline{\Delta}',(A\supset B)^x}\,\mathsf{R}{\supset_{\iota\le}}}{\Gamma^x,\underline{\Gamma}'\Rightarrow\underline{\Delta}',\Delta^x,(A\supset B)^x}\,\mathsf{RW}^*$$

(3) Suppose the proof ends in an instance of LQ. The rule LQ can be derived using dir (where active formulae are emphasised for clarity):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1^z,\Gamma_2^z,\underline{\Gamma}\Rightarrow\underline{\Delta}}{\Gamma_1^x,\Gamma_2^y,\Gamma_1^z,\Gamma_2^z,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,(\mathsf{EW})^+}{w\le x,w\le y,\mathbf{x}\le\mathbf{z},\mathbf{y}\le\mathbf{z};\mathbf{\Gamma_1^x},\mathbf{\Gamma_2^y},\mathbf{\Gamma_1^z},\mathbf{\Gamma_2^z},\underline{\Gamma}\Rightarrow\underline{\Delta}}\,(\mathsf{LW}\le)^*}{\mathbf{w}\le\mathbf{x},\mathbf{w}\le\mathbf{y},\mathbf{x}\le\mathbf{z},\mathbf{y}\le\mathbf{z};\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,(\mathsf{L}\le)^*}{\mathbf{w}\le\mathbf{x},\mathbf{w}\le\mathbf{y};\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,\mathsf{dir}}{\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta}}\,(\mathsf{root})$$

where the labels $w,x,y,z\#\underline{\Gamma},\underline{\Delta}$. (The relational context Σ has been omitted, because it is a translation of a labelled sequent rule, which has no relational context.)

(4) Suppose the proof ends in an instance of $\mathsf{Com_m}$. The rule $\mathsf{Com_m}$ can be derived using lin (where active formulae are emphasised for clarity):

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\Gamma_1^x,\Gamma_1^y,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_2^x,\Delta_2^y}{\mathbf{x\le y};\mathbf{\Gamma_1^x},\mathbf{\Gamma_1^y},\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_2^x,\Delta_2^y}\;(\mathsf{LW}\le)}{\mathbf{x\le y};\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\mathbf{\Delta_2^x},\mathbf{\Delta_2^y}}\;(\mathsf{L}\le)^*}{\mathbf{x\le y};\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_2^y}\;(\mathsf{R}\le)^*
\qquad
\dfrac{\dfrac{\dfrac{\Gamma_1^x,\Gamma_2^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_1^y,\Delta_2^y}{\mathbf{y\le x};\Gamma_1^x,\mathbf{\Gamma_2^x},\mathbf{\Gamma_2^y},\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_1^y,\Delta_2^y}\;(\mathsf{LW}\le)}{\mathbf{y\le x};\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\mathbf{\Delta_1^x},\mathbf{\Delta_1^y},\Delta_2^y}\;(\mathsf{L}\le)^*}{\mathbf{y\le x};\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_2^y}\;(\mathsf{R}\le)^*
}{\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_2^y}\;\mathsf{lin}
$$

where the labels $x,y\#\underline{\Gamma},\underline{\Delta}$.

(5) Suppose the proof ends in an instance of S. The rule S can be derived using sym (where active formulae are emphasised for clarity):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1^z,\Gamma_2^z,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^z,\Delta_2^z}{\Gamma_1^x,\Gamma_2^y,\Gamma_1^z,\Gamma_2^z,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_2^y,\Delta_1^z,\Delta_2^z}\;(\mathsf{EW})^+}{z\le x,\mathbf{x\le z},z\le y,\mathbf{y\le z};\mathbf{\Gamma_1^x},\mathbf{\Gamma_2^y},\mathbf{\Gamma_1^z},\mathbf{\Gamma_2^z},\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_2^y,\Delta_1^z,\Delta_2^z}\;(\mathsf{LW}\le)^+}{\mathbf{z\le x},x\le z,\mathbf{z\le y},y\le z;\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\mathbf{\Delta_1^x},\mathbf{\Delta_2^y},\mathbf{\Delta_1^z},\mathbf{\Delta_2^z}}\;(\mathsf{L}\le)^*}{\mathbf{z\le x},\mathbf{x\le z},\mathbf{z\le y},\mathbf{y\le z};\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_2^y}\;(\mathsf{R}\le)^*}{\mathbf{z\le x},\mathbf{z\le y};\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_2^y}\;\mathsf{sym}^+}{\Gamma_1^x,\Gamma_2^y,\underline{\Gamma}\Rightarrow\underline{\Delta},\Delta_1^x,\Delta_2^y}\;(\mathsf{root})
$$

where the labels $x,y,z\#\underline{\Gamma},\underline{\Delta}$.

□

LEMMA G.16. *The following rule*

$$
\dfrac{(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^x \qquad (A\supset B)^x,B^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}{(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\;\mathsf{L}{\supset_\iota}
$$

is admissible in **LG3ipm**.

Proof. By derivation.

$$
\dfrac{\dfrac{\dfrac{(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^x}{(A\supset B)^x,(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta},A^x}\;\mathsf{LW} \qquad (A\supset B)^x,B^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}{(A\supset B)^x,(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\;\mathsf{L}{\supset}}{(A\supset B)^x,\underline{\Gamma}\Rightarrow\underline{\Delta}}\;\mathsf{LC}
$$

□

LEMMA G.17. *The rule*

$$\frac{A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{L}\underset{\sim}{\subseteq}$$

is admissible in **LG3ipm**+S.

Proof. By derivation.

$$\dfrac{\dfrac{\dfrac{A^x, A^y, \underline{\Gamma} \Rightarrow \underline{\Delta}}{A^x, A^x, [x/y]\underline{\Gamma} \Rightarrow [x/y]\underline{\Delta}}\ (\mathsf{M})}{A^x, [x/y]\underline{\Gamma} \Rightarrow [x/y]\underline{\Delta}}\ (\mathsf{LC})}{A^x, \underline{\Gamma} \Rightarrow \underline{\Delta}}\ \mathsf{S}$$

□

LEMMA G.18. *The following rule*

$$\frac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y \quad \underline{\Gamma} \Rightarrow \underline{\Delta}, B^x, B^y}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^x, A^y}\ \mathsf{R}\wedge_\bullet$$

where $\underline{\Gamma} /\!\!/ x \underset{\sim}{\subseteq} \underline{\Gamma} /\!\!/ y$*, is admissible in* **LG3ipm**.

Proof. By derivation.

$$\dfrac{\dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^x, A^y}{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^y}\ (\mathsf{R}\underset{\sim}{\subseteq})}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^x, A^y}\ \mathsf{RW}$$

Note that the right premiss is irrelevant, but given for symmetry with the $\mathsf{L}\vee_\bullet$ rule. □

COROLLARY G.19. *The following rule*

$$\frac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^{x_1}, \ldots, A^{x_n} \quad \underline{\Gamma} \Rightarrow \underline{\Delta}, B^{x_1}, \ldots, B^{x_n}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^{x_1}, \ldots, (A \wedge B)^{x_n}}\ \mathsf{R}\wedge_\bullet^*$$

where $\underline{\Gamma} /\!\!/ x_i \underset{\sim}{\subseteq} \underline{\Gamma} /\!\!/ x_n$ *(for* $1 \leq i \leq n$*), is admissible in* **LG3ipm**.

Proof. By derivation. Apply the $\mathsf{R}\wedge_\bullet$ rule $n-1$ times to each of the premisses and then use the $\mathsf{R}\wedge$ rule:

$$\frac{\underline{\Gamma} \Rightarrow \underline{\Delta}, A^{x_1}, \ldots, A^{x_n}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^{x_1}, \ldots, (A \wedge B)^{x_{n-1}}, A^{x_n}}\ (\mathsf{R}\wedge_\bullet)^{n-1} \tag{151}$$

$$\dfrac{\vdots\ (151) \qquad \dfrac{\underline{\Gamma} \Rightarrow \underline{\Delta}, B^{x_1}, \ldots, B^{x_n}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^{x_1}, \ldots, (A \wedge B)^{x_{n-1}}, B^{x_n}}\ (\mathsf{R}\wedge_\bullet)^{n-1}}{\underline{\Gamma} \Rightarrow \underline{\Delta}, (A \wedge B)^{x_1}, \ldots, (A \wedge B)^{x_n}}\ \mathsf{R}\wedge$$

□

LEMMA G.20. *The following rule*

$$\frac{\underline{\Gamma} \Rightarrow B^x, (A \supset B)^x, \underline{\Delta}}{\underline{\Gamma} \Rightarrow (A \supset B)^x, \underline{\Delta}} \; \mathsf{RC}\supset$$

is admissible in **LG3ipm***.

Proof. By induction on the derivation depth. The interesting cases are shown below:

(1) Note that the premiss can not be an instance of Ax, as it contains no side formulae. However, suppose the premiss is a generalised axiom (Proposition 7.22 on page 159), derived by an axiom and a set of 0 or more instances of weakening and contractions: If B^x is the principal formula, then the conclusion can be derived,

$$\frac{\dfrac{B^x \Rightarrow B^x}{\underline{\Gamma}', A^x, B^x \Rightarrow B^x} \; \mathsf{LW}^+}{\underline{\Gamma}', B^x \Rightarrow (A \supset B)^x, \underline{\Delta}} \; \mathsf{R}\supset$$

where $\underline{\Gamma} = \underline{\Gamma}', B^x$. Otherwise, the conclusion is also a generalised axiom.

(2) Suppose the premiss is a result of an instance of the $\mathsf{R}\supset$ rule. There are the following subcases:

   (a) $B^x = (C \supset D)^x$ is the principal formula. Then $(A \supset B)^x$ in the premiss is the result of weakening in the $\mathsf{R}\supset$ rule. The following can be derived instead:

$$\frac{\dfrac{\dfrac{\dfrac{\underline{\Gamma}, C^x \Rightarrow D^x, \underline{\Delta}'}{\underline{\Gamma}, A^x, C^x \Rightarrow D^x, \underline{\Delta}'} \; \mathsf{LW}}{\underline{\Gamma}, A^x \Rightarrow (C \supset D)^x, \underline{\Delta}'} \; \mathsf{R}\supset}{\underline{\Gamma}, A^x \Rightarrow B^x, \underline{\Delta}'} \; \mathsf{B = C \supset D}}{\underline{\Gamma} \Rightarrow (A \supset B)^x, \underline{\Delta}} \; \mathsf{R}\supset$$

   where $\underline{\Delta}' = \underline{\Delta} \setminus (\underline{\Delta} /\!\!/ x)$.

   (b) $(A \supset B)^x$ is the principal formula. Then B^x is the result of weakening in the $\mathsf{R}\supset$ rule, so the conclusion can be derived instead.

   (c) $(C \supset D)^x \in \underline{\Delta}$ is the principal formula. Then both B^x and $(A \supset B)^x$ are the result of weakening in the $\mathsf{R}\supset$ rule, so the conclusion can be derived instead.

   (d) Otherwise by the the induction hypothesis, the conclusion is derivable.

□

APPENDIX H

Translating from Relational Sequents to Labelled Sequents

In this appendix, we examine a potential method to extend the translation to weaker logics than **GD**, at least for a subset of proofs, by tracking the significant labels of a derivation. Below are the partial results, which have not been successful. We believe the ideas are worth including here, in particular that a form of $\mathsf{L}{\vee_\bullet}$ is admissible (Lemma H.9 on page 293) in a calculus for **Int** when certain conditions hold.

Definition H.1 (Annotated Simply Labelled Sequent). An annotated simply labelled sequent $\underline{\Gamma} \Rightarrow^{M} \underline{\Delta}$ is a simply labelled sequent where the sequent arrow is annotated with a set of labels M that denotes the **significant label(s)** of the sequent.

Remark H.2. It is assumed that an annotated simply labelled sequent is derivable.

Remark H.3. The purpose of the M (mix) rule in **LG3ipm$_\bullet$** (Figure H.1 on the following page) is to account for how the same sequent to be derived with different significant labels, for example,

$$\frac{A^x \Rightarrow^{\{x\}} A^x \quad A^y \Rightarrow^{\{y\}} A^y}{A^x, A^y \Rightarrow^{\{x,y\}} A^x, A^y}\ \mathsf{M}$$

The end sequent can be derived from either premiss by weakening, but without the combined significant labels.

The mix rules also allow the the inverted forms of the logical rules to be admissible.

Theorem H.4 (Interderivability). **LG3ipm$_\bullet$** $\vdash \underline{\Gamma} \Rightarrow^{M} \underline{\Delta}$ *iff* **LG3ipm** $\vdash \underline{\Gamma} \Rightarrow \underline{\Delta}$, *where* $M \subseteq \mathtt{lab}(\underline{\Gamma}, \underline{\Delta})$.

Proof. Straightforward, by induction on the derivation height. (Note that the M rule is admissible in **LG3ipm** by GW.) $\square$

Corollary H.5. *If a rule ρ*

$$\frac{\underline{\Gamma}'_1 \Rightarrow \underline{\Delta}'_1 \quad \underline{\Gamma}'_2 \Rightarrow \underline{\Delta}'_2}{\underline{\Gamma} \Rightarrow \underline{\Delta}}\ \rho$$

$$\frac{}{P^x \Rightarrow^{\{x\}} P^x}\ \mathsf{Ax} \qquad \frac{}{\bot^x \Rightarrow^{\{x\}}}\ \mathsf{L}\bot$$

$$\frac{\underline{\Gamma}, A^x, B^x \Rightarrow^M \underline{\Delta}}{\underline{\Gamma}, A \wedge B^x \Rightarrow^M \underline{\Delta}}\ \mathsf{L}\wedge \qquad \frac{\underline{\Gamma} \Rightarrow^M A^x, \underline{\Delta} \quad \underline{\Gamma} \Rightarrow^{M'} B^x, \underline{\Delta}}{\underline{\Gamma} \Rightarrow^{M \cup M'} A \wedge B^x, \underline{\Delta}}\ \mathsf{R}\wedge$$

$$\frac{\underline{\Gamma}, A^x \Rightarrow^M \underline{\Delta} \quad \underline{\Gamma}, B^x \Rightarrow^{M'} \underline{\Delta}}{\underline{\Gamma}, A \vee B^x \Rightarrow^{M \cup M'} \underline{\Delta}}\ \mathsf{L}\vee \qquad \frac{\underline{\Gamma} \Rightarrow^M A^x, B^x, \underline{\Delta}}{\underline{\Gamma} \Rightarrow^M A \vee B^x, \underline{\Delta}}\ \mathsf{R}\vee$$

$$\frac{(A \supset B)^x, \underline{\Gamma} \Rightarrow^M \underline{\Delta}, A^x \quad B^x, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}}{(A \supset B)^x, \underline{\Gamma} \Rightarrow^{M \cup M'} \underline{\Delta}}\ \mathsf{L}\supset \qquad \frac{A^x, \underline{\Gamma}' \Rightarrow^M \underline{\Delta}', B^x}{\underline{\Gamma}' \Rightarrow^M \underline{\Delta}', \Delta^x, (A \supset B)^x}\ \mathsf{R}\supset$$

$$\frac{\underline{\Gamma} \Rightarrow^M \underline{\Delta}}{\underline{\Gamma}, A^x \Rightarrow^M \underline{\Delta}}\ \mathsf{LW} \qquad \frac{\underline{\Gamma} \Rightarrow^M \underline{\Delta}}{\underline{\Gamma} \Rightarrow^M A^x, \underline{\Delta}}\ \mathsf{RW}$$

$$\frac{\underline{\Gamma}, A^x, A^x \Rightarrow^M \underline{\Delta}}{\underline{\Gamma}, A^x \Rightarrow^M \underline{\Delta}}\ \mathsf{LC} \qquad \frac{\underline{\Gamma} \Rightarrow^M A^x, A^x, \underline{\Delta}}{\underline{\Gamma} \Rightarrow^M A^x, \underline{\Delta}}\ \mathsf{RC}$$

$$\frac{\underline{\Gamma}_1 \Rightarrow^M \underline{\Delta}_1 \quad \underline{\Gamma}_2 \Rightarrow^{M'} \underline{\Delta}_2}{\underline{\Gamma}_1, \underline{\Gamma}_2 \Rightarrow^{M \cup M'} \underline{\Delta}_1, \underline{\Delta}_2}\ \mathsf{M}$$

FIGURE H.1. The annotated simply labelled calculus **LG3ipm$_\bullet$**.

is admissible in **LG3ipm***, then the corresponding rule*

$$\frac{\underline{\Gamma}_1' \Rightarrow^M \underline{\Delta}_1' \quad \underline{\Gamma}_2' \Rightarrow^M \underline{\Delta}_2'}{\underline{\Gamma} \Rightarrow^{M \cup M'} \underline{\Delta}}\ \rho$$

is admissible in **LG3ipm$_\bullet$**.

Proof. Using the same steps as the proof for **LG3ipm**. □

LEMMA H.6 (Label substitution). *The following label substitution rule*

$$\frac{\underline{\Gamma} \Rightarrow^M \underline{\Delta}}{[y/x]\underline{\Gamma} \Rightarrow^{[y/x]M} [y/x]\underline{\Delta}}\ [\mathsf{y/x}]$$

where x is explicit, is admissible in **LG3ipm$_\bullet$**.

Proof. By induction on the derivation depth. □

REMARK H.7. Although Lemma H.6 on the preceding page follows from Corollary H.5 on page 291, it is given explicitly to emphasise that the set of significant labels is also affected by the substitution.

LEMMA H.8. *If* $\mathbf{LG3ipm}_{\bullet} \vdash A^x, \underline{\Gamma} \Rightarrow^M \underline{\Delta}$ *or* $\mathbf{LG3ipm}_{\bullet} \vdash \underline{\Gamma} \Rightarrow^M \underline{\Delta}, A^x$*, where* $x \notin M$*, then* $\mathbf{LG3ipm}_{\bullet} \vdash \underline{\Gamma} \Rightarrow^M \underline{\Delta}$.

Proof. By induction on the derivation depth. □

LEMMA H.9. *The rule*

$$\frac{A^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta} \quad B^x, B^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}}{A^x, (A \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}} \ (\mathsf{L}\vee_{\bullet})$$

where $M \cap M' \neq \emptyset$ *and* $\underline{\Delta} /\!/ y \precsim \underline{\Delta} /\!/ x$*, is admissible in* $\mathbf{LG3ipm}_{\bullet}$.

Proof. By induction on the rank determined by the sum of the sizes of A and B, and the derivation depth of the left premiss. The cases are given below:

(1) The base case is a generalised axiom of derivation depth 0, where A is atomic.

Suppose the principal formula is A^y. Then $A^y \in \underline{\Delta}$. But by the constraint on succedent, $A^x \in \underline{\Delta}$. So A^x can be considered the principal formula of the axiom, and the conclusion is also a generalised axiom.

For all other subcases, the conclusion is also a generalised axiom.

Note that this case applies to generalised axioms of higher derivation depths as well.

(2) Suppose $A = \bot$. Then the conclusion is trivially derivable.

(3) Suppose $A = C \wedge D$. Then the following can be derived, by applying the $\mathsf{L}\vee_{\bullet}$ rule to C and D:

$$\frac{\dfrac{A^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{C^x, C^y, D^x, D^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}} \ (\mathsf{L}\wedge^{-1})^+ \quad \dfrac{B^x, B^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}}{B^x, B^y, D^x, D^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}} \ \mathsf{LW}^+}{C^x, (C \vee D)^y, D^x, D^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}} \ (\mathsf{L}\vee_{\bullet}) \tag{152}$$

$$\frac{\dfrac{\dfrac{\vdots \ (152)}{D^x, D^y, C^x, (C \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}} \quad \dfrac{B^x, B^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}}{B^x, B^y, C^x, (C \vee B)^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}} \ \mathsf{LW}^+}{D^x, (D \vee B)^y, C^x, (C \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}} \ (\mathsf{L}\vee_{\bullet})}{(C \wedge D)^x, (D \vee B)^y, (C \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}} \ \mathsf{L}\wedge \tag{153}$$

$$\dfrac{\dfrac{\dfrac{\vdots\ (153)}{(C\wedge D)^x, C^y, D^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_1^{-1})^+}{(C\wedge D)^x, (C\wedge D)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ \mathsf{L}\wedge \qquad \dfrac{\dfrac{\vdots\ (153)}{(C\wedge D)^x, B^y, B^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_2^{-1})^+}{(C\wedge D)^x, B^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ \mathsf{LC}}{\dfrac{(C\wedge D)^x, ((C\wedge D)\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{A^x, (A\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ \mathsf{A = C\wedge D}}\ \mathsf{L}\vee$$

(4) Suppose $A = C \vee D$. Then the following can be derived, by applying the $\mathsf{L}\vee_\bullet$ rule to C and D:

$$\dfrac{\dfrac{A^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{D^x, D^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_2^{-1})^+ \qquad B^x, B^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}}{D^x, (D\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_\bullet) \tag{154}$$

$$\dfrac{\dfrac{B^x, B^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta} \qquad \dfrac{A^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{D^x, D^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_2^{-1})^+}{B^x, (B\vee D)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_\bullet)}{B^x, (D\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L\vee com}) \tag{155}$$

$$\dfrac{\dfrac{\dfrac{A^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{C^x, C^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_1^{-1})^+ \qquad \dfrac{\vdots\ (154) \qquad \vdots\ (155)}{(D\vee B)^x, (D\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ \mathsf{L}\vee}{C^x, (C\vee(D\vee B))^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_\bullet)}{C^x, ((C\vee D)\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L\vee com/ass}) \tag{156}$$

$$\dfrac{\dfrac{A^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{C^x, C^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_1^{-1})^+ \qquad B^x, B^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}}{C^x, (C\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_\bullet) \tag{157}$$

$$\dfrac{\dfrac{B^x, B^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta} \qquad \dfrac{A^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{C^x, C^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_1^{-1})^+}{B^x, (B\vee C)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_\bullet)}{B^x, (C\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L\vee com}) \tag{158}$$

$$\dfrac{\vdots\ (156) \qquad \dfrac{\dfrac{\dfrac{A^x, A^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{D^x, D^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_2^{-1})^+ \qquad \dfrac{\vdots\ (157) \qquad \vdots\ (158)}{(C\vee B)^x, (C\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ \mathsf{L}\vee}{D^x, (C\vee(D\vee B))^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L}\vee_\bullet)}{D^x, ((C\vee D)\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ (\mathsf{L\vee com/ass})}{\dfrac{(C\vee D)^x, ((C\vee D)\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{A^x, (A\vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\ \mathsf{A = C\vee D}}\ \mathsf{L}\vee$$

(5) Suppose $A = C \supset D$. Then the following can be derived, by applying the $\mathsf{L}\vee_\bullet$ rule to C and D:

$$\dfrac{\dfrac{(C \supset D)^x, (C \supset D)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{D^x, D^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\,(\mathsf{L}\supset_2^{-1})^+ \qquad B^x, B^y, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}}{D^x, (D \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\,(\mathsf{L}\vee_\bullet) \tag{159}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(C \supset D)^x, (C \supset D)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{(C \supset D)^x, (C \supset D)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}, C^y}\,(\mathsf{L}\supset_1^{-1})}{(C \supset D)^x, ((C \supset D) \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}, C^y}\,(\mathsf{IH})}{(C \supset D)^x, (C \supset (D \vee B))^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}, C^y}\,(\mathsf{L}\supset\vee^{-1})}{D^x, (C \supset (D \vee B))^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}, C^y}\,(\mathsf{L}\supset_2^{-1}) \qquad \vdots\,(159)}{D^x, (C \supset (D \vee B))^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\,\mathsf{L}\supset}{D^x, ((C \supset D) \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\,(\mathsf{L}\supset\vee) \tag{160}$$

$$\dfrac{\dfrac{\dfrac{(C \supset D)^x, (C \supset D)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{(C \supset D)^x, (C \supset D)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}, C^x}\,(\mathsf{L}\supset_1^{-1})}{(C \supset D)^x, ((C \supset D) \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}, C^x}\,(\mathsf{IH}) \qquad \vdots\,(160)}{(C \supset D)^x, ((C \supset D) \vee B)^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}\,\mathsf{L}\supset$$

□

REMARK H.10. It is important for the premisses of the $\mathsf{L}\vee_\bullet$ rule to share a significant label. Otherwise we have the following counterexample:

$$\dfrac{A^x, A^y, (A \supset C)^y \Rightarrow^{\{y\}} C^x, C^y, B^x \qquad B^x, B^y, (A \supset C)^y \Rightarrow^{\{x\}} C^x, C^y, B^x}{A^x, (A \vee B)^y, (A \supset C)^y \Rightarrow^{\{y\}} C^x, C^y, B^x}$$

Both premisses are derivable, but the conclusion is not.

While we have a potentially useful result on when the $\mathsf{L}\vee_\bullet$ rule is admissible in a weaker logic, we have not found a way to apply the technique to relational sequent calculi so as to identify proofs that can be translated without the use of $\mathsf{Com_m}$.

In particular, it's not clear how the significant label in an instance of $\mathsf{L} \preceq_0$ is handled:

$$\dfrac{\Sigma; P^x, P^y, \underline{\Gamma} \Rightarrow^M \underline{\Delta}}{\Sigma; P^x, \underline{\Gamma} \Rightarrow^{M'} \underline{\Delta}}\,\mathsf{L} \preceq_0$$

If $y \in M'$, it is not obvious whether $y \in M$ or $x \in M$. In particular, the relational proofs in Examples 9.50 on page 218 and 9.51 suggest conflicting answers to this.

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